Nonlinear degenerate elliptic equations with measure data

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Abstract. In this paper we prove existence results for some nonlinear degenerate elliptic equations with data in the space of bounded Radon measures and we improve the results already obtained in Cirmi G.R., *On the existence of solutions to non-linear degenerate elliptic equations with measure data*, Ricerche Mat. **42** (1993), no. 2, 315–329.

Keywords: nonlinear degenerate elliptic equations, existence, measure data

Classification: 35A35, 35J70

1. Introduction

This paper deals with the following problem

$$(P)\begin{cases} -\operatorname{div}(a(x,u,Du)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a bounded Radon measure on Ω, and Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$. $\nu(x)$ is a nonnegative function on $\overline{\Omega}$ such that

$$a(x,s,\xi)\xi \geq \nu(x)|\xi|^p$$

holds for almost every $x \in \Omega$ and every $(s,\xi) \in \mathbb{R}^{N+1}$.

If we assume that $\nu(x)$ is a positive constant, the existence of solution for the corresponding problem (P) involving measure data was studied in [1]–[5]. The recent progress in this area can be found in [6] (see also the bibliography of [6]).

The existence of distributional solutions of problem (P) belonging to classical weighted-Sobolev space has been proved in [7], assuming $\nu \in L^s(\Omega)$, $s > \frac{Nt}{t(p-1)-N}$ and $p > \frac{2sN-s-N}{N(s-1)}(1+\frac{1}{t})$. In [8] the author studied the existence of the solution of the corresponding problem (P) with a lower-order term if $f \in L^1(\Omega)$, but here the lower-order term played an importance role. In this paper we will discuss existence results of problem (P) without the assumption that $p > \frac{2sN-s-N}{N(s-1)}(1+\frac{1}{t})$. In order to overcome this difficulty, we introduce a new type of sets similar to the so-called T-set, already used in [4] and [8]. Our proof is based on the method used in [4]–[5].

Project partially supported by NSFC(No:10401009) and NCET of China.

Thus we improve those results for [7] and we extend some results for [1]–[5] to the nonlinear degenerate case. Similar results to parabolic equations have been obtained in [14].

The paper is organized as follows. In Section 2 we will give a new type of set and specify the link with the classical weighted-Sobolev space. In Section 3 assumptions and statements of the main results will be given. In Section 4 we will prove the main results.

2. A new type of set

In this paper Ω is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, for $1 < p < \infty$, let $\nu(x)$ be a locally integrable, nonnegative function on $\mathbb{R}^N$ and also be $p-$admissible. The definition of $p-$admissible can be seen in [9]. We need to introduce a new type of sets similar to a space defined in [4] and [8].

Let $L^0(\nu, \Omega)$ denote the set of all ν-measurable functions on Ω. Let

$$\mathrm{Lip}_p(\mathbb{R}) = \{\phi \in W^{1,\infty}(\mathbb{R}) : |\phi'|^p \in L^1(\mathbb{R}), \phi(0) = 0\}. \tag{2.1}$$

For $k > 0$, we set $T_k(\sigma) = \max\{-k, \min\{k, \sigma\}\}$, $\forall \sigma \in \mathbb{R}$. Let

$$\begin{aligned} L_0^{1,p}(\nu,\Omega) = \Big\{ & u \in L^0(\nu,\Omega) : \forall \phi \in \mathrm{Lip}_p(\mathbb{R}), \phi(u) \in H_0^{1,p}(\nu,\Omega), \\ & \sup_{k>0} \int_\Omega \nu(x) \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{1+\delta}}\,dx \ \text{ is finite for all } \ \delta > 0 \Big\} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} L_{01}^{1,p}(\nu,\Omega) = \Big\{ & u \in L_0^{1,p}(\nu,\Omega) : \exists\, C > 0 : \\ & \int_\Omega \nu(x) \frac{|DT_k(u)|^p}{(1+|T_k(u)|)}\,dx \leq C \int_\Omega (1+|T_k(u)|)\,dx, \forall\, k > 0 \Big\}. \end{aligned} \tag{2.3}$$

For any given $m > 1$, let

$$\begin{aligned} L_{0m}^{1,p}(\nu,\Omega) = \Big\{ & u \in L_0^{1,p}(\nu,\Omega) : \exists\, C > 0 : \int_\Omega \nu(x) \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda}\,dx \\ & \leq C\Big(\int_\Omega (1+|T_k(u)|)^{(1-\lambda)m/(m-1)}\,dx \Big)^{1-\frac{1}{m}}, \\ & \forall\, k > 0, \forall\, 0 < \lambda < 1 \Big\}. \end{aligned} \tag{2.4}$$

If $\nu(x)$ is a positive constant, then $L_0^{1,p}(\nu, \Omega)$ is called a T-set in [4]. It is easy to prove that $H_0^{1,p}(\nu, \Omega) \subset L_0^{1,p}(\nu, \Omega)$. Here $L_{01}^{1,p}(\nu, \Omega)$ and $L_{0m}^{1,p}(\nu, \Omega)$ are new type of sets even if $\nu(x)$ is a positive constant. If we assume that $\nu(x)$ is a nonnegative function on $\overline{\Omega}$ such that

$$\nu(x) \in L^s(\Omega), \frac{1}{\nu(x)} \in L^t(\Omega), \tag{2.5}$$

where s and t are real numbers such that

$$t > N, s > \frac{Nt}{t(p-1)-N}, \frac{2sN-s-N}{N(s-1)}(1+\frac{1}{t}) < p < N\Big(1+\frac{1}{t}\Big), \tag{2.6}$$

then we have better regularity results.

Proposition 2.1. *If* (2.5) *and* (2.6) *hold, then* $L_0^{1,p}(\nu,\Omega) \subset H_0^{1,q}(\nu,\Omega)$, $\forall\, q \in [1+\frac{1}{t}, \bar{q})$, $\bar{q} = \frac{N[ts(p-1)-tp-s]}{t[s(N-1)-N]}$.

Proof: Working as in the proof of Proposition I.1 of [4] we can prove that for any given $u \in L_0^{1,p}(\nu,\Omega)$, $Du(x)$ exists almost everywhere in Ω. For any given $k > 0$, Hölder's inequality implies that

$$\begin{aligned} &\int_\Omega \nu(x)|DT_k(u)|^q\,dx \\ &\le \Big(\int_\Omega \nu(x)\frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{1+\delta}}\,dx\Big)^{q/p}\Big(\int_\Omega \nu^s(x)\,dx\Big)^{(p-q)/(sp)} \\ &\quad\times\Big(\int_\Omega (1+|T_k(u)|)^{\frac{(1+\delta)qs'}{p-q}}\,dx\Big)^{(p-q)/(ps')} \\ &\le C_1\Big(\int_\Omega (1+|T_k(u)|)^{\frac{(1+\delta)qs'}{p-q}}\,dx\Big)^{(p-q)/(ps')}, \end{aligned} \tag{2.7}$$

where C_1 denotes a positive constant independent of u, k. From now on, C_i will denote analogous constants, which can vary from line to line. By Theorem 3.1 and Corollary 3.5 in [10], (2.7) yields

$$\Big(\int_\Omega |T_k(u)|^{q^\#}\,dx\Big)^{\frac{q}{q^\#}} \le C_2\Big[1+\Big(\int_\Omega (1+|T_k(u)|)^{\frac{(1+\delta)qs'}{p-q}}\,dx\Big)^{(p-q)/(ps')}\Big], \tag{2.8}$$

where $q^\# = \frac{Nqt}{N(1+t)-qt}$. We can choose $\delta > 0$ such that $\frac{(1+\delta)qs'}{p-q} = q^\#$, thus we can get

$$\int_\Omega |T_k(u)|^{q^\#}\,dx \le C_2 \quad \text{(note that assumption (2.6) implies } \bar{q} < p), \tag{2.9}$$

and we obtain

$$\int_\Omega \nu(x)|DT_k(u)|^q\,dx \le C_2. \tag{2.10}$$

Let $k \to \infty$ in (2.10). By Fatou's lemma, we can deduce that

$$\int_\Omega \nu(x)|Du|^q\,dx \le C_2. \tag{2.11}$$

□

Proposition 2.2. *If* (2.5) *and* (2.6) *hold, then* $L^{1,p}_{01}(\nu,\Omega) \subset H^{1,\bar q}_0(\nu,\Omega)$, *where* $\bar q$ *is the number defined in Proposition* 2.1.

Proof: For any given $u \in L^{1,p}_{01}(\nu,\Omega)$, Hölder's inequality implies that

$$\begin{aligned}
&\int_\Omega \nu(x)|DT_k(u)|^{\bar q}\,dx \\
&\le \Big(\int_\Omega \nu(x)\frac{|DT_k(u)|^p}{(1+|T_k(u)|)}\,dx\Big)^{\bar q/p}\Big(\int_\Omega \nu^s(x)\,dx\Big)^{(p-\bar q)/(sp)} \\
&\quad\times\Big(\int_\Omega (1+|T_k(u)|)^{\frac{\bar q s'}{p-\bar q}}\,dx\Big)^{(p-\bar q)/(ps')} \\
&\le C_3\Big(\int_\Omega (1+|T_k(u)|)\,dx\Big)^{\bar q/p}\Big(\int_\Omega (1+|T_k(u)|)^{\frac{\bar q s'}{p-\bar q}}\,dx\Big)^{(p-\bar q)/(ps')} \\
&\le C_3\Big(\int_\Omega (1+|u|)\,dx\Big)^{\bar q/p}\Big(\int_\Omega (1+|T_k(u)|)^{\frac{\bar q s'}{p-\bar q}}\,dx\Big)^{(p-\bar q)/(ps')}.
\end{aligned} \tag{2.12}$$

Proposition 2.1 implies that

$$\int_\Omega (1+|u|)\,dx \le C_4. \tag{2.13}$$

Let $\frac{\bar q s'}{p-\bar q} = \bar q^{\#}$. Then it follows from (2.12)–(2.13) that

$$\begin{aligned}
\Big(\int_\Omega |T_k(u)|^{\bar q^{\#}}\,dx\Big)^{\frac{\bar q}{\bar q^{\#}}} &\le C\int_\Omega \nu(x)|DT_k(u)|^{\bar q}\,dx \\
&\le C_5\Big[1+\Big(\int_\Omega |T_k(u)|^{\bar q^{\#}}\,dx\Big)^{(p-\bar q)/(ps')}\Big].
\end{aligned} \tag{2.14}$$

Noting $\frac{\bar q}{\bar q^{\#}} > (p-\bar q)/(ps')$, we get

$$\int_\Omega \nu(x)|DT_k(u)|^{\bar q}\,dx \le C_6. \tag{2.15}$$

Taking $k \to \infty$ in (2.15), we obtain $u \in H^{1,\bar q}_0(\nu,\Omega)$. □

Proposition 2.3. *If* (2.5) *and* (2.6) *hold then* $L^{1,p}_{0m}(\nu,\Omega)\subset H^{1,\hat q}_0(\nu,\Omega)$, $1<m<\bar m=\frac{Npt}{t(Np-N+p)-N}$, *where* $\hat q=\frac{mN[ts(p-1)-tp-s]}{t[s(N-m)-Nm]}$.

Proof: For any given $u\in L^{1,p}_{0m}(\nu,\Omega)$, by Hölder's inequality we get

$$\begin{aligned}
&\int_\Omega \nu(x)|DT_k(u)|^{\hat q}\,dx\\
&\le\Big(\int_\Omega \nu(x)\frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda}\,dx\Big)^{\hat q/p}\Big(\int_\Omega \nu^s(x)\,dx\Big)^{(p-\hat q)/(sp)}\\
&\quad\times\Big(\int_\Omega (1+|T_k(u)|)^{\frac{\lambda\hat q s'}{p-\hat q}}\,dx\Big)^{(p-\hat q)/(ps')}\\
&\le C_7\Big(\int_\Omega (1+|T_k(u)|)^{\frac{(1-\lambda)m}{m-1}}\,dx\Big)^{(1-\frac1m)\hat q/p}\\
&\quad\times\Big(\int_\Omega (1+|T_k(u)|)^{\frac{\lambda\hat q s'}{p-\hat q}}\,dx\Big)^{(p-\hat q)/(ps')}.
\end{aligned}\tag{2.16}$$

Let $0<\lambda<1$ and

$$\frac{(1-\lambda)m}{m-1}=\hat q^{\#},\ \ \frac{\lambda\hat q s'}{p-\hat q}=\hat q^{\#}.\tag{2.17}$$

By Theorem 3.1 and Corollary 3.5 in [10], it follows from (2.16) and (2.17) that

$$\begin{aligned}
\Big(\int_\Omega |T_k(u)|^{\hat q^{\#}}\,dx\Big)^{\frac{\hat q}{\hat q^{\#}}}&\le C\int_\Omega \nu(x)|DT_k(u)|^{\hat q}\,dx\\
&\le C_8\Big[1+\Big(\int_\Omega |T_k(u)|^{\hat q^{\#}}\,dx\Big)^{1-\frac1s+\frac{\hat q}{ps}-\frac{\hat q}{mp}}\Big].
\end{aligned}\tag{2.18}$$

Using the assumptions on $m(m<\bar m)$ and (2.5)–(2.6), we have

$$\frac{\hat q}{\hat q^{\#}}>1-\frac1s+\frac{\hat q}{ps}-\frac{\hat q}{mp}.\tag{2.19}$$

The inequalities (2.18) and (2.19) yield

$$\int_\Omega \nu(x)|DT_k(u)|^{\hat q}\,dx\le C_9\tag{2.20}$$

and letting $k\to\infty$ in (2.20) we deduce that $u\in H^{1,\hat q}_0(\nu,\Omega)$. □

3. Assumptions and statements of the main results

Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying for almost every $x \in \Omega$ and every $(s, \xi) \in \mathbb{R}^{N+1}$, $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$, $\xi \neq \xi'$,

$$a(x,s,\xi)\xi \geq \nu(x)|\xi|^p, \tag{3.1}$$

$$|a(x,s,\xi)| \leq \beta(a_0(x) + |s|^{p-1} + |\xi|^{p-1})\nu(x), \tag{3.2}$$

$$[a(x,s,\xi) - a(x,s,\xi')][\xi - \xi'] > 0, \tag{3.3}$$

where $\nu(x)$ is a nonnegative function on $\overline{\Omega}$ such that (2.5) and

$$s,t > 1,\ \frac{1}{t} + \frac{1}{s} < \frac{p}{N},\ 1 + \frac{1}{t} < p < N\Big(1 + \frac{1}{t}\Big) \tag{3.4}$$

hold, β is a positive constant, a_0 is a nonnegative function belonging to $L^{p'}(\nu, \Omega)$ and $p' = \frac{p}{p-1}$.

Definition 3.1. A function $u \in L_0^{1,p}(\nu, \Omega)$ is called a *solution of problem* (P) if $a(x, u, Du) \in L^1(\Omega)$ and

$$\int_\Omega a(x,u,Du)D\psi\, dx = \int_\Omega \psi\, df, \quad \forall\, \psi \in D(\Omega). \tag{3.5}$$

We denote by $M_b(\Omega)$ the space of bounded Radon measures on Ω. Now, we state the main results of this paper.

Theorem 3.1. *Let hypotheses* (2.5), (3.1)–(3.4) *be satisfied and* $f \in M_b(\Omega)$. *Then there exists a solution of problem* (P) *in the sense of Definition* 3.1.

Remark 3.1. The upper bound $p < N(1 + \frac{1}{t})$ is not a limitation. In fact, Theorem 3.1 is also true when $p \geq N(1 + \frac{1}{t})$ because in the case of $p > N(1 + \frac{1}{t})$, existence of a solution $u \in H_0^{1,p}(\nu, \Omega)$ is a consequence of the result of [11] or [12], and in the case of $p = N(1 + \frac{1}{t})$, we only need to modify simply the proof of Lemma 4.2–4.3 and Theorem 3.1.

Theorem 3.2. *Let hypotheses* (2.5), (3.1)–(3.4) *be satisfied and* $|f| \log(1+|f|) \in L^1(\Omega)$. *Then problem* (P) *admits a solution* $u \in L_{01}^{1,p}(\nu, \Omega)$.

Theorem 3.3. *Let hypotheses* (2.5), (3.1)–(3.4) *be satisfied and* $f \in L^m(\Omega)$, $1 < m < \bar{m}$[1]. *Then problem* (P) *admits a solution* $u \in L_{0m}^{1,p}(\nu, \Omega)$.

Remark 3.2. Theorems 3.1–3.3 improve Theorems 1–3 of [7].

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[1] $\bar{m}$ is the number defined in Proposition 2.3.

4. Proofs of the main results

In order to prove Theorem 3.1, we consider the approximate problems

$$(P_n) \begin{cases} -\operatorname{div}(a(x,u_n,Du_n)) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\{f_n\} \subset D(\Omega)$ and satisfy

$$\text{(4.1)} \quad \|f_n\|_{L^1(\Omega)} \le C\|f\|_{M_b(\Omega)} \text{ and } f_n \to f \text{ in the weak* topology of measures.}$$

By the well-known result of [11] and [12], there exists at least a solution $u_n \in H_0^{1,p}(\nu,\Omega)$ of problem (P_n) such that

$$\text{(4.2)} \qquad \int_\Omega a(x,u_n,Du_n)D\psi\,dx = \int_\Omega f_n\psi\,dx,\ \forall\,\psi \in H_0^{1,p}(\nu,\Omega).$$

Lemma 4.1. *There exists a positive constant C, independent of n, such that*

$$\text{(4.3)} \quad \int_\Omega \nu(x)|D\phi(u_n)|^p\,dx \le C\|f\|_{M_b(\Omega)}\||\phi'|^p\|_{L^1(\mathbb{R})},$$

for any given $\phi \in \mathrm{Lip}_p(\mathbb{R})$.

Proof: Taking $\psi(x) = \int_0^{u_n(x)} |\phi'|^p d\sigma$ as test function in (4.2), we have

$$\text{(4.4)} \qquad \begin{aligned} \int_\Omega a(x,u_n,Du_n)Du_n|\phi'(u_n)|^p\,dx &= \int_\Omega f_n\psi(u_n)\,dx \\ &\le \|f_n\|_{L^1(\Omega)}\||\phi'|^p\|_{L^1(\mathbb{R})}. \end{aligned}$$

Thus, inequality (4.3) follows from (3.1), (4.1) and (4.4). □

As a consequence of the previous lemma we have the following corollaries:

Corollary 4.1. *There exists a positive constant C_{10}, independent of n, such that*

$$\text{(4.5)} \qquad \int_\Omega \nu(x)|DT_k(u_n)|^p\,dx \le C_{10}k, \quad \textit{for any given } k > 0.$$

Corollary 4.2. *For any given $\delta > 0$, there exists a positive constant $C(\delta)$ independent of n, k such that*

$$\text{(4.6)} \qquad \int_\Omega \nu(x)\frac{|DT_k(u_n)|^p}{(1+|T_k(u_n)|)^{1+\delta}}\,dx \le C(\delta).$$

For any given $k > 0$, Corollary 4.1 implies that $\{T_k(u_n)\}$ is a bounded set in $H_0^{1,p}(\nu,\Omega)$. Thus there exists a subsequence of $\{T_k(u_n)\}$(still be denoted by $\{T_k(u_n)\}$) such that $T_k(u_n)$ converges weakly in $H_0^{1,p}(\nu,\Omega)$, strongly in $L^p(\Omega)$ and almost everywhere in Ω. In order to pass to the limit in (4.2) we have to prove that the sequence (Du_n) converges to Du almost everywhere in Ω.

Lemma 4.2. *There exist two positive constants C_{11}, C_{12} independent of n, k such that*

$$\text{meas}\{|u_n| > k\} \le C_{11} k^{-\frac{Nt(p-1)}{N(1+t)-pt}}, \tag{4.7}$$

$$\text{meas}_\nu\{|u_n| > k\} = \int_{\{|u_n|>k\}} \nu(x)\, dx \le C_{12} k^{-p_1}, \tag{4.8}$$

for all $n \in \mathbb{N}$, $k > 0$, where $p_1 = \frac{Nt(p-1)(s-1)}{[N(1+t)-pt]s}$.

Proof: Let $k > 0$. Working as in [5] and using Theorem 3.1 and Corollary 3.5 of [10] and estimate (4.5), we have

$$\|T_k(u_n)\|_{L^{p^\#}(\Omega)} \le C_{13} \|DT_k(u_n)\|_{L^p(v,\Omega)} \le C_{14} k^{\frac{1}{p}}, \tag{4.9}$$

where $p^\# = \frac{Npt}{N(1+t)-pt}$. For $0 < \varepsilon \le k$, we have $\{|u_n| \ge \varepsilon\} = \{|T_k(u_n)| \ge \varepsilon\}$. Hence

$$\text{meas}\{|u_n| > \varepsilon\} \le \left(\frac{\|T_k(u_n)\|_{L^{p^\#}(\Omega)}}{\varepsilon} \right)^{p^\#} \le C_{15} k^{\frac{p^\#}{p}} \varepsilon^{-p^\#}. \tag{4.10}$$

Setting $\varepsilon = k$, we obtain (4.7). Moreover, Hölder's inequality implies that

$$\begin{aligned} \text{meas}_\nu\{|u_n| > k\} &= \int_{\{|u_n|>k\}} \nu(x)\, dx \\ &\le \Big(\int_\Omega \nu(x)^s\, dx \Big)^{\frac{1}{s}} \text{meas}\{|u_n| > k\}^{\frac{s-1}{s}} \\ &\le C_{16}\, \text{meas}\{|u_n| > k\}^{\frac{s-1}{s}}. \end{aligned} \tag{4.11}$$

Thus one can deduce (4.8) from (4.11) and (4.7). □

Lemma 4.3. *There exists a positive constant C_{17} independent of n such that*

$$\text{meas}_\nu\{|Du_n| > h\} = \int_{\{|Du_n|>h\}} \nu(x)\, dx \le C_{17} h^{-p_2}, \quad \text{for all } n \in \mathbb{N} \text{ and } h > 0 \tag{4.12}$$

where $p_2 = \frac{Ntp(p-1)(s-1)}{(N-pt+Ntp)s-Nt(p-1)}$.

Proof: Let $h, k > 0$. Then

$$\text{meas}_\nu\{|DT_k(u_n)| > \frac{h}{2}\} \le \int_\Omega \Big[\frac{|DT_k(u_n)|}{\frac{h}{2}}\Big]^p \nu(x)\, dx \le C_{18} \frac{k}{h^p}. \tag{4.13}$$

Thus, from (4.8) and (4.13) it follows that

$$
\begin{aligned}
\operatorname{meas}_\nu\{|D(u_n)|>h\} &\le \operatorname{meas}_\nu\{|Du_n - DT_k(u_n)| > \frac{h}{2}\} \\
&\quad + \operatorname{meas}_\nu\{|DT_k(u_n)| > \frac{h}{2}\} \\
&\le \operatorname{meas}_\nu\{|u_n|>k\} + \operatorname{meas}_\nu\{|DT_k(u_n)| > \frac{h}{2}\} \\
&\le C_{11}k^{-p_1} + C_{18}\frac{k}{h^p}.
\end{aligned} \tag{4.14}
$$

Setting $k = h^{p_2/p_1}$ in (4.14), we get (4.12). □

Remark 4.1. Lemmas 4.2–4.3 generalize Lemmas 4.1–4.2 of [5].

Proof of Theorem 3.1: Lemma 4.2 implies that $\forall\,\varepsilon>0$, there is a large k such that

$$\operatorname{meas}\{|u_n|>k\} + \operatorname{meas}\{|u_m|>k\} < \frac{\varepsilon}{2} \tag{4.15}$$

holds for all m and n. For the above $k>0$, since $\{T_k(u_n)\}$ converges almost everywhere in Ω, $\{T_k(u_n)\}$ is a Cauchy sequence in measure in Ω, $\forall\,\sigma>0$, $\forall\,\varepsilon>0$, there is a large N such that

$$\operatorname{meas}\{|T_k(u_n) - T_k(u_m)|>\sigma\} < \frac{\varepsilon}{2} \tag{4.16}$$

holds for all $m,n>N$. (4.15) and (4.16) yield

$$
\begin{aligned}
\operatorname{meas}\{|u_n-u_m|>\sigma\} &\le \operatorname{meas}\{|u_n|>k\} + \operatorname{meas}\{|u_m|>k\} \\
&\quad + \operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\sigma\} \\
&< \varepsilon, \quad \forall\, m,n>N.
\end{aligned} \tag{4.17}
$$

Thus (4.17) implies that $\{u_n\}$ is a Cauchy sequence in measure in Ω, hence we get

$$u_n \longrightarrow u \ \text{ a.e. in } \Omega \text{ (up to a subsequence).} \tag{4.18}$$

It follows that

$$T_k(u_n) \longrightarrow T_k(u) \ \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega \tag{4.19}$$

and

$$T_k(u_n) \rightharpoonup T_k(u) \ \text{ weakly in } \ H_0^{1,p}(\nu,\Omega). \tag{4.20}$$

Working as in the proof of Theorem 2.1 in [13] we can prove that $\{Du_n\}$ is a Cauchy sequence in measure in Ω. Hence there is a measurable function $w(x)$ in Ω such that

$$Du_n \longrightarrow w \ \text{ in measure or a.e. in } \Omega. \tag{4.21}$$

Using (4.18), (4.20) and (4.21), we can deduce that $Du = w$ and

$$Du_n \longrightarrow Du \ \text{ a.e. in } \Omega \text{ (or up to a subsequence).} \tag{4.22}$$

Letting $n \to \infty$ in (4.3) and (4.6), Fatou's lemma yields

$$\int_\Omega \nu(x)|D\phi(u)|^p \, dx \le C(\phi), \tag{4.23}$$

$$\int_\Omega \nu(x)\frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{1+\delta}} \, dx \le C(\delta). \tag{4.24}$$

Thus we obtain $u \in L_0^{1,p}(\nu,\Omega)$. Moreover, (4.18) and (4.22) imply that

$$a(x,u_n,Du_n) \longrightarrow a(x,u,Du) \ \text{ a.e. in } \ \Omega. \tag{4.25}$$

By Lemma 4.2–4.3 we can get

$$|u|^{p-1} \in M^{\frac{Nt(s-1)}{[N(1+t)-pt]s}}(\nu,\Omega)\,^2,$$
$$|Du|^{p-1} \in M^{\frac{Ntp(s-1)}{(N-pt+Ntp)s-Nt(p-1)}}(\nu,\Omega),$$

hence we have

$$\frac{1}{\nu(x)}a(x,u_n,Du_n) \in M^{\frac{Ntp(s-1)}{(N-pt+Ntp)s-Nt(p-1)}}(\nu,\Omega) \subset L^q(\nu,\Omega),$$

$q \in [1, \frac{Ntp(s-1)}{(N-pt+Ntp)s-Nt(p-1)})$. Vitali's theorem implies that

$$\frac{1}{\nu}a(x,u_n,Du_n) \longrightarrow \frac{1}{\nu}a(x,u,Du) \ \text{ strongly in } \ L^q(\nu,\Omega). \tag{4.26}$$

By virtue of (3.4) it follows

$$a(x,u_n,Du_n) \longrightarrow a(x,u,Du) \ \text{ strongly in } \ L^1(\Omega). \tag{4.27}$$

[2] The definition of $\mathcal{M}^r(\nu,\Omega)$ is as follows: For $0 < r < +\infty$, $\mathcal{M}^r(\nu,\Omega)$ can be defined as the set of measurable function $v : \Omega \to \mathbb{R}$ such that $\mathrm{meas}_\nu\{|v| > k\} = \int_{\{|v|>k\}} \nu(x)\,dx \le Ck^{-r}$, $\forall\, k > 0$, $C < +\infty$.

Taking $\psi \in D(\Omega)$ in (4.2) and letting $n \to \infty$, we get

$$\int_\Omega a(x,u,Du)D\psi\,dx = \int_\Omega \psi\,df,\ \forall\,\psi \in D(\Omega), \tag{4.28}$$

which concludes our proof. □

Proof of Theorem 3.2: Let $\{f_n\} \subset D(\Omega)$ be such that

$$f_n \longrightarrow f \ \text{ strongly in } \ L^1\log L^1(\Omega). \tag{4.29}$$

Since $f \in L^1\log L^1(\Omega) \subset L^1(\Omega)$, by Theorem6 3.1, we obtain $u \in L_0^{1,p}(\nu,\Omega)$ and (3.5) holds. Hence we only need to prove that $\forall\,k > 0$, there is a positive constant C_{19} independent of k such that

$$\int_\Omega \nu(x)\frac{|DT_k(u)|^p}{(1+|T_k(u)|)}\,dx \le C_{19}\int_\Omega (1+T_k(u))\,dx. \tag{4.30}$$

As a matter of fact, taking $\psi(x) = \int_0^{T_k(u_n(x))}\frac{1}{1+|t|}\,dt$ as test function in (4.2), we have

$$\begin{aligned}\int_\Omega \frac{a(x,u_n,Du_n)}{(1+|T_k(u_n)|)}DT_k(u_n)\,dx &\le \int_\Omega |f_n|\log(1+|T_k(u_n)|)\,dx\\ &\le \int_\Omega |f_n|\log(1+|f_n|)\,dx\\ &\quad + \int_\Omega (1+|T_k(u_n)|)\,dx.\end{aligned} \tag{4.31}$$

The assumption (3.1) and inequality (4.29) imply that

$$\int_\Omega \nu(x)\frac{|DT_k(u_n)|^p}{(1+|T_k(u_n)|)}\,dx \le C_{20}\int_\Omega (1+T_k(u_n))\,dx. \tag{4.32}$$

Letting $n \to \infty$ in (4.32), Fatou's lemma yields (4.30). □

Proof of Theorem 3.3: Let $\{f_n\} \subset D(\Omega)$ be such that

$$f_n \longrightarrow f \ \text{ strongly in } \ L^m(\Omega). \tag{4.33}$$

By Theorem 3.1, we get $u \in L_0^{1,p}(\nu,\Omega)$ and (3.5). Hence we only need to prove that $\forall\,k > 0$, $\forall\,0 < \lambda < 1$, there is a positive constant C_{21} independent of k such that

$$\int_\Omega \nu(x)\frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda}\,dx \le C_{21}\Big(\int_\Omega (1+|T_k(u)|)^{(1-\lambda)m/(m-1)}\,dx\Big)^{1-\frac{1}{m}}. \tag{4.34}$$

Set $\psi(x) = \int_0^{T_k(u_n(x))} \frac{1}{(1+|t|)^\lambda}\,dt$ as test function in (4.2). We have

(4.35)
$$\begin{aligned}\int_\Omega \nu(x)\frac{|DT_k(u_n)|^p}{(1+|T_k(u_n)|)^\lambda}\,dx &\le \frac{1}{1-\lambda}\|f_n\|_{L^m(\Omega)} \\ &\quad\times \Big(\int_\Omega (1+|T_k(u_n)|)^{(1-\lambda)m/(m-1)}\,dx\Big)^{1-\frac{1}{m}} \\ &\le C_{22}(\int_\Omega (1+|T_k(u_n)|)^{(1-\lambda)m/(m-1)}\,dx)^{1-\frac{1}{m}}.\end{aligned}$$

Letting $n \to \infty$ in (4.35) we obtain (4.34). □

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(*Received* January 13, 2007, *revised* August 20, 2007)