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On some soluble groups in which U-subgroups form a lattice

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Abstract. The article is dedicated to groups in which the set of abnormal and normal subgroups (U-subgroups) forms a lattice. A complete description of these groups under the additional restriction that every counternormal subgroup is abnormal is obtained.

Keywords: abnormal subgroups, U-subgroups, counternormal subgroups

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R.W. Carter [C] introduced the abnormal subgroups in connection with his famous investigation of the nilpotent self-normalizing subgroups in soluble finite groups. According to the definition, a subgroup A is called *abnormal in a group* G if $g \in \langle A, A^g \rangle$ for each element g of G. Maximal non-normal subgroups of arbitrary groups are obviously abnormal. Other well-known examples of abnormal subgroups in finite groups are Carter subgroups and normalizers of Sylow subgroups (see [BB, Section 6]). The famous Tits example provides us with a non-trivial abnormal subgroup of the complete linear group $GL_{(n,K)}$ over an arbitrary skew field K (see, for example, [BB, Section 1]). Infinite groups saturated with abnormal subgroups have been studied in [S], [KS], [DeFKS], and [KS2].

It follows immediately from the definition that abnormal subgroups are self-normalizing, and every subgroup containing an abnormal subgroup is also abnormal. Therefore, we can consider abnormality as a kind of strong opposite to normality. So, it seems logical to describe the groups, all proper non-normal subgroups of which are abnormal. It means that all proper subgroups of such groups are separated by two classes with an empty intersection: the class of normal subgroups and the class of abnormal subgroups. Following [KS1], we will call normal and abnormal subgroups U-normal (from "union" and "U-turn"). Finite groups with only U-normal subgroups have been considered in [F]. Locally soluble (in the periodic case locally graded) infinite groups with U-subgroups have been studied in [S1]. In [KS1] the groups with all U-normal subgroups and the groups with transitivity of U-normality have been described completely.

In this article, we will consider a natural question regarding the structure of *groups in which* U*-normal subgroups form a lattice*. We will denote these groups as $\#U$*-groups*. It is easy to see that the groups with no abnormal subgroups are $\#U$-groups. In particular, all locally-nilpotent groups have this property (see Lemma 0 below).

Recall that a subgroup H is *counternormal in* a group G if the normal closure A^G of A in G coincides with G [R].

The following lemma collects all specific facts necessary for the proofs. Some of them (like (i) and (ii)) are really obvious or very easy to prove. Others (like (iii) and (iv)) are not so immediate but can be proved in a few lines.

Lemma 0. (i) *If B is an abnormal subgroup in a group G, then $G = G'B$.*

(ii) *In a soluble group an abnormal subgroup R is exactly a subgroup that is counternormal in all subgroups containing R* ([DeFKS, Lemma 4]).

(iii) *If $FC = B$ is abnormal in G, F is normal in G, and C is abnormal in $B = FC$, then C is abnormal in G* (*see, for example* [BB, Theorem 3]).

(iv) *A locally nilpotent group does not have proper abnormal subgroups* ([KNS, Lemma 4]).

Observe that a union of any two U-normal subgroups is U-normal. However, this assertion is false for intersections. In connection with this, we have the following simple lemma.

Lemma 1. *Let G be a $\#U$-group. Then:*

(1) *an intersection of any two normal subgroups of G is normal in G;*

(2) *an intersection of a proper normal subgroup and an abnormal subgroup of G is normal in G.*

Proof: We only need to prove the second assertion. Let A be a normal subgroup in G and B be an abnormal subgroup in G. Since G is a $\#U$-group, the intersection $K = A \cap B$ could be normal in G (and in this case everything is clear) or abnormal in G. The second case means that K is abnormal in the proper normal subgroup A of G. This is impossible, since every subgroup containing an abnormal subgroup of a group is abnormal. $\square$

Corollary. *Let G be a $\#U$-group and $G \neq G'$. Then for any abnormal subgroup B the intersection $B \cap G'$ is normal in G.*

Observe that in an arbitrary group the abnormality is not transitive. For example, in S_4 there is an abnormal subgroup of order 6 which includes an abnormal in it but non-abnormal in S_4 subgroup of order 2 (see [BB, Section 1]). However, if a group G has a normal subgroup A satisfying the normalizer condition and the factor-group G/A does not have abnormal subgroups, then in G abnormality is transitive (i.e. G is a TA-group) [KS2, Theorem 1.2].

In connection with this, the following assertion seems to be interesting.

Lemma 2. *Let G be a $\#U$-group and $G \neq G'$. Then in G abnormality is a transitive relation (i.e. G is a TA-group).*

Proof: Let B be an abnormal subgroup in G, and C be an abnormal subgroup in B. By Lemma 0(i), $G'B = G$, and $B'C = B$. Then $G = G'B = G'(B'C) = G'C$. Let $F = B \cap G'$. By Lemma 1, F is normal in G. It is clear that $B = B'C = FC$. Therefore, $FC = B$ is abnormal in G, F is normal in G, and C is abnormal in $B = FC$. By Lemma 0(iii), C is abnormal in G, i.e. G is a TA-group. □

Lemma 3. *Any factor group of a $\#U$-group is a $\#U$-group.*

This lemma is a direct consequence of the following simple observation: if N is a normal subgroup of G, then any subgroup $A > N$ is abnormal (normal) in G if and only if A/N is abnormal (normal) in G/N.

Lemma 4. *Let G be a $\#U$-group and $G \neq G'$. If G contains an abnormal proper subgroup, then G/G' is a cyclic subgroup of prime power order* (*or a primary subgroup* [K, p. 179]).

Moreover, if B is a proper abnormal subgroup in G, $F = (G' \cap B) \lhd G$, $G^* = G/F \cong G'^* \ltimes B^*$, where G'^* and B^* are the images of G' and B respectively, B^* is a proper abnormal cyclic primary subgroup in G^*.

Proof: Let B be a proper abnormal subgroup of G. Then $G = G'B$. Corollary of Lemma 1 implies that $F = G' \cap B \lhd G$. By Lemma 3, without loss of generality, we can assume that $F = \langle 1 \rangle$. Let b be a non-identical element of B. Assume that $B \neq \langle b \rangle$. Consider in G a proper normal subgroup $K = G' \ltimes \langle b \rangle$. It is clear that $K \cap B = \langle b \rangle$. Lemma 1 implies that $\langle b \rangle$ is normal in G. Since every proper cyclic subgroup of B is normal in G, every proper subgroup of G is also normal in G. Observe that if B is not cyclic, then its every (finite or infinite) set of generators consists of more than one element. In this case B is generated by its normal in G cyclic subgroups and therefore B is normal in G. However, B is abnormal in G. This contradiction shows that B is a cyclic group.

Assume that $\langle b \rangle$ is infinite. Consider two proper normal subgroups $G' \ltimes \langle b^2 \rangle$ and $G' \ltimes \langle b^3 \rangle$ of G. Lemma 1 implies that $\langle b^2 \rangle$ and $\langle b^3 \rangle$ are normal in G. So their product, i.e. $\langle b \rangle$ itself, is also a normal subgroup in G. This contradiction shows that $\langle b \rangle$ is a finite subgroup.

Repeating almost the same arguments we can prove that B is a subgroup of prime power order. □

Note that from Lemma 4 it follows that there is a metabelian group that is not a $\#U$-group.

J.S. Rose has introduced in [R] the counternormal subgroups. A subgroup H is *counternormal* in a group G if the normal closure A^G of A in G coincides with G. It is easy to observe that in a soluble group an abnormal subgroup R is exactly a subgroup that is counternormal in all subgroups containing R (see

Lemma 0(ii)). The condition that every counternormal subgroup is abnormal (the CA-property) is an amplification of the TA-property. However, the class of TA-groups is wider then the class of CA-groups. The following example supports this statement and confirms the existence of TA-groups that are not $\#U$-groups. Let $Q = \langle s_1, s_2 \rangle \ltimes \langle q \rangle$, $q^3 = 1$, $\langle s_1, s_2 \rangle$ — a quaternion group, $s_1^q = s_2, s_2^q = s_1^{-1}s_2$. Then $\langle q \rangle$ is a counternormal subgroup of Q but it is not abnormal since the center $Z(Q) = \langle s_1^2 \rangle$ does not belong to $\langle q \rangle$. So Q is nilpotent-by-nilpotent, and hence Q is a TA- but not a CA-group. Moreover, Q is a $\#U$-group. Indeed, since a subgroup of index 2 is normal in a group, every proper abnormal subgroup of Q is a direct product of $Z(Q) = \langle s_1^2 \rangle = \langle s_2^2 \rangle$ and a conjugate to $\langle q \rangle$ subgroup. This implies that Q is a $\#U$-group.

The following simple examples of metabelian groups that are not a CA-groups are interesting in connection with this.

1. Let $G = P_2^\infty \ltimes \langle x \rangle$, P_2^∞ be a Prüfer 2-subgroup, $x^2 = 1$, p transfers every element of P_2^∞ into its inverse. Then G is a 2-group, G is locally nilpotent (even hypercentral), and $\langle p \rangle$ is a counternormal but not abnormal subgroup.

2. Let $G = P_2^\infty \langle x \rangle$, P_2^∞ be a Prüfer 2-subgroup, $x^4 = 1$, $x^2 \in P_2^\infty, x$ transfers every element of P_2^∞ into its inverse. Then G is a 2-group, G is locally nilpotent, and $\langle x \rangle$ is a counternormal but not abnormal subgroup. This is an example of a metabelian locally nilpotent group which is not a CA-group (even though $\langle x \rangle$ contains $Z(G)$). Note that P_2^∞ contains infinitely many G-central chief factors.

3. Let $G = R \ltimes \langle x \rangle, b^2 = 1$, where R be an additive group of rational numbers, x transfers every element of R into its inverse. Then G is an non-periodic metabelian group with the counternormal but not abnormal subgroup $\langle x \rangle$.

4. $G = L \ltimes \langle x \rangle$, $x^2 = 1$, where L is an infinite cyclic group, x transfers every element of L into its inverse. Then G is a ZD-group with the counternormal but not abnormal subgroup $\langle x \rangle$.

5. $G = (\langle x \rangle \times P_2^\infty) \ltimes \langle b \rangle$, where P_2^∞ is a Prüfer 2-group, $x^3 = 1$, $y^2 = 1$, y transfers every element from $\langle x \rangle \times P_2^\infty$ into its inverse. In this metabelian non-locally nilpotent $\#U$-group the subgroup $\langle x \rangle \times P_2^\infty$ is an abelian derived subgroup, $\langle b \rangle$ is a counternormal but not an abnormal subgroup.

Recall that a subgroup A is called *a supplement* to a subgroup B in a group G if $G = AB$ (see, for example, [LR, p. 220]).

The following proposition is interesting in connection with above examples.

Proposition. *Let G be a soluble group, A an abelian normal subgroup of G having no central chief G-factors and defining the quotient group G/A with no proper counternormal subgroups. Then:*

(1) *a subgroup B is abnormal in G if and only if B is a supplement to A in G;*
(2) *G is a CA-group.*

PROOF: (1) Let G be a soluble group, A its abelian normal subgroup having no central chief G-factors, and the quotient group G/A does not have counternormal (and therefore no abnormal) subgroups. Let B be a supplement to A in G, i.e. $G = AB$. For any subgroup $M \leq B$, we can write $M = A_M B$, where $A_M = A \cap M$. Since A is an abelian normal subgroup in G and $G = AM$, A_M is normal in G. Since A does not have central chief factors, $[M, A_M] = A_M$. Indeed, if $K = [M, A_M] \neq A_M$, then AM/K is a G-central factor. This contradicts the conditions of our proposition. So $[M, A_M] \leq M'$ and $M = M'B$. By Lemma 0(ii), B is abnormal in G.

(2) is a direct consequence of (1). $\square$

Following [RD, p. 429], we will call a maximal p-subgroup of an infinite group G a *Sylow p-subgroup* of G.

Lemma 5. *Let a soluble group G be a $\#U$-group and let $\langle b\rangle$ be a proper abnormal proper cyclic subgroup of G. Then $G = G'\langle b\rangle$ and one of the following assertions holds.*

(i) *b is an element of order p^n, $n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, and $Z(G) \leq \left\langle b^p\right\rangle \lhd G$. If the center $Z(G)$ is trivial, then $b^p \in G'$, i.e. G' has index p in G. If the center $Z(G)$ is nontrivial and G is periodic, then $Z(G) = \left\langle b^p\right\rangle$.*

(ii) *$|b| = \infty$, and there are a prime number p and a natural number n such that $b^{p^{n-1}} \in G'$, but $b^{p^n} \notin G'$; $\left\langle b^{p^n}\right\rangle$ is a normal subgroup of G with the factor-group G^*, $p \notin \pi([G^*, G^*])$, and $Z(G) \leq \left\langle b^p\right\rangle \lhd G$. If the center $Z(G)$ is trivial, then at $p \neq 2$, $b^p \in G'$, and at $p = 2$, $\left|G : G'\right| \leq 4$.*

PROOF: I. First of all, we will prove that $G = G'\langle b\rangle$, where

(i) b is an element of order p^n, $n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, or

(ii) $|b| = \infty$, and there are a prime number p and a natural number n such that $b^{p^{n-1}} \in G'$, but $b^{p^n} \notin G'$; $\left\langle b^{p^n}\right\rangle$ is a normal subgroup of G with the factor-group G^* and $p \notin \pi([G^*, G^*])$.

By Lemma 4, G/G' is a cyclic subgroup of prime power order. It means that either $\langle b\rangle$ is a primary subgroup itself, or $\langle b\rangle$ is infinite cyclic. Consider the first case where $\langle b\rangle$ is a cyclic primary subgroup such that $G = G'\langle b\rangle$, and therefore it is a counternormal subgroup in G (see Lemma 0(ii)). It follows that $\langle b\rangle$ is abnormal in a Sylow p-subgroup S of G, $S \geq \langle b\rangle$. Since G is soluble, S is locally nilpotent [RD, p. 363]. Lemma 0(iv) implies that there is no proper abnormal subgroups in a locally nilpotent group. So $S = \langle b\rangle$.

Let $|b| = \infty$. Then $\langle b\rangle$ is counternormal and therefore abnormal in G, $\langle 1\rangle \neq G' \cap \langle b\rangle$, and by Lemma 4, there are a prime number p and a natural number n such that $b^{p^{n-1}} \in G'$, but $b^{p^n} \notin G'$. By Lemma 1, $\left\langle b^{p^{n-1}}\right\rangle$ as the intersection of an abnormal subgroup $\langle b\rangle$ and a proper normal subgroup G', is normal in G.

Thus, $G/\langle b^{p^{n-1}}\rangle$ is a group from case (i) above. By Lemma 3, the factor group $G^* = G/\langle b^{p^{n-1}}\rangle = [G^*, G^*] \ltimes \langle b\rangle^*$ is a $\#U$-group from case (i). It follows that $\langle b\rangle^*$ is a Sylow p-subgroup of G^*, so $p \notin \pi([G^*, G^*])$.

Note that in the first case under the additional restriction that G being periodic, all Sylow p-subgroups of G are conjugate. This is a direct consequence of [RD, 14.3.4].

II. Let us consider the case when b is an element of finite order. By the above, $G = G'\langle b\rangle$, b is an element of order p^n, $n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G. We will show that

if the center $Z(G)$ is trivial, then $b^p \in G'$, i.e. G' has index p in G;

if the center $Z(G)$ is nontrivial, then $Z(G) \leq \langle b^p\rangle$.

First of all observe that $\langle b\rangle$ as an abnormal subgroup contains $Z(G)$. Consider the subgroup $G'\langle b^p\rangle$. If $G = G'\langle b^p\rangle$, then $b \in G'\langle b^p\rangle$ and $b = gb^{kp^n}$ where $g \in G'$, $k, n \in \mathbb{N}$, $n \geq 1$. Then $g = b^{1-kp^n}$ and $(1 - kp^n, p) = 1$. So $\langle b\rangle \leq G'$. This contradiction shows that $G'\langle b^p\rangle$ is a proper subgroup in G. By Lemma 1, the subgroup $\langle b^p\rangle = \langle b\rangle \cap G'\langle b^p\rangle$ is a normal subgroup in $G'\langle b^p\rangle$.

If $Z(G)$ is trivial, then there is an element $x \in G$ such that $\langle (b^p)^x b^{-p}\rangle = \langle b^p\rangle$ and therefore it is a subgroup of G'.

Let $Z(G) \neq \langle 1\rangle$. Since $\langle b^p\rangle$ is a normal subgroup in G and $Z(G) \leq \langle b\rangle$, for any element c of G such that $\langle c\rangle \cap \langle b\rangle = \langle 1\rangle$ the subgroup $\langle b^p\rangle \ltimes \langle c\rangle$ is hypercentral and even nilpotent. By I, if G is periodic, we come to the conclusion that $\langle b^p\rangle = Z(G)$.

III. Let now G be a soluble $\#U$-group, G contains an abnormal proper subgroup $\langle b\rangle$, $G = G'\langle b\rangle$, where $|b| = \infty$. In this case we will also consider two possibilities and prove that

if $Z(G)$ is trivial, then if $p \neq 2$, $b^p \in G'$, and if $p = 2$, then $|G : G'| \leq 4$;

if $Z(G)$ is nontrivial, then $Z(G) = \langle b^p\rangle$.

By II, there are a prime number p and a natural number n such that $b^{p^{n-1}} \in G'$, but $b^{p^n} \notin G'$; $\langle b^{p^n}\rangle$ is a normal subgroup of G with the factor-group G^* and $p \notin \pi([G^*, G^*])$. Let us consider the subgroup $G'\langle b^p\rangle$. If $G = G'\langle b^p\rangle$, then $b \in G'\langle b^p\rangle$ and $b = gb^{kp^n}$ where $g \in G'$, $k, n \in \mathbb{N}$, $n \geq 1$. Then $g = b^{1-kp^n}$ and $(1-kp^n, p) = 1$. Therefore, there exist integers v and u such that $up+v(1-kp^n) = 1$. So $\langle b\rangle \leq G'$. This contradiction shows that $G'\langle b^p\rangle$ is a proper subgroup in G. By Lemma 1, the subgroup $\langle b^p\rangle = \langle b\rangle \cap G'\langle b^p\rangle$ is a normal subgroup in $G'\langle b\rangle$.

Let $Z(G)$ be trivial. Then there is an element x of G such that $(b^p)^x = b^{-p}$; so $[x, b^p] = b^{-2p} \in G'$. If $p = 2$, it follows that $|G : G'| \leq 4$. If $p \neq 2$, then since $b^{p^n} \in G'$ and $b^{2p} \in G'$, $b^p \in G'$, and $|G : G'| = p$.

Let now $Z(G)$ be nontrivial. Since $\langle b^p\rangle$ is the intersection of the proper normal subgroup $G'\langle b^p\rangle$ and the abnormal subgroup $\langle b\rangle \geq Z(G)$, $\langle b^p\rangle$ is a normal subgroup in G and $Z(G) \leq \langle b^p\rangle$. □

Quite naturally, the next result will be a description of soluble CA-groups having $\#U$-property. From Lemma 5, we can easily derive the following theorem.

Theorem A. *Let a soluble CA-group G containing an abnormal proper subgroup be a $\#U$-group. Then $G = G'\langle b\rangle$, and one of the following assertions (1)–(2) holds.*

(1) *b is an element of order p^n, $n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, and $Z(G) \leq \left\langle b^p\right\rangle \lhd G$. If the center $Z(G)$ is trivial, then $b^p \in G'$, i.e. G' has index p in G. If the center $Z(G)$ is nontrivial and G is periodic, then $Z(G) = \left\langle b^p\right\rangle$.*

(2) *$|b| = \infty$, and there are a prime number p and a natural number n such that $b^{p^{n-1}} \in G'$, but $b^{p^n} \notin G'$; $\left\langle b^{p^n}\right\rangle$ is a normal subgroup of G with the factor-group G^*, $p \notin \pi([G^*, G^*])$, and $Z(G) \leq \left\langle b^p\right\rangle \lhd G$. If the center $Z(G)$ is trivial, then at $p \neq 2$, $b^p \in G'$, and at $p = 2$, $|G : G'| \leq 4$.*

Proof: By Lemma 4, $G = G'\langle b\rangle$, where G/G' is a primary cyclic subgroup. It means that either $\langle b\rangle$ is a primary itself, or $\langle b\rangle$ is infinite cyclic. In any case, $\langle b\rangle$ is a supplement to G' in a soluble group G. Therefore, $\langle b\rangle$ is counternormal in G. Since G is a CA-group, $\langle b\rangle$ is an abnormal subgroup in G. The rest follows from Lemma 5. □

Theorem B. *A soluble periodic CA-group G containing an abnormal proper subgroup is a $\#U$-group if and only if $G = G'\langle b\rangle$, b is an element of order p^n, $n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, $Z(G) \leq \left\langle b^p\right\rangle \lhd G$, and every abnormal subgroup B of G intersects G' by a normal in G' subgroup.*

Moreover, the following assertions hold.

(1) *If the center $Z(G)$ is trivial, then $b^p \in G'$, i.e. G' has index p in G.*

(2) *If the center $Z(G)$ is nontrivial then $Z(G) = \left\langle b^p\right\rangle$.*

In both mentioned cases, $|G : G'Z(G)| = p$.

Proof: Necessity.

Let G be a soluble periodic $\#U$-group having CA-property and containing an abnormal proper subgroup. By Theorem A(1), $G = G'\langle b\rangle$, b is an element of order p^n, $n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, and $Z(G) \leq \left\langle b^p\right\rangle \lhd G$. If the center $Z(G)$ is trivial, then $b^p \in G'$, i.e. G' has index p in G. If the center $Z(G)$ is nontrivial, then $Z(G) = \left\langle b^p\right\rangle$. Evidently, $|G : G'Z(G)| = p$. Since G' is a proper normal subgroup, Corollary from Lemma 1 implies that every abnormal subgroup B of G intersects G' by a normal in G' subgroup.

Sufficiency.

Let G be a soluble CA-group containing an abnormal proper subgroup, $G = G'\langle b\rangle$, and assume that every abnormal subgroup B of G intersects G' by a normal in G' subgroup, b is an element of order p^n, $n \geq 1$, $\langle b\rangle$ is a Sylow p-subgroup of G, $Z(G) \leq \left\langle b^p\right\rangle \lhd G$ and the following assertions hold.

(1) If the center $Z(G)$ is trivial, then $b^p \in G'$, i.e. G' has index p in G.

(2) If the center $Z(G)$ is nontrivial then $Z(G) = \langle b^p \rangle$.

We will prove that G is a $\#U$-group. Observe that since G is periodic, all Sylow p-subgroups of G are conjugate (this is a direct consequence of [RD, 14.3.4]). From the equation $G = G'\langle b\rangle$ and the property $\langle b^p \rangle \lhd G$, it follows that any abnormal subgroup of G as a supplement to G' contains $\langle b\rangle$ or some its conjugate. On the other hand, since all Sylow p-subgroups are conjugate with $\langle b\rangle$ and hence abnormal in G, we can state that any subgroup of G containing a Sylow p-subgroup is abnormal in G. So, with respect to the choice of a Sylow p-subgroup of G, any abnormal subgroup H in G can be viewed as a product $H = L \ltimes \langle b\rangle$, $L = H \cap G'$. The latter is normal in G' and in $H \geq \langle b\rangle$. Therefore it is normal in $G = G'\langle b\rangle$.

It is easy to see that every proper normal subgroup of G is contained in the subgroup $F = G'\langle b^p \rangle$. In fact, if M is a normal subgroup in G and $M \nleq F$, then $G = FM = G'(\langle b^p \rangle M)$. Since G is soluble, it is clear that there is no a proper normal subgroup $X \lhd G$ such that $G = G'X$. So, it follows that $G = \langle b^p \rangle M$. In the factor group $G^* = G/(\langle b^p \rangle \cap M) = \langle b^p \rangle^* \times M^*$ the subgroup $\langle b^p \rangle^*$ is complemented. It means that $\langle b^p \rangle^*$ is complemented in $\langle b\rangle^*$. It follows that $\langle b^p \rangle^* = \langle 1\rangle$ and $M = G$.

Let us consider two proper abnormal subgroups H and R in G. If they contain the same Sylow p-subgroup of G, let say $\langle b\rangle$, then the intersection $H \cap R$ also contains $\langle b\rangle$ and hence is abnormal in G. If the intersection of H and R does not contain a Sylow p-subgroup of G, the subgroups H and R contain at least two different conjugate Sylow p-subgroups of G, let say $\langle b\rangle$ and $\langle b^x \rangle$. Since $\langle b^p \rangle \lhd G$, $H \cap R \geq \langle b^p \rangle$. So, we can write $H = H_1 \ltimes \langle b\rangle$, $R = R_1 \ltimes \langle b^x \rangle$, $H_1 = H \cap G' \trianglelefteq G$, $R_1 = R \cap G' \trianglelefteq G$, $H \cap R \geq \langle b^p \rangle \trianglelefteq G$. It follows that $H \cap R \leq H_2 = H_1 \times \langle b^p \rangle \trianglelefteq G$, $H \cap R \leq R_2 = R_1 \times \langle b^p \rangle \trianglelefteq G$, $H \cap R \leq H_2 \cap R_2$. On the other hand, it is obvious that $H \cap R \geq H_2 \cap R_2$. So $H \cap R = H_2 \cap R_2$ and $(H_2 \cap R_2) \trianglelefteq G$.

Let now H be an abnormal subgroup in G and M be a proper normal subgroup in G. Without loss of generality, we can write $H = H_1 \ltimes \langle b\rangle$ and $M = M_1 \ltimes \langle b^{p^k} \rangle$, $k \geq 1$, $H_1 = H \cap G' \trianglelefteq G, M_1 = M \cap G' \trianglelefteq G$. Since $\langle b\rangle$ is a Sylow p-subgroup in G, $H \cap M = H_1 \ltimes \langle b^{p^k} \rangle \cap M$ and $H_1 \ltimes \langle b^{p^k} \rangle \trianglelefteq G$, $M \trianglelefteq G$. Hence, $H \cap M \trianglelefteq G$. □

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