Interior regularity of weak solutions to the equations of a stationary motion of a non-Newtonian fluid with shear-dependent viscosity. The case $q = \frac{3d}{d+2}$

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Abstract. In this paper we consider weak solutions $\mathbf{u} : \Omega \to \mathbb{R}^d$ to the equations of stationary motion of a fluid with shear dependent viscosity in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$). For the critical case $q = \frac{3d}{d+2}$ we prove the higher integrability of $\nabla\mathbf{u}$ which forms the basis for applying the method of differences in order to get fractional differentiability of $\nabla\mathbf{u}$. From this we show the existence of second order weak derivatives of u.

Keywords: non-Newtonian fluids, weak solutions, interior regularity

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1. Introduction. Statement of the main result

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) be a domain. The stationary motion of an incompressible fluid through Ω is governed by the following two equations

$$-\operatorname{div}\mathbf{S} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mathbf{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

where

$$\begin{aligned} \mathbf{S} = \{S_{ij}\} = {} & \text{ deviatoric stress tensor}^{(1)}, \\ p = {} & \text{ pressure}, \\ \mathbf{u} = \{u_1, \dots, u_d\} = {} & \text{ velocity}, \\ \mathbf{f} = \{f_1, \dots, f_d\} = {} & \text{ external force}. \end{aligned}$$

On the boundary of Ω we assume the following condition of adherence

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

(1) Throughout Latin subscripts take the values 1 to d. Repeated subscripts imply summation over 1 to d.

In addition, $\mathbf{S}$ may depend on the "rate of strain tensor" $D = \{D_{ij}\}$, which is defined by

$$D_{ij} = D_{ij}(\mathbf{u}) := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, \dots, d$$

(for the continuum mechanical background cf. [2], [3], [10]).

To motivate the conditions on $\mathbf{S}$ let us mention the following constitutive laws which are often used in engineering practice

$$\mathbf{S} = \nu(D_{II})^{(q-2)/2} D, \quad 1 < q < 2$$
$$\mathbf{S} = \nu(1 + D_{II})^{(q-2)/2} D, \quad 1 < q < 2 \quad (\nu = \text{const} > 0),$$

where

$$D_{II} = \frac{1}{2} D_{ij} D_{ij} = \text{ second invariant of } D$$

(cf. [2], [4], [13]). A fluid which is determined by the first of these constitutive laws is said "pseudoplastic" or "shear thinning". Having in mind these constitutive laws as special cases we impose the following conditions on the components of the deviatoric stress $\mathbf{S}$. Let μ denote either the number 1 or 0.

(I) $$S_{ij} \in C(\mathbf{M}^{d^2}_{\text{sym}})^{(2)};$$

(II) $$|S_{ij}(\boldsymbol{\xi})| \le c_0(\mu + |\boldsymbol{\xi}|^{q-1}) \quad \forall\, \boldsymbol{\xi} \in \mathbf{M}^{d^2}_{\text{sym}};$$

(III) $$\begin{cases} (S_{ij}(\boldsymbol{\xi}) - S_{ij}(\boldsymbol{\eta}))(\xi_{ij} - \eta_{ij}) \ge \nu_0(\mu + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{q-2}|\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \\ \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbf{M}^{d^2}_{\text{sym}} \end{cases}$$

($c_0 > 0$, $\nu_0 > 0$ and $1 < q < 2$).

Weak solution to (1.1)–(1.3). Before we introduce the notion of a weak solution to (1.1), (1.2) let us provide some notations and function spaces which will be used in sequence of the paper. By $W^{k,q}(\Omega)$, $W^{k,q}_0(\Omega)$ $(k \in \mathbb{N}; 1 \le q \le +\infty)$ we denote the usual Sobolev spaces. By $C^\infty_0(\Omega)$ we denote the space of all smooth functions having compact support in Ω. Then we set

$$\mathcal{D}_\sigma(\Omega) := \Big\{\boldsymbol{\varphi} \in C^\infty_0(\Omega)^d \;\Big|\; \operatorname{div} \boldsymbol{\varphi} = 0\Big\},$$
$$D^{1,q}_0(\Omega) := \text{ closure of } \mathcal{D}_\sigma(\Omega) \text{ in } W^{1,q}(\Omega).$$

(2) $\mathbf{M}^{d^2}_{\text{sym}}$ = vector space of all symmetric $d \times d$ matrices $\boldsymbol{\xi} = \{\xi_{ij}\}$. We equip $\mathbf{M}^{d^2}_{\text{sym}}$ with scalar product $\boldsymbol{\xi} : \boldsymbol{\eta} := \xi_{ij}\eta_{ij}$ and norm $|\boldsymbol{\xi}| := (\boldsymbol{\xi} : \boldsymbol{\xi})^{1/2}$. By $|\mathbf{a}|$ we denote the norm of $\mathbf{a} \in \mathbb{R}^d$.

Definition 1.1. Let $\frac{2d}{d+2} \le q < 2$. Assume (II). Let $\mathbf{f} \in L^1(\Omega)^d$. A vector-valued function $\mathbf{u} \in D_0^{1,q}(\Omega)$ is called a *weak solution to* (1.1)–(1.3) if the following integral identity is fulfilled for all $\boldsymbol{\varphi} \in \mathcal{D}_\sigma(\Omega)$:

$$\int_\Omega S_{ij}(D(\mathbf{u}))D_{ij}(\boldsymbol{\varphi})\,\mathrm{d}\,x - \int_\Omega u_i u_j \frac{\partial \varphi_i}{\partial x_i}\,\mathrm{d}\,x = \int_\Omega f_j \varphi_j\,\mathrm{d}\,x. \tag{1.4}$$

$\square$

Remarks. If $q \ge \frac{3d}{d+2}$ by Sobolev's imbedding theorem we have $u_i u_j \frac{\partial v_i}{\partial x_i} \in L^1(\Omega)$ for all $\mathbf{u}, \mathbf{v} \in W^{1,q}(\Omega)^d$. Thus, in (1.4) the test function $\boldsymbol{\varphi} \in \mathcal{D}_\sigma(\Omega)$ can be replaced by $\boldsymbol{\varphi} \in D_0^{1,q}(\Omega)$. Then applying the theory of pseudo-monotone operators provides the existence of a weak solution to (1.1)–(1.3).

In case $\frac{2d}{d+1} < q < \frac{3d}{d+2}$ the existence of weak solutions to (1.1)–(1.3) ($\mathbf{f} \in L^1(\Omega)^d$) has been proved independently by Frehse, Málek and Steinhauer [5] and Růžička [12]. Afterwards Frehse, Málek and Steinhauer [6] obtained weak solutions to (1.1)–(1.3) for all $\frac{2d}{d+2} < q < \infty$ by using the Lipschitz truncation method.

The interior regularity of any weak solution to (1.1)-(1.3) $\left(\frac{3d}{d+2} < q < 2\right)$ has been proved in [11]. This result has been achieved by the method of differences. The existence of the second weak derivatives are proved by a standard bootstrap argument using fractional differentiability of $\nabla \mathbf{u}$ together with Sobolev's embedding theorem. However in the special case $q = \frac{3d}{d+2}$ we first have to prove the higher integrability of $\nabla \mathbf{u}$ (see Theorem 1 below) in order to start an similar bootstrap argument as in [11].

Furthermore we wish to mention that the method of difference quotient fails if the force $\mathbf{f}$ has not sufficient integrability.

Statement of the Main Result. The aim of the present paper is to prove the interior regularity of any weak solution to (1.1)–(1.3) for the special case $q = \frac{3d}{d+2}$. This will be achieved by an analogous reasoning as in [11] after having established the higher integrability of $\nabla \mathbf{u}$, which will be our first main result.

Theorem 1. *Let* $\mathbf{S} = \{S_{ij}\}$ *fulfill conditions* (I), (II) *and* (III). *Assume*

$$q = \frac{3d}{d+2}\,.$$

Let $\mathbf{f} \in L^\sigma_{\mathrm{loc}}(\Omega)^d \left(\sigma > \dfrac{3d}{2d+1}\right)$. *Let* $\mathbf{u} \in W^{1,q}_{\mathrm{loc}}(\Omega)^d$ *with* $\operatorname{div} \mathbf{u} = 0$ *in* Ω *satisfy*

$$\int_\Omega S_{ij}(D(\mathbf{u}))D_{ij}(\boldsymbol{\varphi})\,\mathrm{d}\,x + \int_\Omega u_i \frac{\partial u_j}{\partial x_i} \varphi_j\,\mathrm{d}\,x = \int_\Omega f_i \varphi_i\,\mathrm{d}\,x \tag{1.5}$$

for all $\varphi \in \mathcal{D}_\sigma(\Omega)$. *Then there exists* $\tilde q > q$, *such that*

$$\mathbf{u} \in W^{1,\tilde q}_{\mathrm{loc}}(\Omega)^d.$$

□

As a consequence of Theorem 1 we may apply the method of differences to get fractional differentiability of $\nabla \mathbf{u}$, which by Sobolev's embedding theorem improves the integrability of $\nabla \mathbf{u}$ iteratively. Then arguing similarly as in [11] one gets the existence of the second derivatives of $\mathbf{u}$. Thus, we have

Corollary 2. *Let all assumption of Theorem* 1 *be fulfilled. Furthermore, suppose* $\mathbf{f} \in L^\sigma_{\mathrm{loc}}(\Omega)^d$, *where*

$$\sigma > \frac{27}{13} \quad \textit{if} \quad n = 3, \qquad \sigma > 2 \quad \textit{if} \quad n = 2.$$

Then

$$(1 + |D(\mathbf{u})|)^{\frac{q-2}{2}} \nabla D_{ij}(\mathbf{u}) \in L^2_{\mathrm{loc}}(\Omega)^d \quad (i, j = 1, \ldots, d), \tag{1.6}$$

$$\begin{cases} \mathbf{u} \in W^{2,t}_{\mathrm{loc}}(\Omega)^2 \quad \forall\, 1 \le t < 2 & \textit{if} \quad d = 2, \\ \mathbf{u} \in W^{2,\frac{3q}{1+q}}_{\mathrm{loc}}(\Omega)^3 & \textit{if} \quad d = 3. \end{cases} \tag{1.7}$$

In particular, by Sobolev's embedding theorem we have

$$\begin{aligned} &\mathbf{u} \in C^\alpha(\Omega)^2 \quad \forall\, 0 < \alpha < 1 \quad \textit{if} \quad n = 2, \\ &\mathbf{u} \in C^{1-1/q}(\Omega)^3 \quad \textit{if} \quad n = 3. \end{aligned}$$

□

2. Higher integrability. Proof of Theorem 1

The proof of Theorem 1 relies essentially on the following result of higher integrability which is due to Giaquinta and Modica (cf. [9]).

Lemma 2.1. *Let* $F \in L^t_{\mathrm{loc}}(\Omega)$ *and* $G \in L^s_{\mathrm{loc}}(\Omega)\,(1 < t < s < +\infty)$ *be given non-negative functions. Suppose there are constants* $K_0 \ge 1$, $0 < \varepsilon_0 < 1$ *and* $r_0 > 0$ *such that*

$$⨍_{B_{r/2}(x_0)} F^t \,\mathrm{d}\,x \le K_0 \left(⨍_{B_r(x_0)} F \,\mathrm{d}\,x \right)^t + \varepsilon_0 ⨍_{B_r(x_0)} F^t \,\mathrm{d}\,x + ⨍_{B_r(x_0)} G^t \,\mathrm{d}\,x \tag{2.1}$$

for all $x_0 \in \Omega$, $0 < r < \min\{r_0, \mathrm{dist}(x_0, \partial\Omega)\}$. *Then there exists* $t < \tau_0 \le s$, *such that*

$$F \in L^{\tau}_{\mathrm{loc}}(\Omega) \quad \forall \tau \in [1, \tau_0[\,. \tag{2.2}$$

□

Throughout this section let

$$q = \frac{3d}{d+2}\,.$$

PROOF OF THEOREM 1: 1° *Pressure estimate*. Let $B_r \subset \Omega$ with $\overline{B}_{2r} \subset \Omega$. It is readily seen that the mapping $F_{(B_r)} : W^{1,q}_0(B_r)^d \to \mathbb{R}$ defined by

$$\begin{aligned}\varphi \mapsto \int_{B_r} S_{ij}(D(\mathbf{u}))D_{ij}(\varphi)\,\mathrm{d}\,x \\ + \int_{B_r} u_i \frac{\partial}{\partial x_i}(u_j - (u_j)_{B_r})\varphi_j\,\mathrm{d}\,x - \int_{B_r} f_i\,\varphi_i\,\mathrm{d}\,x\end{aligned} \tag{2.3}$$

is a linear continuous functional on $W^{1,q}_0(B_r)^d$ (3) which vanishes for all $\varphi \in D^{1,q}_0(B_r)$. Appealing to [7, III 3.1, Theorem III 5.2] there exists $\hat{p} \in L^{q'}(B_r)/\mathbb{R}$ such that for any $p \in \hat{p}$:

$$\int_{B_r} \Big(S_{ij}(D(\mathbf{u})) - u_i(u_j - (u_j)_{B_r})\Big)\frac{\partial \varphi_j}{\partial x_i}\,\mathrm{d}\,x - \int_{B_r} f_i\,\varphi_i\,\mathrm{d}\,x = \int_{B_r} p\ \mathrm{div}\,\varphi\,\mathrm{d}\,x \tag{2.4}$$

for all $\varphi \in W^{1,q}_0(B_r)^d$. In addition, by means of Sobolev's embedding theorem we have the estimate

$$\begin{aligned}&\int_{B_r} |p - p_{B_r}|^{q'}\,\mathrm{d}\,x \\ &\le c\left\{\int_{B_r} |\mathbf{S}(D(\mathbf{u}))|^{q'}\,\mathrm{d}\,x + \int_{B_r} |\mathbf{u}|^{q'}|\mathbf{u} - \mathbf{u}_{B_r}|^{q'}\,\mathrm{d}\,x\right\} + c\bigg(\int_{B_r} |\mathbf{f}|^{q^{*\prime}}\,\mathrm{d}\,x\bigg)^{\frac{q'}{q^{*\prime}}},\end{aligned}$$

(3) Note that from $q = \frac{3d}{d+2}$ it follows that

$$2q' = q^*, \quad 2dq - \frac{d}{q'} = 2, \quad \frac{d}{q}\frac{q^* - q}{q^*} = \frac{1}{q}\Big(1 - \frac{q}{q^*}\Big) = 1.$$

where $c =$ const independent of r. Thus, observing (II) applying Hölder's inequality gives

$$
\begin{aligned}
&\int_{B_r} |p - p_{B_r}|^{q'} \,\mathrm{d}\,x \\
(2.5) \quad &\le c \int_{B_r} (1 + |D(\mathbf{u})|)^q \,\mathrm{d}\,x + c\Big(\int_{B_r} |\mathbf{u}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{2}} \Big(\int_{B_r} |\mathbf{u} - \mathbf{u}_{B_r}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{2}} \\
&\quad + c\Big(\int_{B_r} |\mathbf{f}|^{q^{*\prime}} \,\mathrm{d}\,x\Big)^{\frac{q'}{q^{*\prime}}},
\end{aligned}
$$

where $c =$ const > 0 depending on d only.

2° *Caccioppoli-type inequality.* Let $\zeta \in C_0^\infty(B_r)$ be a cut-off function, such that $0 \le \zeta \le 1$ in $B_r, \zeta \equiv 1$ on $B_{3r/4}$ and $|\nabla\zeta| \le \dfrac{c_1}{r}$ ($c_1 =$ const). Clearly, $\boldsymbol{\varphi} = (\mathbf{u} - \mathbf{u}_{B_r})\zeta^2$ is an admissible test function in (2.4). Inserting this function into (2.4) using (III) gives

$$
\begin{aligned}
&\nu_0 \int_{B_r} (1 + |D(\mathbf{u})|)^{q-2} |D(\mathbf{u})|^2 \zeta^2 \,\mathrm{d}\,x \\
&\le -2 \int_{B_r} (S_{ij}(D(\mathbf{u})) - S_{ij}(0))(u_i - (u_i)_{B_r})\zeta \frac{\partial\zeta}{\partial x_j} \,\mathrm{d}\,x \\
(2.6) \quad &\quad - \int_{B_r} u_i\Big(\frac{\partial}{\partial x_i}(u_j - (u_j)_{B_r})\Big)(u_j - (u_j)_{B_r})\zeta^2 \,\mathrm{d}\,x \\
&\quad + 2\int_{B_r} p(u_i - (u_i)_{B_r})\zeta \frac{\partial\zeta}{\partial x_i} \,\mathrm{d}\,x + \int_{B_r} \mathbf{f} \cdot (\mathbf{u} - \mathbf{u}_{B_r})\zeta^2 \,\mathrm{d}\,x \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

1) Applying Hölder's and Young's inequality implies

$$
\begin{aligned}
I_1 &\le \frac{c}{r}\Big(\int_{B_r} (1 + |D(\mathbf{u})|)^q \,\mathrm{d}\,x\Big)^{\frac{1}{q'}} \Big(\int_{B_r} |\mathbf{u} - \mathbf{u}_{B_r}|^q \,\mathrm{d}\,x\Big)^{\frac{1}{q}} \\
&\le \varepsilon \int_{B_r} |D(\mathbf{u})|^q \,\mathrm{d}\,x + \frac{c}{r^q} \int_{B_r} |\mathbf{u} - \mathbf{u}_{B_r}|^q \,\mathrm{d}\,x + c\,r^d.
\end{aligned}
$$

2) Taking into account (I) and using integration by parts together with the

Sobolev-Poincaré's inequality and Hölder's inequality one obtains

$$
\begin{aligned}
I_2 &= \int_{B_r} |\mathbf{u}-\mathbf{u}_{B_r}|^2 u_i \zeta \frac{\partial \zeta}{\partial x_i} \,\mathrm{d}\,x \\
&\le \frac{c}{r}\Big(\int_{B_r} |\mathbf{u}-\mathbf{u}_{B_r}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{q'}}\Big(\int_{B_r} |\mathbf{u}|^q \,\mathrm{d}\,x\Big)^{\frac{1}{q}} \\
&\le \frac{c}{r}\Big(\int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x\Big)^{\frac{2}{q}}\Big(\int_{B_r} |\mathbf{u}|^q \,\mathrm{d}\,x\Big)^{\frac{1}{q}} \\
&\le c\, r^{\frac{d}{q}-\frac{d}{2q'}-1}\Big(\int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x\Big)^{\frac{2}{q}}\Big(\int_{B_r} |\mathbf{u}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{2q'}} \\
&\le c\,\Theta_1(r) \int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x,
\end{aligned}
$$

where

$$
\Theta_1(r) := \Big(\int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x\Big)^{\frac{2-q}{q}}\Big(\int_{B_r} |\mathbf{u}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{2q'}}.
$$

3) First, applying Hölder's inequality along with (2.5) one gets

$$
\begin{aligned}
I_3 \le{}& \frac{c}{r}\Big(\int_{B_r} (1+|D(\mathbf{u})|)^q \,\mathrm{d}\,x\Big)^{\frac{1}{q'}}\Big(\int_{B_r} |\mathbf{u}-\mathbf{u}_{B_r}|^q \,\mathrm{d}\,x\Big)^{\frac{1}{q}} \\
&+\frac{c}{r}\Big(\int_{B_r} |\mathbf{u}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{2q'}}\Big(\int_{B_r} |\mathbf{u}-\mathbf{u}_{B_r}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{2q'}} \\
&\times \Big(\int_{B_r} |\mathbf{u}-\mathbf{u}_{B_r}|^q \,\mathrm{d}\,x\Big)^{\frac{1}{q}} \\
&+\frac{c}{r}\Big(\int_{B_r} |\mathbf{f}|^{q^{*\prime}} \,\mathrm{d}\,x\Big)^{\frac{1}{q^{*\prime}}}\Big(\int_{B_r} |\mathbf{u}-\mathbf{u}_{B_r}|^q \,\mathrm{d}\,x\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Then, by the aid of Sobolev-Poincaré's inequality and Young's inequality one arrives at

$$
\begin{aligned}
I_3 \le{}& \varepsilon \int_{B_r} |D(\mathbf{u})|^q \,\mathrm{d}\,x + \frac{c}{r^q}\int_{B_r} |\mathbf{u}-\mathbf{u}_{B_r}|^q \,\mathrm{d}\,x + c\, r^d \\
&+ c\,\Theta_2(r)\int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x + c\Big(\int_{B_r} |\mathbf{f}|^{q^{*\prime}} \,\mathrm{d}\,x\Big)^{\frac{q'}{q^{*\prime}}},
\end{aligned}
$$

where

$$
\Theta_2(r) := \Big(\int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x\Big)^{\frac{2-q}{q-1}}\Big(\int_{B_r} |\mathbf{u}|^{2q'} \,\mathrm{d}\,x\Big)^{\frac{1}{2}}.
$$

4) Finally, with help of Hölder's inequality and Sobolev-Poincaré's inequality one obtains

$$I_4 \le c \Big(\int_{B_r} |\mathbf{f}|^{q^{*'}} \,\mathrm{d}\,x \Big)^{\frac{1}{q^{*'}}} \Big(\int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x \Big)^{\frac{1}{q}}.$$

By means of Young's inequality from the estimate above it follows that

$$I_4 \le \varepsilon \int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x + c \Big(\int_{B_r} |\mathbf{f}|^{q^{*'}} \,\mathrm{d}\,x \Big)^{\frac{q'}{q^{*'}}}.$$

Inserting the estimates of I_1, I_2, I_3 and I_4 into (2.6) gives

$$\begin{aligned} &\int_{B_{3r/4}} |D(\mathbf{u})|^q \,\mathrm{d}\,x \\ &\le \frac{c}{r^q} \int_{B_r} |\mathbf{u} - \mathbf{u}_{B_r}|^q \,\mathrm{d}\,x + c\,r^d + c\,(\varepsilon + \Theta_1(r) + \Theta_2(r)) \int_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x \\ &\quad + c \int_{B_r} |\mathbf{f}|^{q^{*'}} \,\mathrm{d}\,x^{(4)}. \end{aligned}$$

Next, we divide the inequality above by $\mathrm{meas}_d(B_r)$ and then apply Sobolev-Poincaré's inequality. This shows that

$$\begin{aligned} \fint_{B_{3r/4}} |D(\mathbf{u})|^q \,\mathrm{d}\,x &\le c \Big(\fint_{B_r} (1 + |\nabla \mathbf{u}|)^{dq/d+q} \,\mathrm{d}\,x \Big)^{\frac{d+q}{d}} \\ &\quad + c\,(\varepsilon + \Theta_1(r) + \Theta_2(r)) \fint_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x + c \fint_{B_r} |\mathbf{f}|^{q^{*'}} \,\mathrm{d}\,x. \end{aligned} \tag{2.7}$$

Now, let $\widetilde{\zeta} \in C_{\mathrm{c}}^{\infty}(B_{3r/4})$ be a cut-off function, such that $0 \le \widetilde{\zeta} \le 1$ in $B_{3r/4}$, $\widetilde{\zeta} \equiv 1$ on $B_{r/2}$ and $|\nabla \widetilde{\zeta}| \le \dfrac{c_1}{r}$. Then by means of Korn's inequality we estimate

$$\begin{aligned} \int_{B_{r/2}} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x &\le \int_{B_{3r/4}} |\nabla((\mathbf{u} - \mathbf{u}_{B_r})\widetilde{\zeta})|^q \,\mathrm{d}\,x \\ &\le c \int_{B_{3r/4}} |D((\mathbf{u} - \mathbf{u}_{B_r})\widetilde{\zeta})|^q \,\mathrm{d}\,x \\ &\le \int_{B_{3r/4}} |D(\mathbf{u})|^q \,\mathrm{d}\,x + \frac{c}{r^q} \int_{B_r} |\mathbf{u} - \mathbf{u}_{B_r}|^q \,\mathrm{d}\,x. \end{aligned}$$

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(4) Note that by $\frac{q'}{q^{*'}} > 1$ we have $\Big(\int_{B_r} |\mathbf{f}|^{q^{*'}} \,\mathrm{d}\,x \Big)^{\frac{q'}{q^{*'}}} \le c \int_{B_r} |\mathbf{f}|^{q^{*'}} \,\mathrm{d}\,x$.

As before we divide both sides of this inequality by $\mathrm{meas}_d(B_r)$; applying Sobolev-Poincaré's inequality yields

$$\fint_{B_{r/2}} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x \le \fint_{B_{3r/4}} |D(\mathbf{u})|^q \,\mathrm{d}\,x + c\left(\fint_{B_r} |\nabla \mathbf{u}|^{dq/(d+q)} \,\mathrm{d}\,x\right)^{\frac{d+q}{d}}. \tag{2.8}$$

Estimating the first integral on the right of (2.8) by (2.7) gives

$$\begin{aligned} &\fint_{B_{r/2}} (1+|\nabla \mathbf{u}|)^q \,\mathrm{d}\,x \\ &\le c\left(\fint_{B_r} (1+|\nabla \mathbf{u}|)^{dq/d+q} \,\mathrm{d}\,x\right)^{\frac{d+q}{d}} + (c\,\varepsilon + \Theta(r))\fint_{B_r} |\nabla \mathbf{u}|^q \,\mathrm{d}\,x \\ &\quad + c\fint_{B_r} |\mathbf{f}|^{q^{*\prime}} \,\mathrm{d}\,x, \end{aligned} \tag{2.9}$$

where $\Theta(r)$ goes to 0 as $r \to 0$. Here $c = \mathrm{const} > 0$ depending on d only. Choosing $0 < \varepsilon < 1$ and $r_0 > 0$ sufficiently small, our desired result of higher integrability is an immediate consequence of Lemma 2.1. □

References

[1] Adams R.A., *Sobolev Spaces*, Academic Press, Boston, 1978.

[2] Astarita G., Marrucci G., *Principles of Non-Newtonian Fluid Mechanics*, Mc Graw-Hill, London, New York, 1974.

[3] Batchelor G.K., *An Introduction to Fluid Mechanics*, Cambridge Univ. Press, Cambridge, 1967.

[4] Bird R.B., Armstrong R.C., Hassager O., *Dynamics of Polymer Liquids. Vol. 1: Fluid Mechanics*, 2nd ed., J. Wiley & Sons, New York, 1987.

[5] Frehse J., Málek J., Steinhauer M., *An existence result for fluids with shear dependent viscosity-steady flows*, Nonlinear Anal. **30** (1997), no. 5, 3041–3049; [Proc. 2nd World Congress Nonlin. Analysts].

[6] Frehse J., Málek J., Steinhauer M., *On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method*, SIAM J. Math. Anal. **34** (2004), no. 5, 1064–1083.

[7] Galdi G.P., *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. I: Linearized Steady Problems*, Springer, New York, 1994.

[8] Giaquinta M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals Math. Studies, no. 105, Princeton Univ. Press, Princeton, N.J., 1983.

[9] Giaquinta M., Modica G., *Almost-everywhere regularity results for solutions of nonlinear elliptic systems*, Manuscripta Math. **28** (1979), 109–158.

[10] Lamb H., *Hydrodynamics*, 6th ed., Cambridge Univ. Press, Cambridge, 1945.

[11] Naumann J., Wolf J., *Interior differentiability of weak solutions to the equations of stationary motion of a class of non-Newtonian fluids*, J. Math. Fluid Mech. **7** (2005), no. 2, 298–313.

[12] Růžička M., *A note on steady flow of fluids with shear dependent viscosity*, Nonlinear Anal. **30** (1997), no. 5, 3029–3039; [Proc. 2nd World Congress Nonlin. Analysts].

[13] Wilkinson W.L., *Non-Newtonian Fluids. Fluid Mechanics, Mixing and Heat Transfer*, Pergamon Press, London, New York, 1960.

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