# Antichains in the homomorphism order of graphs

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*Abstract.* Let $\mathbb{G}$ and $\mathbb{D}$, respectively, denote the partially ordered sets of homomorphism classes of finite undirected and directed graphs, respectively, both ordered by the homomorphism relation. Order theoretic properties of both have been studied extensively, and have interesting connections to familiar graph properties and parameters. In particular, the notion of a duality is closely related to the idea of splitting a maximal antichain. We construct both splitting and non-splitting infinite maximal antichains in $\mathbb{G}$ and in $\mathbb{D}$. The splitting maximal antichains give infinite versions of dualities for graphs and for directed graphs.

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## 1. Introduction

For any fixed type of finite relational structure, homomorphisms induce an ordering of the set of all structures. In particular, given two graphs (respectively, directed graphs) $G$ and $H$ write $G \leq H$ or $G \to H$ provided that there is a homomorphism from $G$ to $H$, that is, a map $f : V(G) \to V(H)$ such that for all $\{x, y\} \in E(G)$, $\{f(x), f(y)\} \in E(H)$ (respectively, for all $\langle x, y\rangle \in E(G)$, $\langle f(x), f(y)\rangle \in E(H)$). Then the relation $\leq$ is a quasi-order and so it induces an equivalence relation: we say that $G$ and $H$ are *homomorphism-equivalent* or *hom-equivalent* and write $G \sim H$ if and only if $G \leq H$ and $H \leq G$. The *homomorphism posets* $\mathbb{G}$ and $\mathbb{D}$ are the partially ordered sets of all equivalence classes of finite undirected and directed graphs, respectively, ordered by $\leq$. We will often abuse notation by replacing the classes that comprise $\mathbb{G}$ and $\mathbb{D}$ with their members.

These partially ordered sets are of significant intrinsic interest and are useful tools in the study of graph and directed graph properties. For instance, it is easily seen that both are countable distributive lattices: the supremum, or

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join, of any pair is their disjoint sum, and the infimum, or meet, is their categorical or relational product. Both $\mathbb{G}$ and $\mathbb{D}$ are "predominantly" dense — the former shown by Welzl [20] and the latter, by Nešetřil and Tardif [16]. Both also embed all countable partially ordered sets — see [19] for a presentation. Basic order-theoretic properties, such as the existence of suprema and infima for several natural families in $\mathbb{G}$, are considered in [13].

The maximal chains and antichains of an ordered set are subobjects of interest. In this case, maximal antichains are particularly relevant because of their relationship to the notion of a *homomorphism duality*, introduced by Nešetřil and Pultr [14]: say that an ordered pair $\langle F, D\rangle$ of graphs, or directed graphs, is a *duality pair* if

$$F\rightarrow \; = \; \nrightarrow D \tag{1}$$

where $F\rightarrow\, = \{G : F \rightarrow G\}$ and $\nrightarrow D = \{G : G \nrightarrow D\}$. Equivalently, the set of all structures is partitioned by the *upset* (or *final segment*) $F\rightarrow$ and the *downset* (or *initial segment*) $\rightarrow D$. (Here we also use the other common notation $F^{\uparrow}$ and $D^{\downarrow}$ for upsets and downsets, respectively.)

One important motivation for consideration of duality pairs is that of an "obstruction" to a graph property. For instance, the possibility of a homomorphism of a graph $G$ to $K_2$, a 2-coloring, is obstructed by the existence of a homomorphism of some odd cycle to $G$. While there are no nontrivial duality pairs in $\mathbb{G}$, in $\mathbb{D}$, each tree can play the role of $F$ in (1). In fact, in [16], Nešetřil and Tardif obtain a correspondence between duality pairs and gaps in the homomorphism order for general relational structures. They use this to characterize duality pairs and generalize this by describing exactly when the left handside of (1) can be replaced by a finite union of final segments. They further note in [17] that the 2-element maximal antichains in $\mathbb{D}$ are exactly the duality pairs $\langle F, D\rangle$ where $F$ is a tree and $D$ is its dual.

Foniok, Nešetřil and Tardif [10] are concerned with the most general circumstance. Let $\mathcal{F}$ and $\mathcal{D}$ both be finite antichains of structures of fixed type $\Delta$. Call $\langle \mathcal{F}, \mathcal{D}\rangle$ a *generalized duality* if

$$\bigcup_{F\in\mathcal{F}} F\rightarrow \; = \bigcap_{D\in\mathcal{D}} \nrightarrow D. \tag{2}$$

Equivalently, with $\mathbb{S}$ denoting the homomorphism poset of $\Delta$-structures, $\mathbb{S}$ is partitioned by

$$\mathbb{S} = \left(\bigcup_{F\in\mathcal{F}} F\rightarrow\right) \cup \left(\bigcup_{D\in\mathcal{D}} \rightarrow D\right). \tag{3}$$

The generalized dualities are characterized in [10]. They also show that when $\Delta$ consists of one $k$-ary relation, which contains the graph cases, every finite maximal antichain in the lattice of $\Delta$-structures is of the form $\mathcal{F} \cup \mathcal{D}$. Conversely, for all but three exceptional cases, the generalized dualities $\langle \mathcal{F}, \mathcal{D} \rangle$ yield a maximal antichain $\mathcal{F} \cup \mathcal{D}$.

It is quite natural to ask, in more general circumstances, if maximal antichains possess these sorts of partitions. Indeed, Ahlswede, Erdős and Graham [1] introduced the notion of "splitting" a maximal antichain. Say that a maximal antichain $A$ of a poset $P$ *splits* if $A$ can be partitioned into two subsets $B$ and $C$ such that $P = B^{\uparrow} \cup C^{\downarrow}$; say that $P$ has the *splitting property* if all of its maximal antichains split. They obtained sufficient conditions for the splitting property, from which they proved, in particular, that all finite Boolean lattices possess it. The property is also a useful tool in combinatorial investigations of posets, particularly distributive lattices; see, for instance [2], [3]. It is also a natural notion for infinite posets; see [4], [8], [9].

The correspondence between generalized dualities and maximal antichains obtained in [10] and the partition in (3) demonstrate that for $\Delta = (k)$, essentially all finite maximal antichains in the lattice $\mathbb{S}$ of $\Delta$-structures split.

This paper is motivated by two goals. First, we would like to obtain general order theoretic conditions on countable posets that ensure antichains split and, thereby, obtain some of the duality results that had been restricted to finite maximal antichains, as described above. See Section 4 for applications to $\mathbb{G}$ and Section 5 for results on $\mathbb{D}$. Second, we obtain splitting and non-splitting results for infinite maximal antichains; in particular, these results underscore differences between the structures $\mathbb{G}$ and $\mathbb{D}$. The necessary results on splitting and related notions are given in Section 3, which is preceded in Section 2 by a directed version of what is known as the Sparse Incomparability Lemma.

In addition to the selected papers cited in this section, we refer the reader to the book [11] by Hell and Nešetřil that is devoted to graph homomorphisms. Chapter 3 gives a thorough introduction and many of the key results on maximal antichains and dualities in $\mathbb{G}$ and $\mathbb{D}$.

## 2. A Directed Sparse Incomparability Lemma

Recall that the *girth* of a graph, $\operatorname{girth}(G)$, is the length of a shortest cycle contained in the graph. In case $G$ is directed, its girth is that of the underlying undirected graph, that is, of the symmetric version of $G$. In one of the first applications of the probabilistic method, in 1959 Paul Erdős [5] showed the existence of graphs with independently prescribed girth and chromatic number. More precisely, for all natural numbers $k$ and $\ell$ there is a graph $G$ such that $\chi(G) > k$ and $\operatorname{girth}(G) > \ell$.

Based on another probabilistic argument due to Erdős and Hajnal [6], Nešetřil and Rödl [15] obtained an interesting generalization, referred to as the "Sparse Incomparability Lemma": for every pair of graphs $H$ and $G$ such that $G \to H$ but $H \nrightarrow G$, and for every positive integer $\ell$ there exists a graph $H'$ with $\operatorname{girth}(H') > \ell$

such that $H' \to H$ and $H' \nrightarrow G$.

Here is a formulation from which the Sparse Incomparability Lemma follows, itself a special case of a more far-reaching generalization.

**Theorem 2.1** (Nešetřil-Zhu [18]). *For every graph $H$ and for all positive integers $k$ and $\ell$ there exists a graph $G$ with the following properties:*

(i) $\mathrm{girth}(G) > \ell$, *and*

(ii) *for every graph $H_0$ with at most $k$ vertices, $G \to H_0$ if and only if $H \to H_0$.*

Here, we require a directed graph version of Theorem 2.1. The following is a special case of a Sparse Incomparability Lemma for finite relational structures [12].

**Theorem 2.2** (Directed Sparse Incomparability Lemma). *For each directed graph $H = (W, F)$ and for all integers $m, \ell \in \mathbb{N}$ there is a directed graph $H'$ such that*

(i) $\mathrm{girth}(H') > \ell$,

(ii) *for each directed graph $G$ with $|V(G)| < m$ we have $H' \to G$ if and only if $H \to G$, and*

(iii) *$H$ and $H'$ have the same numbers of connected components. In particular, if $H$ is connected then so is $H'$.*

Regarding the proof of Theorem 2.2, there are both probabilistic and deterministic arguments available. For instance, it is straightforward to adapt the probabilistic proof of Nešetřil-Rödl. We found an alternative approach based on what appears to be a new graph parameter. Here is a brief outline of the argument.

Given a graph $G = (V, E)$ let the *bipartite stability number* $\alpha_b(G)$ be the maximum integer $\beta$ such that:

$$\exists\, A, B \in [V]^\beta \ \text{ with } \ A \cap B = \emptyset \ \text{ and no edge between } A \text{ and } B.$$

Clearly $\alpha_b(G) \geq \alpha(G)/2$, where $\alpha(G)$ denotes the usual stability or independence number of $G$. The following result is obtained by adjusting the Erdős-Rényi proof [7] that there are graphs of large girth and small independence number.

**Lemma 2.3.** *For all $k, \ell \in \mathbb{N}$ and for all but finitely many $n \in \mathbb{N}$ there exists a connected graph $G' = (V, E)$ with $|V| = n$, $\mathrm{girth}(G') > \ell$ and $\alpha_b(G') < n/k$.*

Let $H$, $m$ and $\ell$ be as in the statement of Theorem 2.2. Let $k = 3m|W|$ and $n = kj$ for sufficiently large $j$. By Lemma 2.3, there exists a graph $G' = (V, E)$ such that

- $V = W \times [3mj]$,
- $\mathrm{girth}(G') > \ell$,
- $\alpha_b(G') < n/k = j$.

In effect, we "blow up" each vertex of $H$ into a class of $3mj$ vertices.

Define a directed graph $H^* = (V, E^*)$ as follows: if $\langle h, i\rangle, \langle h', i'\rangle \in V$ then $\langle \langle h, i\rangle, \langle h', i'\rangle\rangle \in E^*$ if and only if $(\langle h, i\rangle, \langle h', i'\rangle) \in E$ and $\langle h, h'\rangle \in F$.

One now argues that if $H$ is connected then $H^*$ has a large enough connected component that satisfies (i), (ii) and (iii) of the theorem.

## 3. The splitting property

In the forthcoming sections we would like to apply some results from [9] to the posets $\mathbb{G}$ and $\mathbb{D}$ to obtain antichains with certain properties related to dualities and partitions of $\mathbb{G}$ and $\mathbb{D}$. Concerning $\mathbb{G}$ it would be enough just to quote some theorems from [9], but concerning $\mathbb{D}$ we should reformulate them a bit to make them applicable here.

Let $\mathcal{P} = (P, \leq)$ be a poset. (We find it useful sometimes to maintain a distinction between $\mathcal{P}$ and the underlying set $P$.) We say that a subset $A \subset P$ is *cut-free in* $\mathcal{P}$ provided there are no $y \in A$ and $x, z \in P$ such that $x < y < z$ and $A \cap [x, z] = A \cap ([x, y] \cup [y, z])$. An element $y \in P$ is called *cut-point* iff there are $x, z \in P$ such that $x < y < z$ and $[x, z] = [x, y] \cup [y, z]$. Clearly there is no cut-point in a cut-free set.

If $\mathcal{P} = (P, <)$ is a poset and $A \subset P$ then recall that the *upset* $A^{\uparrow}$ and the *downset* $A^{\downarrow}$ of $A$ are the sets

$$A^{\uparrow} = \{p \in P : \exists\, a \in A\ a \leq p\}, \quad A^{\downarrow} = \{p \in P : \exists a \in A\ p \leq a\};$$

also, use this natural extension of the notation,

$$A^{\updownarrow} = A^{\uparrow} \cup A^{\downarrow}.$$

As usual, we drop the braces and write $a^{\uparrow}$, $a^{\downarrow}$, and $a^{\updownarrow}$ in place of $\{a\}^{\uparrow}$, $\{a\}^{\downarrow}$ and $\{a\}^{\updownarrow}$, respectively.

A *maximal antichain* $A$ in $P$ is a set of pairwise incomparable elements (an antichain) that is maximal with respect to containment. We say that a maximal antichain $A$ *split*s if there is a partition $(B, C)$ of $A$ such that $P = B^{\uparrow} \cup C^{\downarrow}$. We say that $A$ *strongly splits* if and only if there is a partition $(B, C)$ of $A$ such that for each $p \in P \setminus A$ either the set $B \cap p^{\downarrow}$ or the set $C \cap p^{\uparrow}$ is infinite.

To construct maximal antichains with desired properties (for instance, splitting or non-splitting), it is useful to be able to extend existing finite antichains to maximal ones in certain special ways. This motivated Erdős and Soukup [9] to formulate this definition: call $\mathcal{P}$ *loose* if for all $x \in P$ and $F \in [P]^{<\omega}$, if $x \notin F^{\uparrow}$ there is $y \in x^{\uparrow} \setminus \{x\}$ that is incomparable to all elements in $F$. This property is the key in showing that very familiar infinite distributive lattices, such as $([\omega]^{<\omega}, \subseteq)$, the lattice of finite subsets of a countably infinite set, do not have the splitting property. We shall see that the nontrivial part of $\mathbb{G}$ has this property (see Theorem 4.1) but that $\mathbb{D}$ requires a sharpening of the definition (see Theorem 5.1).

**Definition 3.1.** Let $\mathcal{P} = \langle P, \leq\rangle$ be a poset and let $P' \subset P$. We say that $P'$ is a *loose kernel* in $\mathcal{P}$ if

(LK) for all finite subsets $F \subseteq P'$ and $x \in P \setminus F^{\uparrow}$ there is $y \in [x^{\uparrow} \cap P']$, $y \neq x$, such that each element of $F$ is incomparable to $y$.

Of course, $\mathcal{P}$ is *loose* if $P$ itself is a loose kernel in $\mathcal{P}$, just as in [9].

**Remarks. (1)** A loose kernel $P'$ in a poset $\mathcal{P}$ does not have maximal elements — just take $F = \emptyset$ in (LK) in the definition and any $x$ in $P'$ to produce $y \in P'$ such that $y > x$. In particular, $P'$ is infinite. Also, if $\mathcal{P}$ contains a loose kernel then there is a loose kernel of $\mathcal{P}$ that is maximal, with respect to containment. This is easily shown using Zorn's Lemma.

**(2)** Regarding the homomorphism poset $\mathbb{D}$, it is not loose since it has finite maximal antichains — a finite maximal antichain as $F$ in (LK) shows that (LK) fails. Moreover, $\mathbb{D}$ has infinitely many finite maximal antichains, so we cannot obtain a loose kernel for $\mathbb{D}$ by deleting finitely many elements, as we can for $\mathbb{G}$.

Here is a condition that allows extension of a finite antichain in a particular special way.

**Definition 3.2.** Let $\mathcal{P} = \langle P, \leq\rangle$ be a poset and $P' \subset P$. We say that $P'$ has the *finite antichain extension property* (in $\mathcal{P}$) provided

(FAE) for all finite antichains $F \subseteq P'$ and $x \in P \setminus F$ there is $y \in [x^{\updownarrow} \cap P']$ such that each element of $F$ is incomparable to $y$.

**Observation 3.3.** *If $P' \subset P$ is both a loose kernel in $\mathcal{P} = (P, \leq)$ and a loose kernel in the dual $\mathcal{P}^d = (P, \geq)$ then $P'$ has the finite antichain extension property in $\mathcal{P}$.*

The following observation is a sharpening of [9, Theorem 3.9]. We include the straightforward proof to illustrate how the FAE property can be applied.

**Theorem 3.4.** *Let $\mathcal{P} = \langle P, \leq\rangle$ be a countably infinite poset, let $P' \subset P$ have the finite antichain extension property in $\mathcal{P}$, and let $A_1 \subset P'$ be a finite antichain. Then there is a strongly splitting $\mathcal{P}$-maximal antichain $A$ such that $A_1 \subset A \subset P'$.*

Proof: Let $\{p_n : n < \omega\}$ be an $\omega$-abundant enumeration of $P$, that is, the set $\{n : p_n = p\}$ is infinite for each $p \in P$. Let $A_1 = \{a_0, a_1, \ldots, a_{r-1}\}$. Proceed by induction on $i \geq r$ to construct an infinite antichain $A = \{a_i : i < \omega\} \subset P'$:

- if $p_i \notin \{a_j : j < i\}$ then let $a_i$ be comparable to $p_i$;
- if $p_i \in \{a_j : j < i\}$ then let $n_i = \min\{n : p_n \notin \{a_m : m < i\}\}$ and let $a_i$ be comparable to $p_{n_i}$.

This construction can be carried out because $P'$ has the finite antichain extension property.

Let $p \in P \setminus A$. Then the set $A_p = \{a_i : p_i = p\}$ is infinite and for each $a \in A_p$ the element $a$ and $p$ are comparable. Let $(B, C)$ be a partition of $A$ such that $|B \cap A_p| = |C \cap A_p| = \omega$ for each $p \in P \setminus A$.

Then the partition $(B, C)$ has the required property. □

The following results show that the existence of a loose kernel guarantees an infinite non-splitting maximal antichain. The first is a slight generalization of [9, Theorem 3.6].

**Theorem 3.5.** *Let $\mathcal{P} = \langle P, \leq \rangle$ be a countably infinite poset, and let $P' \subset P$ be a loose kernel in $\mathcal{P}$. Then there exists an infinite non-splitting $\mathcal{P}$-maximal antichain $A \subset P'$.*

PROOF: See [9, Theorem 3.6]. □

**Theorem 3.6.** *Let $\mathcal{P} = \langle P, \leq \rangle$ be a countably infinite poset, let $P' \subset P$ be a loose kernel in $\mathcal{P}$, and let $A_1 \subset P'$ be a non-maximal antichain in $P$. Then there is an infinite non-splitting $\mathcal{P}$-maximal antichain $A$ such that $A_1 \subset A \subset P'$.*

PROOF: The set $P' \setminus A_1^{\updownarrow}$ is a loose kernel in $P \setminus A_1^{\updownarrow}$, and $P \setminus A_1^{\updownarrow} \neq \emptyset$ because $A_1$ was not a maximal antichain. Hence by Theorem 3.5 there is a $P \setminus A_1{}^{\updownarrow}$-maximal antichain $A' \subset P' \setminus A_1^{\updownarrow}$ which does not split in $P \setminus A_1^{\updownarrow}$. Then $A = A_1 \cup A'$ is a maximal antichain in $P$ having the required properties. □

## 4. The homomorphism poset $\mathbb{G}$

The partially ordered set $\mathbb{G}$ of hom-equivalence classes of finite undirected graphs is known to have only two finite maximal antichains — $\{K_1\}$ and $\{K_2\}$. Consequently, there are no nontrivial dualities. However, in studying the ordered set $\mathbb{G}$, it is interesting to know whether maximal antichains split and whether antichains extend to maximal ones that split.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$. For any bipartite graph $G$, $G \to K_2$, so we know that all graphs in $\mathbb{G}'$ are hom-equivalent to graphs all of whose connected components contain odd cycles. The *odd girth* of a graph $G$, oddgirth$(G)$, is the length of the shortest odd cycle contained in the graph. As with girth, if graph does not contain any odd cycles, its oddgirth is regarded as infinite.

The notion of odd girth is useful in dealing with homomorphism questions because of this: for graphs $G$ and $H$, if oddgirth$(G) <$ oddgirth$(H)$ then $G \nrightarrow H$. Also, it is straightforward to construct graphs of prescribed odd girth and chromatic number using shift graphs — for instance, see [11, Theorem 2.23]. Alternatively, the original Erdős result could be used in the first part of the proof below.

**Theorem 4.1.** *$\mathbb{G}'$ is both a loose kernel in $\mathbb{G}$ and a loose kernel in $\mathbb{G}^d$. Hence, $\mathbb{G}'$ has the finite antichain extension property in $\mathbb{G}$.*

PROOF: Let $\mathcal{F} \subseteq \mathbb{G}'$ be finite. To see that $\mathbb{G}'$ is a loose kernel in $\mathbb{G}$, let $X \in \mathbb{G} \setminus \mathcal{F}^{\uparrow}$, that is, $F \nrightarrow X$ for all $F \in \mathcal{F}$. Let $Y'$ be a graph such that for all $F \in \mathcal{F}$,

(i) oddgirth$(Y') >$ oddgirth$(F')$ for all components $F'$ of $F$, and

(ii) $\chi(Y') > \chi(F)$.

Now let $Y = X + Y'$ and let $F \in \mathcal{F}$. By (i), $F \nrightarrow Y$, since no component of $F$ has a homomorphism to $Y$ and $F \nrightarrow X$. By (ii), $Y' \nrightarrow F$, so $Y \nrightarrow F$. Hence, (LK) holds and $\mathbb{G}'$ is loose in $\mathbb{G}$.

Now let us show that $\mathbb{G}'$ is loose in the dual. Let $H \in \mathbb{G} \setminus \mathcal{F}^{\downarrow}$, that is, $H \nrightarrow F$ for all $F \in \mathcal{F}$. Let $k = \max\{|V(H)|, |V(F)| : F \in \mathcal{F}\}$ and $\ell = \max\{\text{oddgirth}(F) : F \in \mathcal{F}\} + 1$. Here $\ell$ is finite because $\mathcal{F} \subset \mathbb{G}'$.

By Theorem 2.1 there is a graph $G \in \mathbb{G}$ such that $\text{girth}(G) > \ell$ and for all $K \in \mathbb{G}$ where $|V(K)| \leq k$,

$$G \to K \iff H \to K.$$

Therefore $G \to H$ but for all $F \in \mathcal{F}$ we have $G \nrightarrow F$. Since $\text{girth}(G) > \text{oddgirth}(F)$ we have $F \nrightarrow G$ for each $F \in \mathcal{F}$. Furthermore $H \nrightarrow K_2$ therefore $H \in \mathbb{G}'$, therefore $G \nrightarrow K_2$ and so $G \in \mathbb{G}'$. Thus (LK) holds for $\mathbb{G}'$ in $\mathbb{G}^d$, and we can apply Observation 3.3. $\square$

As noted above, it is well-known that $\mathbb{G}'$ has no finite maximal antichains. We include a short proof to illustrate the relationship between loose kernels and maximal antichains.

**Corollary 4.2.** *There is no finite maximal antichain in* $\mathbb{G}'$.

Proof: Indeed, let $\mathcal{F} \subset \mathbb{G}'$ be a finite antichain. Then $K_1 < F_i$ (for each $i$) therefore $K_1 \notin \mathcal{F}^{\uparrow}$. The application of Theorem 4.1 gives us an element of $\mathbb{G}'$, which is incomparable to $\mathcal{F}$. $\square$

Since there are no finite maximal antichains and every finite antichain extends to a maximal one, each finite antichain can be extended to an infinite maximal antichain. The following shows that quite different behavior can be found in the various extensions.

**Corollary 4.3.** *Let $A \subseteq \mathbb{G}'$ be a finite antichain. Then*

(i) *there exists a non-splitting maximal antichain $A_1 \subset \mathbb{G}'$ such that $A \subset A_1$, and*

(ii) *there exists a strongly splitting maximal antichain $A_2 \subset \mathbb{G}'$ such that $A \subset A_2$.*

Proof: (i) This is a direct consequence of Theorem 3.6, applied to the poset $\mathbb{G}$ and the loose kernel $\mathbb{G}'$.

(ii) We can apply Theorem 3.4 because $\mathbb{G}'$ has the finite antichain extension property in $\mathbb{G}$. $\square$

The notions of a *cut-point* and a *cut-free* subset are closely tied to the splitting property: see [1] and [9]. They have also been studied independently in the context of homomorphism orders of graphs: see [12]. We provide a short proof that $\mathbb{G}'$ is cut-free, both to illustrate an application of the Sparse Incomparability Lemma and to highlight differences between $\mathbb{G}$ and $\mathbb{D}$ that we shall see again in the next section.

**Proposition 4.4.** $\mathbb{G}'$ *is cut-free.*

Proof: We need to show that for all triples $F < G < H$, if $G \in \mathbb{G}'$ (and therefore $H \in \mathbb{G}'$) then there is a $G' \in \mathbb{G}'$ such that $F < G' < H$ and $G'$ is incomparable to $G$.

Since oddgirth$(G)$ is finite for $G \in \mathbb{G}'$, we can apply Theorem 2.1 to $H$ with parameters $k = \max(|V(G)|, |V(F)|) + 1$ and $\ell = \text{oddgirth}(G) + 1$ to get a graph $H'$ such that:

- $H' \to H$, since $H \to H$,
- $H' \nrightarrow G$, since $H \nrightarrow G$,
- $H' \nrightarrow F$, since $H \nrightarrow F$, and
- $\text{girth}(H') > \ell$.

Since oddgirth$(G) < \ell$ we have $X \nrightarrow H'$ for each connected component $X$ of $G$. Therefore the graph $G' = F + H'$ satisfies the requirements. □

## 5. The homomorphism poset $\mathbb{D}$

In [10], the complete characterization of finite maximal antichains in the homomorphism poset for finite relational structures with a single relation shows the crucial role of forests. In the study of $\mathbb{G}$, odd cycles play a crucial role. So, one might hope that the investigation the two subsets $\mathbb{D}'$ and $\mathbb{D}^*$ of $\mathbb{D}$ defined below would lead to the construction of interesting antichains by verifying the loose kernel or FAE properties, then employing results such as Theorems 3.4–3.6. It turns out to be a bit more complicated.

Before defining these, it is useful to recall that a finite directed graph $X$ is a *core* if every homomorphism of $X$ to itself is bijective. Every directed graph is homomorphically equivalent to a unique core, and, so, every directed graph class contains exactly one core (cf. [11]). For the rest of this section, we shall use "graph" for "directed graph" and, given a directed graph $X$, $\overline{X}$ denotes its undirected version.

We now define the two subsets of $\mathbb{D}$:

- $\mathbb{D}' \subseteq \mathbb{D}$ consists of all graph classes with core $X$ such that every connected component of $\overline{X}$ contains an odd cycle;
- $\mathbb{D}^* \subseteq \mathbb{D}$ consists of all graph classes with core $Y$ such that at least one connected component of $\overline{Y}$ has contains an odd cycle.

Of course, $\mathbb{D}' \subseteq \mathbb{D}^*$, while the graph $H$ defined below in the proof of Proposition 5.2 is in $\mathbb{D}^*$ and not in $\mathbb{D}'$.

The following result collects some straightforward observations about these subsets in $\mathbb{D}$. The proofs have been omitted since the methods are not very different from those encountered in the undirected case.

**Proposition 5.1.** *In the partially ordered set $\mathbb{D}$:*

(i) $\mathbb{D}'$ *is a loose kernel in* $\mathbb{D}$*;*
(ii) $\mathbb{D}^*$ *is a cut-free subset of* $\mathbb{D}$*; and,*
(iii) $\mathbb{D}^*$ *is loose in* $(\mathbb{D}^*)^d$.

Unfortunately, we can also prove that

**Proposition 5.2.** *In the partially ordered set $\mathbb{D}$:*

(i) $\mathbb{D}'$ *does not have the finite antichain extension property in* $\mathbb{D}$*; and*
(ii) $\mathbb{D}^*$ *does not have the finite antichain extension property in* $\mathbb{D}$.

Proof: (i) Let $\vec{T_3}$ be the transitive tournament on three vertices and let $\vec{P_3} = (W, F)$ be the directed path on four vertices: $W = \{x_0, x_1, x_2, x_3\}$ and $F = \{\langle x_i, x_{i+1}\rangle : i = 0, 1, 2\}$. Now let $H = \vec{T_3} + \vec{P_3}$, the disjoint union of $T$ and $P_3$. Then $H$ is a core in $\mathbb{D}$. Also, $\vec{T_3} \in \mathbb{D}'$, is a core and $\vec{T_3} < H$. Regard $\{\vec{T_3}\}$ as an antichain. If $\mathbb{D}'$ had the (FAE) property there would exist a core $H' \in \mathbb{D}'$ such that $H'$ is incomparable to $\vec{T_3}$ and $H' < H$. However, every connected component of $\overline{H'}$ contains an odd cycle, so $H' < H$ implies that $H' \leq \vec{T_3}$ since no component of $H'$ can be mapped by a homomorphism into $\vec{P_3}$.
(ii) We can base an example on any oriented tree $T$ but to be a bit more specific let $T = \vec{P_k}$ be a directed path on $k$ vertices. Then the dual $D(\vec{P_k})$ is the transitive tournament $\vec{T_k}$ on $k$ vertices (see, for instance [11, Proposition 1.20]). Let $H = \vec{T_k} + \vec{P_k}$. As long as $k \geq 3$, $H \in \mathbb{D}^*$. Also, $H$ and $\vec{T_k}$ are cores in $\mathbb{D}$ and $\vec{T_k} < H$. Regard $\{H\}$ as an antichain. If $\mathbb{D}^*$ had the (FAE) property there would exist a core $H' \in \mathbb{D}^*$ such that $H'$ is incomparable to $H$ and $\vec{T_k} < H'$. But $(\vec{P_k}, \vec{T_k})$ is a dual pair, so the fact that $H'$ is not below $\vec{T_k}$ implies $\vec{P_k} < H'$ — just apply equation (3) in this special case. From this it would follow that $H = \vec{T_k} + \vec{P_k} < H'$, a contradiction. □

Fortunately there is another subset $\mathbb{D}^c$ of $\mathbb{D}$ which is both an upward loose kernel in $\mathbb{D}$ and has the finite antichain extension property in $\mathbb{D}$. To discuss it, first we need an easy observation. A finite directed graph $\vec{C}$ is a *directed cycle* if it is connected and each vertex has indegree and outdegree 1. It is easily seen that each directed cycle is a core. Use $\vec{C_k}$ to denote the directed cycle on $k$ vertices.

**Proposition 5.3.** *Let $\vec{C}$ be a directed cycle and $T$ be a graph such that $\overline{T}$ is an arbitrary tree. Then $T \to \vec{C}$.*

Proof: Map a vertex $v$ of $T$ to any vertex of the cycle. Next the vertices adjacent to $v$ in $\overline{T}$ can be mapped into vertices of $\vec{C}$ so that directed edges are preserved. Since there is no cycle in $\overline{T}$ we can finish the process easily. □

Let $\mathbb{D}^c$ be the set of all homomorphism classes in $\mathbb{D}$ whose core $X$ has the property that for some $\vec{C}$, $\vec{C} \to X$. Here is a direct consequence of Proposition 5.3.

**Observation 5.4.** *Let $G \in \mathbb{D}^c$ and let $T \in \mathbb{D}$ be an oriented tree. Then $G + T \sim G$.*

Hence we can assume that no component of an element of $\mathbb{D}^c$ can be embedded into a tree. Therefore from now on we assume that each component $X$ of each element of $\mathbb{D}^c$ has the property that $\overline{X}$ contains a cycle.

**Theorem 5.5.** *$\mathbb{D}^c$ is a loose kernel in $\mathbb{D}$.*

PROOF: Let $\mathcal{F} \subseteq \mathbb{D}^c$ be finite, and $X \in \mathbb{D}$ but $X \notin \mathcal{F}^{\uparrow}$. We are going to find a $Y \in \mathbb{D}^c$ such that $X \to Y$, $Y \nrightarrow X$, and $Y$ is incomparable to each $F \in \mathcal{F}$.

Let $n := \max\{|X|, |F| : F \in \mathcal{F}\}$. Using the Erdős theorem, obtain a graph $Z$ such that $\chi(\overline{Z}) > n$, $\mathrm{girth}(\overline{Z}) > n$, $\overline{Z}$ is connected, and $Z$ contains at least one directed cycle. Then $Z \in \mathbb{D}^c$. Let $Y = X + Z$. Since $Z \in \mathbb{D}^c$ therefore $Y \in \mathbb{D}^c$ as well. Clearly $X \to Y$ while $Y \nrightarrow X$ because $\chi(Y) > |X|$.

Assume that $f$ is a homomorphism of $F$ to $Y$. Then there is a component $K$ of $F$ such that $f[K] \subseteq Z$. But $|V(K)| \leq n$ while $\mathrm{girth}(Z) > n$, hence the image $f[K]$ is a tree, which contradicts the assumption that no component of an element of $\mathbb{D}^c$ can be mapped into a tree. □

**Theorem 5.6.** *Let $A_1 \subseteq \mathbb{D}^c$ be a finite antichain. Then there is a non-splitting antichain $A$ such that $A_1 \subseteq A \subseteq \mathbb{D}^c$ and $A$ is maximal in $\mathbb{D}$.*

PROOF: Since $\mathbb{D}^c$ is an upward loose kernel it can be used in Theorem 3.5 to extend a non-maximal antichain into a non-splitting antichain, maximal in $\mathbb{D}$. □

**Theorem 5.7.** *$\mathbb{D}^c$ has the finite antichain extension property in $\mathbb{D}$.*

PROOF: Let $\mathcal{F} \subseteq \mathbb{D}^c$ be a finite antichain and $X \in \mathbb{D}$. We need to find $Y \in (X^{\updownarrow} \cap \mathbb{D}^c) \setminus \mathcal{F}^{\updownarrow}$. In case $X \notin \mathcal{F}^{\uparrow}$ then Theorem 5.5 provides the required $Y$.

Assume now that $X \in \mathcal{F}^{\uparrow}$. Then $X \in \mathbb{D}^c$ because there exists $F \in \mathcal{F}$ with $F \to X$ and the image of its directed cycle of $F$ is a directed cycle in $X$. Let us assume that $X$ contains the directed cycle $\vec{C}_k$.

Let $n = \max\{|X|, |F| : F \in \mathcal{F}\}$. Apply Theorem 2.2 with $H = X$ and $m = \ell = n$ to obtain $X' = H'$. Now let $Y = X' + \vec{C}_{kn}$. Then $X' \to X$ and $\vec{C}_{kn} \to \vec{C}_k$ therefore $Y \to X$. At the same time $X \nrightarrow Y$ since $\mathrm{girth}(Y) > \ell \geq |X|$. Thus, the cycle $\vec{C}_k$ of $X$ cannot be embedded into $Y$. The same applies for the directed cycles in each $F \in \mathcal{F}$. Therefore, $F \nrightarrow Y$. Finally we have $X \nrightarrow F$ and so $Y \nrightarrow F$. □

**Corollary 5.8.** *$\mathbb{D}^c$ does not contain finite maximal antichains.*

We recall that a full description of the finite maximal antichains in $\mathbb{D}$ is given in [10].

**Corollary 5.9.** *Let $A_1 \subseteq \mathbb{D}^c$ be a finite antichain in $\mathbb{D}$. Then there is a strongly splitting $\mathbb{D}$-maximal antichain $A_1 \subset A \subset \mathbb{D}^c$.*

Proof: This is just the direct application of Theorem 3.4 to the posets $\mathbb{D}$ and $\mathbb{D}^c$. □

Here is a final use of our methods in describing the order structure of $\mathbb{D}$.

**Theorem 5.10.** *$\mathbb{D}^c$ is cut-free in $\mathbb{D}$.*

Proof: Let $F < G < H$ where $G \in \mathbb{D}^c$ (and therefore $H \in \mathbb{D}^c$ as well). We need a $G' \in \mathbb{D}^c$, which is incomparable to $G$ but $F < G' < H$.

Let $n = \max\{|F|, |G|, |H|\}$. Apply Theorem 2.2 to $H$ with parameters $m = \ell = n$ to obtain the directed graph $H'$. Since $H \in \mathbb{D}^c$, there is $k$ such that $\vec{C}_k$ is a subgraph of $H$. Let $G' = F + H' + \vec{C}_{kn}$.

Then $F \to G'$ since $F$ is a subgraph of $G'$. Furthermore $H' \to H$ due to the fact that $|H| = n \leq m$ and $H \to H$. Due to our assumption on $\mathbb{D}^c$, each component of the graph $G$ contains cycles, and at least one of them, say $K$, cannot be embedded into $F$. Therefore if $G \to Y$ then for this component we have $K \to H' + C_{nk}$. However, $\mathrm{girth}(H' + C_{nk}) > |K|$, hence $K$ is embedded into a tree, a contradiction. □

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