MAD families and P-points

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Abstract. The *Katětov ordering* of two maximal almost disjoint (MAD) families $\mathcal{A}$ and $\mathcal{B}$ is defined as follows: We say that $\mathcal{A} \leq_K \mathcal{B}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}(A) \in \mathcal{I}(\mathcal{B})$ for every $A \in \mathcal{I}(\mathcal{A})$. In [Garcia-Ferreira S., Hrušák M., *Ordering MAD families a la Katětov*, J. Symbolic Logic **68** (2003), 1337–1353] a MAD family is called *K-uniform* if for every $X \in \mathcal{I}(\mathcal{A})^+$, we have that $\mathcal{A}|_X \leq_K \mathcal{A}$. We prove that CH implies that for every K-uniform MAD family $\mathcal{A}$ there is a P-point p of ω^* such that the set of all *Rudin-Keisler* predecessors of p is dense in the boundary of $\bigcup \mathcal{A}^*$ as a subspace of the remainder $\beta(\omega) \setminus \omega$. This result has a nicer topological interpretation:

The symbol $\mathcal{F}(\mathcal{A})$ will denote the Franklin compact space associated to a MAD family $\mathcal{A}$. Given an ultrafilter $p \in \beta(\omega) \setminus \omega$, we say that a space X is a *$FU(p)$-space* if for every $A \subseteq X$ and $x \in cl_X(A)$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x = p$-$\lim_{n\to\infty} x_n$ (that is, for every neigborhood V of x, we have that $\{n < \omega : x_n \in V\} \in p$). **[CH]** For every K-uniform MAD family $\mathcal{A}$ there is a P-point p of ω^* such that $\mathcal{F}(\mathcal{A})$ is a FU(p)-*space*. We also establish the following.

[CH] For two P-points $p, q \in \omega^*$, the following are equivalent.

(1) $q \leq_{\mathrm{RK}} p$.
(2) For every *MAD* family $\mathcal{A}$, the space $\mathcal{F}(\mathcal{A})$ is a FU(p)-space whenever it is a FU(q)-space.

Keywords: Franklin compact space, P-point, FU(p)-space, maximal almost disjoint family, Katětov ordering, Rudin-Keisler ordering

Classification: Primary 03E05; Secondary 54B99

1. Introduction

All spaces are assumed to be Tychonoff. If X is a set, then $[X]^\omega = \{A \subseteq X : |A| = \omega\}$ and the definition of $[X]^{<\omega}$ should be clear. The Stone-Čech compactification $\beta(\omega)$ of the countable discrete space ω is identified with the set of all ultrafilters on ω and its remainder $\omega^* = \beta(\omega) \setminus \omega$ is identified with the set of all free ultrafilters on ω. For $A \subseteq \omega$, $\hat{A} = \{p \in \beta(\omega) : A \in p\} = \mathrm{cl}_{\beta(\omega)} A$ and $A^* = \hat{A} \cap \omega^*$. If $\mathcal{A} \subseteq [\omega]^\omega$, then $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$. Recall that an infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is called *almost disjoint* (AD) if for every two distinct $A, B \in \mathcal{A}$

The first author acknowledges support by CONACYT grant no. 40057-F and PAPIIT grants no. IN106103. The second author acknowledges support by NSERC grant 238944 and from Instituto de Matematicas, UNAM (Morelia) Mexico.

we have that $A \cap B \in [\omega]^{<\omega}$. An AD family is said to be *maximal almost disjoint* (MAD) if it is not contained properly in any AD family (a construction of a MAD family of size $\mathfrak{c}$ can be found in [7, 6Q] and [10]). For $A, B \in [\omega]^\omega$, $A \subseteq^* B$ means that $A \setminus B$ is finite, and $A =^* B$ means that $A \subseteq^* B$ and $B \subseteq^* A$. Observe that $A =^* B$ iff $A^* = B^*$. If $\mathcal{A} \subseteq [\omega]^\omega$, then $\bigcup \mathcal{A}^* = \bigcup\{A^* : A \in \mathcal{A}\}$ and we have that $\mathrm{Fr}(\bigcup \mathcal{A}^*) = \omega^* \setminus (\bigcup \mathcal{A}^*)$. It is not difficult to see that an AD family $\mathcal{A}$ is maximal iff $\bigcup \mathcal{A}^*$ is a dense subset of ω^*.

For a MAD-family, $\mathcal{I}(\mathcal{A})$ will denote the ideal consisting of all subsets of ω that can be almost covered by finitely many elements from $\mathcal{A}$ and $\mathcal{I}(\mathcal{A})^+ = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ is the set of positive measure subsets of ω. If $\mathcal{A}$ is a MAD family and $X \in \mathcal{I}(\mathcal{A})^+$, then $\mathcal{A}|_X = \{A \cap X : A \in \mathcal{A} \text{ and } |A \cap X| = \omega\}$ is a MAD family on the set X.

The *Rudin-Keisler* (*pre*)-*order* on ω^* is defined by $p \leq_{\mathrm{RK}} q$ if there is a function $f : \omega \to \omega$ such that $\overline{f}(q) = p$, where $\overline{f} : \beta(\omega) \to \beta(\omega)$ denotes the Stone-Čech extension of the function f. Given $p \in \omega^*$, we let $P_{\mathrm{RK}}(p) = \{q \in \omega^* : q \leq_{\mathrm{RK}} p\}$. The condition $q \leq_{\mathrm{RK}} p$ and $p \leq_{\mathrm{RK}} q$ is equivalent to the existence of a bijection $f : \omega \to \omega$ such that $\overline{f}(q) = p$. In this case, we write $p \approx q$ and say that p and q are RK-equivalent and $T(p) = \{q \in \omega^* : p \approx q\}$.

The *Franklin compact space* associated to a MAD-family $\mathcal{A}$, denoted by $\mathcal{F}(\mathcal{A})$, is the quotient space of $\beta(\omega)$ in which every A^* is identified with a single point, for each $A \in \mathcal{A}$, and $\omega^* \setminus \bigcup \mathcal{A}^*$ is identified to a single point. I.e., $\mathcal{F}$ is the one-point compactification of the Ψ-space $\Psi(\mathcal{A})$. The Franklin compact spaces were introduced by S.P. Franklin [3]. It is well-known that Franklin compact spaces are sequential spaces with degree of sequentiality equal to 2 and are not Fréchet-Urysohn. The notion of a p-limit point, for $p \in \omega^*$ of a countable sequence of a spaces provides a useful tool to study some Fréchet-Urysohn like properties of the Franklin compact spaces. Indeed, following A.R. Bernstein [1], for $p \in \omega^*$, we say that $x \in X$ is the *p-limit point* of a sequence $(x_n)_{n<\omega}$ if for every neighborhood V of x we have that $\{n < \omega : x_n \in V\} \in p$ (we write $x = p\text{-}\lim_{n\to\infty} x_n$). Then, we say that a space X is a $\mathrm{FU}(p)$-space if for every $x \in \mathrm{cl}(A)$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x = p\text{-}\lim_{n\to\infty} x_n$. It is clear that every Fréchet-Urysohn space is a $\mathrm{FU}(p)$-space, for all $p \in \omega^*$.

The following lemma from [6] provides a useful characterization of the non-$\mathrm{FU}(p)$-property of a Franklin compact space.

Lemma 1.1. *Let $p \in \omega^*$ and $\mathcal{A}$ a MAD-family. The space $\mathcal{F}(\mathcal{A})$ is not a FU(p)-space iff there is $C \in [\omega]^\omega$ such that*

$$C^* \setminus \bigcup \mathcal{A}^* \neq \emptyset \ \text{ and } \ P_{\mathrm{RK}}(p) \cap C^* \subseteq \bigcup \mathcal{A}^*.$$

Thus, to prove that a Franklin compact space $\mathcal{F}(\mathcal{A})$ is a $\mathrm{FU}(p)$-space it suffices to show that $P_{\mathrm{RK}}(p)$ is dense in $\mathrm{Fr}(\bigcup A^*)$.

The notion of a P-point of space was introduced by W. Rudin [9]: A point $x \in X$ is called a *P-point* if the intersection of any countable family of neighborhoods of x is again a neighborhood of x. It is known that $p \in \omega^*$ is a P-point iff for every partition $\{A_n : n < \omega\}$ of ω either $A_n \in p$ for some $n < \omega$ or there is $A \in p$ such that $A_n \cap A$ is finite for all $n < \omega$. W. Rudin proved assuming CH that ω^* includes P-points, and S. Shelah [12] constructed a model of ZFC in which there are no P-points in ω^*.

It is shown in [2] that every Franklin compact space is a FU(p)-space for each non-P-point p of ω^*. It is proved in [6] that $\mathfrak{p} = \mathfrak{c}$ implies that for every $p \in \omega^*$ there is a MAD family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is a FU(p)-space, and, under CH, for every P-point $p \in \omega^*$ there is a MAD family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is not a FU(p)-space. More precisely, we have the following result.

Theorem 1.2 [CH]. *For $p \in \omega^*$, the following conditions are equivalent.*

(1) *p is a P-point.*
(2) *There is a MAD family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is not a FU(p)-space.*

PROOF: The implication (1) $\Rightarrow$ (2) follows directly from Theorem 2.6 of [6], and the implication (2) $\Rightarrow$ (1) is a consequence of Theorem 2 from [2]. $\square$

Basing on the previous results, the question whether or not a Franklin space is a FU(p)-space is interesting only for the case when p is a P-point of ω^*. One of the questions posed in [6] that remains open is the following.

Question 1.3. Does CH imply that for every MAD family $\mathcal{A}$ there is a P-point p such that $\mathcal{F}(\mathcal{A})$ is a FU(p)-space ?

In this paper, we show that for some class of MAD families (K-uniform MAD families) the answer to this question is in the positive.

2. Franklin compact spaces and P-points

We start with a construction, under CH, of a MAD family with certain properties.

Theorem 2.1 [CH]. *If $p, q \in \omega^*$ are P-points such that $q \not\leq_{\mathrm{RK}} p$, then there is a MAD family $\mathcal{A}$ such that $\mathcal{F}(\mathcal{A})$ is a FU(q)-space and is not a FU(p)-space.*

PROOF: Assume CH. By repeating elements if it necessary, we may enumerate $P_{\mathrm{RK}}(p)$ by $\{p_\theta : \theta < \omega_1\}$ and $[\omega]^\omega$ by $\{A_\theta : \theta < \omega_1\}$ in such a way that $p_\theta \in A^*_\theta$ for each $\theta < \omega_1$ and $\{A_n : n < \omega\}$ is a partition of ω. It is known that each point of $P_{\mathrm{RK}}(p)$ is a P-point. Now, we proceed by transfinite induction. Assume that for each $\gamma < \theta < \omega_1$ we have defined $B_\gamma \in [\omega]^\omega$ and $q_\gamma \in T(q)$ such that

(1) $\{B_\gamma : \gamma < \theta\}$ is an AD family,
(2) for each $\gamma < \theta$ there is $\delta \leq \gamma$ such that $A_\gamma \cap B_\delta$ is infinite,
(3) $p_\gamma \in B^*_\gamma$, for each $\gamma < \theta$,
(4) $B^*_\gamma \cap \mathrm{cl}(\{q_\delta : \delta < \gamma\}) = \emptyset$, for each $\gamma < \theta$, and
(5) if $A_\gamma \in \mathcal{I}(\{\mathcal{B}_\delta : \delta \leq \gamma\})^+$, then $q_\gamma \in T(q) \cap (A^*_\gamma \setminus (\bigcup_{\delta \leq \gamma} B^*_\delta))$, for each $\gamma < \theta$.

First, we define B_θ. If $p_\theta \in B^*_\gamma$ for some $\gamma < \theta$, then we put $B_\theta = B_\gamma$. Assume that $p_\theta \in \mathrm{Fr}(\bigcup\{B^*_\gamma : \gamma < \theta\})$. Since p_θ is a P-point, we can choose $B_\theta \in p_\theta$ so that $B_\theta \subseteq A_\theta$, $B^*_\theta \cap \mathrm{cl}(\{q_\gamma : \gamma < \theta\}) = \emptyset$ and $\{B_\theta\} \cup \{B_\gamma : \gamma < \theta\}$ is AD. Thus, we have defined B_θ. Now, if $A_\theta \in \mathcal{I}(\{\mathcal{B}_\gamma : \gamma < \theta\})^+$, then we pick $q_\theta \in T(q) \cap (A^*_\gamma \setminus (\bigcup_{\gamma \leq \theta} B^*_\gamma))$. If not, just chose any $q_\theta \in T(q)$. We let $\mathcal{A} = \{B_\gamma : \gamma < \omega_1\}$. It is evident from the construction that $\mathcal{A}$ is a MAD family. As $P_{\mathrm{RK}}(p) \subseteq \bigcup_{\theta < \omega_1} B^*_\theta$, by Lemma 1.1, $\mathcal{F}(\mathcal{A})$ cannot be a FU(p)-space. Take $A \in [\omega]^\omega$ so that $A^* \cap \mathrm{Fr}(\bigcup \mathcal{A}^*) \neq \emptyset$. Choose $\theta < \omega_1$ such that $A = A_\theta$. By the fifth clause, we know that $q_\theta \in T(q) \cap (A^*_\theta \setminus (\bigcup_{\gamma \leq \theta} B^*_\delta))$, and clause (4) guarantees that $q_\theta \notin B^*_\gamma$ for any $\theta < \gamma < \omega_1$. So, $q_\theta \in T(q) \cap A^*_\theta \cap (\mathrm{Fr}(\bigcup \mathcal{A}^*))$. Therefore, according to Lemma 1.1, $\mathcal{F}(\mathcal{A})$ is a FU(q)-space. $\square$

By combining Theorem 2.1 and Corollary 2.2 from [4] we get the following statement.

Corollary 2.2 [CH]. *For two P-points $p, q \in \omega^*$, the following are equivalent.*

(1) $q \leq_{\mathrm{RK}} p$.
(2) *For every MAD family $\mathcal{A}$, the space $\mathcal{F}(\mathcal{A})$ is a FU(p)-space whenever it is a FU(q)-space.*

Given two MAD families $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A} \leq_K \mathcal{B}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}(A) \in \mathcal{I}(\mathcal{B})$ for every $A \in \mathcal{I}(\mathcal{A})$. This relation is called the *Katětov ordering* of MAD families and was studied in [5]. A reformulation of the Katětov ordering that will be used implicity several times below is the following.

Theorem 2.3. *For two MAD families $\mathcal{A}$ and $\mathcal{B}$, the following conditions are equivalent.*

(1) $\mathcal{A} \leq_K \mathcal{B}$.
(2) *There is a function $f : \omega \to \omega$ such that $\overline{f}[\mathrm{Fr}(\bigcup \mathcal{B}^*)] \subseteq \mathrm{Fr}(\bigcup \mathcal{A}^*)$.*

Proof: (1) $\Rightarrow$ (2). Let $p \in \mathrm{Fr}(\bigcup \mathcal{B}^*)$ and put $q = \overline{f}(p)$. Then, we have that $p \subseteq \mathcal{I}(\mathcal{B})^+$. Assume that $q \in \bigcup \mathcal{A}^*$. Then, $q \in B^*$ for some $B \in \mathcal{A}$. Hence, $f^{-1}(B) \in p \subseteq \mathcal{I}(\mathcal{B})^+$, which is impossible.

(2) $\Rightarrow$ (1). Suppose that $f^{-1}(I) \in \mathcal{I}(\mathcal{B})^+$ for some $I \in \mathcal{I}(\mathcal{A})$. Choose $p \in f^{-1}(I)^* \cap \mathrm{Fr}(\bigcup \mathcal{B}^*)$. By assumption, $\overline{f}(p) = q \in \mathrm{Fr}(\bigcup \mathcal{A}^*)$, but this is a contradiction since $q \in I^* \cap \mathrm{Fr}(\bigcup \mathcal{A}^*)$ implies that $I \in \mathcal{I}(\mathcal{A})^+$. $\square$

In the article [5], the authors also considered the following class of MAD families.

Definition 2.4. A MAD family is called K-uniform if for every $X \in \mathcal{I}(\mathcal{A})^+$, we have that $\mathcal{A}|_X \leq_K \mathcal{A}$.

We should understand that the condition $\mathcal{A}|_X \leq_K \mathcal{A}$ means that there is a function $f : \omega \to X$ such that $f^{-1}(A \cap X) \in \mathcal{I}(\mathcal{A})$, for all $A \in \mathcal{A}$. It shows in [5], that the condition $\mathfrak{t} = \mathfrak{c}$ implies the existence of a K-uniform MAD family. The existence of a K-uniform family in ZFC is still unknown. Our next task is to prove the main result of the paper. To do that we need to prove some preliminary lemmas.

The next lemma is due to J. Dočkálková (see [11]) and A.R.D. Mathias [8]. To make the paper self-contained we shall include a proof of the lemma.

Lemma 2.5. *Let $\mathcal{A}$ be a MAD family. If $\{A_n : n < \omega\}$ is a countable family of elements of $\mathcal{I}(\mathcal{A})^+$ with the property that the intersection of every finite subfamily lies in $\mathcal{I}(\mathcal{A})^+$, then there is $A \in \mathcal{I}(\mathcal{A})^+$ for which $A^* \subseteq \bigcap_{n<\omega} A_n^*$.*

Proof: Without loss of generality, we may assume that $A_{n+1}^* \subseteq A_n^*$, for each $n < \omega$. We know that $\operatorname{int}(\bigcap_{n<\omega} A_n^*) \neq \emptyset$ and hence $\operatorname{int}(\bigcap_{n<\omega} A_n^*) \cap (\bigcup \mathcal{A}^*) \neq \emptyset$. Let us assume that the set

$$\mathcal{B} = \Big\{X \in \mathcal{A} : X^* \cap \operatorname{int}(\bigcap_{n<\omega} A_n^*) \neq \emptyset\Big\}$$

consists only of the sets $\{X_i : i < k\}$ for some $k < \omega$. That is,

$$\operatorname{int}\Big(\bigcap_{n<\omega} A_n^*\Big) \cap \Big(\bigcup \mathcal{A}^*\Big) \subseteq \bigcup_{i<k} X_i^*.$$

It is evident that $A_n \setminus (\bigcup_{i<k} X_i)$ is infinite and $(A_{n+1} \setminus (\bigcup_{i<k} X_i))^* \subseteq (A_n \setminus (\bigcup_{i<k} X_i))^*$, for each $n < \omega$. Hence,

$$\begin{aligned}\emptyset \neq \operatorname{int}\Big(\bigcap_{n<\omega} \Big(A_n \setminus \Big(\bigcup_{i<k} X_i\Big)\Big)^*\Big) \cap \Big(\bigcup \mathcal{A}^*\Big) \\ = \operatorname{int}\Big(\bigcap_{n<\omega} \Big(A_n^* \setminus \Big(\bigcup_{i<k} X_i^*\Big)\Big)\Big) \cap \Big(\bigcup \mathcal{A}^*\Big) \\ \subseteq \Big(\operatorname{int}\Big(\bigcap_{n<\omega} A_n^*\Big) \cap \Big(\bigcup \mathcal{A}^*\Big)\Big) \setminus \Big(\bigcup_{i<k} X_i^*\Big),\end{aligned}$$

which is impossible. Thus, we must have that $\mathcal{B}$ is infinite. Take a countable infinite subset $\{X_k : k < \omega\}$ of $\mathcal{B}$. Now, for each $k < \omega$ choose $B_k \in [\omega]^\omega$ so that $B_k^* \subseteq X_k^* \cap \operatorname{int}(\bigcap_{n<\omega} A_n^*)$ and $B_k \subseteq \bigcap_{i<k} A_i$. Define $A = \bigcup_{k<\omega} B_k$. By construction, we have that $A \in \mathcal{I}(\mathcal{A})^+$ and $A \setminus A_n \subseteq \bigcup_{i\leq n}(B_i \setminus A_n)$, for each $n < \omega$. Thus, $A^* \subseteq \bigcap_{n<\omega} A_n^*$. $\square$

Lemma 2.6 [CH]. *For every MAD family $\mathcal{A}$ and each $A \in \mathcal{I}(\mathcal{A})^+$, there is a P-point $p \in \omega^*$ such that $p \in A^* \cap \operatorname{Fr}(\bigcup \mathcal{A}^*)$.*

Proof: Enumerate $[\omega]^\omega$ as $\{B_\xi : \xi < \omega_1\}$. Inductively, we shall construct a family $\{A_\xi : \xi < \omega_1\}$ of sets from $\mathcal{I}(\mathcal{A})^+$ such that

(1) $A_0 = A$,

(2) $A_\xi^* \subseteq A_\zeta^*$ whenever $\zeta < \xi < \omega_1$, and
(3) for each $\xi < \omega_1$, we have that either $A_\xi \subseteq^* B_\xi$ or $A_\xi \subseteq^* \omega \setminus B_\xi$.

Suppose that for each $\xi < \theta < \omega_1$ we have defined $A_\xi \in \mathcal{I}(\mathcal{A})^+$ satisfying the three conditions. Let us consider the family $\mathcal{C} = \{A_\xi : \xi < \theta\}$. Since either $B_\theta \in \mathcal{I}(\mathcal{A})^+$ or $\omega \setminus B_\theta \in \mathcal{I}(\mathcal{A})^+$, without loss of generality, we may assume that $B_\theta \in \mathcal{I}(\mathcal{A})^+$. Let us consider two cases:

Case I. $C \cap B_\theta \in \mathcal{I}(\mathcal{A})^+$ for all $C \in \mathcal{C}$. According to Lemma 2.5, we can find $A_\theta \in \mathcal{I}(\mathcal{A})^+$ so that $A_\theta^* \subseteq (\bigcap_{\xi<\theta} A_\xi^*) \cap B_\theta^*$.

Case II. There is $\xi < \theta$ such that $A_\xi \cap B_\theta \in \mathcal{I}(\mathcal{A})$. In this case it follows that $A_\xi \setminus B_\theta \in \mathcal{I}(\mathcal{A})^+$. Thus, $\omega \setminus B_\theta \in \mathcal{I}(\mathcal{A})^+$. Applying again Lemma 2.5, there is $A_\theta \in \mathcal{I}(\mathcal{A})^+$ so that $A_\theta^* \subseteq (\bigcap_{\xi<\theta} A_\xi^*) \cap (\omega \setminus B_\theta)^*$.

Since the family $\mathcal{B} = \{A_\theta : \theta < \omega_1\}$ has the finite intersection property, there is $p \in \omega^*$ such that $\mathcal{B} \subseteq p$. It follows from (1)–(3) that $\mathcal{B}$ is a base for p, p is a P-point and $p \in A^*$. □

Theorem 2.7 [CH]. *If $\mathcal{A}$ is a K-uniform MAD family, then there is a P-point p such that the set of all RK-predecessors of p is dense in $\mathrm{Fr}(\bigcup \mathcal{A}^*)$. Hence, $\mathcal{F}(\mathcal{A})$ is a FU(p)-space.*

Proof: According to Lemma 2.6, we can find a P-point $p \in \omega^*$ such that $p \in \mathrm{Fr}(\bigcup \mathcal{A}^*)$ since $\omega \in \mathcal{I}(\mathcal{A})^+$. In particular, we have that $p \subseteq \mathcal{I}(\mathcal{A})^+$. Fix $X \in \mathcal{I}(\mathcal{A})^+$ and choose a function $f : \omega \to X$ witnessing the fact $\mathcal{A}|_X \leq_K \mathcal{A}$. Let $q = \overline{f}(p)$. By definition $q \leq_{\mathrm{RK}} p$ and $q \in X^*$. It remains to show that $q \in \mathrm{Fr}(\bigcup \mathcal{A}^*)$. Suppose that $q \in A^*$ for some $A \in \mathcal{A}$. Then $q \in A^* \cap X^* = (A \cap X)^*$. That is, $A \cap X \in q$ and hence $f^{-1}(A \cap X) \in p$, but this is a contradiction. Thus, we must have that $q \in X^* \cap \mathrm{Fr}(\bigcup \mathcal{A}^*)$. This shows that $P_{\mathrm{RK}}(p)$ is dense in $\mathrm{Fr}(\bigcup \mathcal{A}^*)$. □

We can deduce from Theorem 2.3 that if $\mathcal{A} \leq_K \mathcal{B}$, for two MAD families $\mathcal{A}$ and $\mathcal{B}$, then

$$(*) \qquad \text{for every } p \in \mathrm{Fr}(\bigcup \mathcal{B}^*) \text{ there is } q \in \mathrm{Fr}(\bigcup \mathcal{A}^*) \text{ such that } q \leq_{\mathrm{RK}} p.$$

In a model of ZFC without P-points (for a model like this see [12]) condition $(*)$ holds for any two MAD families and we know that for every MAD family $\mathcal{A}$ exists, in ZFC, a MAD family $\mathcal{B}$ with $\mathcal{A} \not\leq_K \mathcal{B}$ ([5, Proposition 3]). Thus, in such a model of ZFC condition $(*)$ is not equivalent to Katětov ordering. This suggests the question whether or not condition $(*)$ implies Katětov ordering in a model of ZFC + CH.

Acknowledgment. We are grateful to the referee for correcting an error which appeared in an earlier version and for his/her suggestions to improve the paper.

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(*Received* April 26, 2007, *revised* July 13, 2007)