A non-Tychonoff relatively normal subspace

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Abstract. This paper presents a new consistent example of a relatively normal subspace which is not Tychonoff.

Keywords: relative topological property, relative normality

Classification: 54B05, 54D15

1. Introduction

In recent years, Arhangel'skii and others have formalized the study of relative topological properties [1]. The relativization of a topological property can often be formulated in a variety of ways. For instance, if one wants to relativize normality, there are many choices about which disjoint closed sets should be separated by which disjoint open sets. While there are other accepted variations, the following has become the standard definition of relative normality:

Definition 1.1. X is (*relatively*) *normal in* Y if whenever C and D are disjoint closed subsets of Y, there are disjoint sets U and V open in Y, such that $C \cap X \subset U$ and $D \cap X \subset V$.

The following are easy consequences of this definition.

(1) Every subspace of a normal space is relatively normal in that space.
(2) Every relatively normal subspace is itself a regular space.
(3) A relatively normal subspace need not be normal.

These facts lead to the following question posed by Arhangel'skii [2]: *Must a relatively normal subspace of a regular space be Tychonoff?* In other words, is there something about being relatively normal that forces a subspace to have stronger separation properties than the space in which it is embedded?

Gartside and Glyn [3] answer this question in the negative assuming Martin's Axiom and $2^{\aleph_0} > \aleph_2$. By using a different example of a regular non-Tychonoff space and adapting some of their techniques, we produce another counterexample which still requires Martin's Axiom but weakens the second assumption to the negation of the Continuum Hypothesis.

2. Mysior's corkscrew space

Mysior has constructed a space which is regular and Hausdorff, but not Tychonoff [4]. We modify Mysior's construction to obtain a space which will serve as our subspace. The proofs in this section are completely analogous to those of Mysior. First, define the point set

$$X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{\infty\}.$$

For each $r \in \mathbb{R}$, define the following sets:

$$U_r = \{\infty\} \cup \{(x, y) : x > r\},$$
$$V_r = \{(r, y) : 0 \leq y < 2\} \cup \{(r + y, y) : 0 < y < 2\}.$$

$U \subseteq X$ is defined to be open in X if

(1) $\infty \in U$ implies that $U_r \subseteq U$ for some $r \in \mathbb{R}$,
(2) $(x, 0) \in U$ implies that $V_x \setminus C \subseteq U$ for some at most countable set $C \subset V_x \setminus \{(x, 0)\}$.

Remark 2.1. In Mysior's original example, only finite sets are removed from V_x. We allow the removal of countable sets as well because this is necessary for our (yet to be defined) larger space to be Hausdorff. Unfortunately, this change will mean that $\neg$ CH is required for the space X to be non-Tychonoff.

Proposition 2.2. *X is a regular Hausdorff space.*

Proof: It is trivial to prove that X is Hausdorff. To see that X is regular, note that the sets $V_x \setminus C$ form a basis of clopen neighborhoods for $(x, 0)$, and $\infty \in U_x \subset \overline{U_x} \subset U_{x-2}$. □

Lemma 2.3. *Let $f \in C(X)$ and $x \in \mathbb{R}$. Then there is an at most countable set C_x such that $f(V_x \setminus C_x) = \{f(x, 0)\}$.*

Proof: Suppose $f \in C(X)$ and $x \in \mathbb{R}$, and let $z = f(x, 0)$. For each $n \in \mathbb{N}$, there exists an at most countable set $C_n \subset X \setminus (\mathbb{R} \times \{0\})$ such that $f(V_x \setminus C_n) \subseteq (r - 1/n, r + 1/n)$. Then $C_x = \bigcup C_n$ is at most countable and $f(V_x \setminus C_x) = \{z\}$. □

Lemma 2.4 ($\neg$ CH). *Suppose $f \in C(X)$, $z \in \mathbb{R}$, and $n \in \mathbb{N}$ such that*

$$f^{\leftarrow}(z) \cap ([n, n + 1] \times \{0\})$$

is uncountable. Then

$$f^{\leftarrow}(z) \cap ([n + 1, n + 2] \times \{0\})$$

is uncountable.

Proof: Let $R = \{y \in \mathbb{R} : f(y,0) = z\}$. Since $R \cap [n, n+1]$ is uncountable, there is a set $S \subseteq R \cap (n, n+1)$ such that $|S| = \aleph_1$.

By Lemma 2.3, for each $x \in S$, there is an at most countable set C_x such that $f(V_x \setminus C_x) = \{z\}$. Now $C = \bigcup\{C_x : x \in S\}$ has cardinality at most $\aleph_1$. In particular,

$$|C \cap ([n+1, n+2] \times (0,2))| \leq \aleph_1.$$

But since we are assuming $\neg$ CH, $|[n+1, n+2]| = \mathfrak{c} > \aleph_1$, so the set

$$T = \{y \in [n+1, n+2] : (\{y\} \times (0,2)) \cap C = \emptyset\}$$

has cardinality $\mathfrak{c}$. For each $y \in T$, let

$$T_y = (\{y\} \times [0,2)) \cap \bigcup\{V_x \setminus C : x \in S\}.$$

Then $|T_y| = |S| = \aleph_1$, $T_y \subset V_y$, and $f(T_y) = \{z\}$, so $f(y,0) = z$. Thus $R \cap [n+1, n+2]$ is uncountable. □

Proposition 2.5 ($\neg$ CH)**.** *X is not Tychonoff.*

Proof: To show that X is not Tychonoff, we will show that ∞ and a closed set of the form $[n, n+1] \times \{0\}$ cannot be separated by a continuous function on X.

Suppose $f \in C(X)$ and $f([n, n+1] \times \{0\}) = \{z\}$. Let $R = \{y \in \mathbb{R} : f(y,0) = z\}$. By Lemma 2.4, $R \cap [n+1, n+2]$ is uncountable. By induction, $R \cap [m, m+1]$ is uncountable for all $m \geq n$. Thus $\infty \in \overline{R \times \{0\}}^X$, and $f(\infty) = z$.

So $[n, n+1] \times \{0\}$ and ∞ cannot be separated by any $f \in C(X)$. □

Remark 2.6. While the proof that X is not Tychonoff requires $\neg$ CH, the proof that Mysior's original space is not Tychonoff requires only ZFC. Furthermore, Bill Fleissner has noted that the property of the space X being Tychonoff is actually equivalent to the Continuum Hypothesis. If CH holds, then a continuous function $f : X \to [0,1]$ such that $f(\infty) = 1$ and $f((-\infty, r] \times [0, \infty)) = \{0\}$ can be constructed.

3. Construction of the larger space

In this section, we show how to embed the space X as a relatively normal subspace of a regular space Y. Some of the techniques and notation are adapted from [3]. Throughout this section, assume MA + $\neg$ CH. Define $\mathcal{S}$ to be the set

$$\{(P,Q) \in [\mathbb{R}]^{\omega_1} \times [\mathbb{R}]^{\omega_1} : \exists N \in \mathbb{Z} \text{ such that } P \cup Q \subset [N, N+4)\}.$$

By our set-theoretic assumptions, $\mathfrak{c} \leq |\mathcal{S}| \leq |[\mathbb{R}]^{\omega_1} \times [\mathbb{R}]^{\omega_1}| = \mathfrak{c}$. Thus, we can index

$$\mathcal{S} = \{(P_\alpha, Q_\alpha) : \alpha < \mathfrak{c}\}.$$

Let $\mathcal{A}$ be a maximal almost disjoint family of countable subsets of $\mathbb{R}$. $|\mathcal{A}| = \mathfrak{c}$, so by induction, for all $\alpha < \mathfrak{c}$, there exist

$$A'_\alpha, B'_\alpha \in \mathcal{A} \setminus \{A'_\beta, B'_\beta : \beta < \alpha\}$$

such that $|A'_\alpha \cap P_\alpha| = \aleph_0$ and $|B'_\alpha \cap Q_\alpha| = \aleph_0$. The induction step is possible because for each α, the collection

$$\{A \cap P_\alpha : A \in \mathcal{A} \;\text{ and }\; |A \cap P_\alpha| = \aleph_0\}$$

forms an infinite maximal almost disjoint family of subsets of P_α. Under MA + $\neg$ CH, this collection has cardinality $\mathfrak{c}$, and thus there are many choices for A'_α and B'_α. (See [3].)

To simplify the notation, let $A_\alpha = A'_\alpha \cap P_\alpha$ and $B_\alpha = B'_\alpha \cap Q_\alpha$. Then, construct a space

$$Y = X \cup \{(A_\alpha, B_\alpha) : \alpha < \mathfrak{c}\},$$

and give Y the following topology $\tau(Y)$:

(1) $X \setminus \{\infty\} \in \tau(Y)$.
(2) $(A_\alpha, B_\alpha) \in U \in \tau(Y)$ implies that
  (a) $U \cap X$ is open in X, and
  (b) $[(A_\alpha \cup B_\alpha) \setminus F] \times \{0\} \subset U$ for some finite F.
(3) $\infty \in U \in \tau(Y)$ implies that for some $r \in \mathbb{R}$,

$$U_r \cup \{(A_\alpha, B_\alpha) : (A_\alpha \cup B_\alpha) \subset (r, \infty)\} \subset U.$$

Proposition 3.1. *Y is a regular Hausdorff space.*

Proof: Y can be easily seen to be Hausdorff. To prove that Y is regular, note that a basic open neighborhood of (A_α, B_α),

$$\{(A_\alpha, B_\alpha)\} \cup \bigcup \{V_x \setminus C_x : x \in (A_\alpha \cup B_\alpha) \setminus F\},$$

where each C_x is countable, is clopen. To see regularity at the point ∞, it is straightforward to show that $\overline{W_r} \subset W_{r-6}$, where

$$W_r = U_r \cup \{(A_\alpha, B_\alpha) : (A_\alpha \cup B_\alpha) \subset (r, \infty)\}.$$

□

Proposition 3.2. *X is relatively normal in Y.*

Proof: Suppose C and D are disjoint closed sets in Y such that $C \cap X$ and $D \cap X$ are nonempty. We need to show that $C \cap X$ and $D \cap X$ can be separated by disjoint open sets in Y. Since the points (x, y), $y > 0$ are isolated, we actually just need to show that

$$C' = C \cap ((\mathbb{R} \times \{0\}) \cup \{\infty\}) \text{ and}$$
$$D' = D \cap ((\mathbb{R} \times \{0\}) \cup \{\infty\})$$

can be separated by disjoint open sets in Y. Before proceeding, it is useful to define

$$V'_y = V_y \setminus \bigcup \{V_y \cap V_x : (x, 0) \in C'\}.$$

The argument breaks down into several cases:

(1) Suppose C' is countable and $\infty \notin C \cup D$. For each $(y, 0) \in D'$,

$$\bigcup \{V_y \cap V_x : (x, 0) \in C'\}$$

is at most countable, so V'_y is open. Then $\bigcup \{V_x : (x, 0) \in C'\}$ and $\bigcup \{V'_y : (y, 0) \in D'\}$ are disjoint open sets separating C' and D'.

Remark 3.3. If $\infty \notin C$ and C is closed, then C is bounded.

(2) If C' is countable and $\infty \in C'$, $D' \subset (-\infty, N] \times \{0\}$ for some $N \in \mathbb{Z}$. Then $\bigcup \{V'_y : (y, 0) \in D'\}$ and $\bigcup \{V_x : (x, 0) \in C'\} \cup W_{N+8}$ are disjoint open sets separating C' and D'.

(3) Suppose C' is countable and $\infty \in D$. There is an N such that $C' \subset (\infty, N] \times \{0\}$. Then $\bigcup \{V_x : (x, 0) \in C'\}$ and $\bigcup \{V'_y : (y, 0) \in D'\} \cup W_{N+8}$ are disjoint open sets separating C' and D'.

(4) Suppose both C' and D' are uncountable, but for all $N \in \mathbb{Z}$, at least one of $C' \cap ((N, N + 4) \times \{0\})$ and $D' \cap ((N, N + 4) \times \{0\})$ is countable. Then C' and D' can be separated by disjoint open sets as in the previous cases.

(5) If there does exist an $N \in \mathbb{Z}$ such that $C' \cap ((N, N + 4) \times \{0\})$ and $D' \cap ((N, N + 4) \times \{0\})$ are both uncountable, then there is an $\alpha < \mathfrak{c}$ such that $P_\alpha \times \{0\} \subset C'$ and $Q_\alpha \times \{0\} \subset D'$. But then

$$(A_\alpha, B_\alpha) \in \overline{C'}^Y \cap \overline{D'}^Y \subset C \cap D.$$

As C and D are disjoint, this is not possible. □

Acknowledgments. I would like to thank the referee for his/her careful reading and many useful suggestions. I am also grateful to Jack Porter, my advisor at the University of Kansas, for his support.

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(*Received* June 26, 2006, *revised* June 28, 2007)