Two types of remainders of topological groups

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Abstract. We prove a Dichotomy Theorem: for each Hausdorff compactification bG of an arbitrary topological group G, the remainder $bG \setminus G$ is either pseudocompact or Lindelöf. It follows that if a remainder of a topological group is paracompact or Dieudonne complete, then the remainder is Lindelöf, and the group is a paracompact p-space. This answers a question in A.V. Arhangel'skii, *Some connections between properties of topological groups and of their remainders*, Moscow Univ. Math. Bull. 54:3 (1999), 1–6. It is shown that every Tychonoff space can be embedded as a closed subspace in a pseudocompact remainder of some topological group. We also establish some other results and present some examples and questions.

Keywords: remainder, compactification, topological group, p-space, Lindelöf p-space, metrizability, countable type, Lindelöf space, pseudocompact space, π-base, compactification

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1. Introduction

By a space we understand a Tychonoff topological space. By a *remainder* of a space X we understand the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. We consider how properties of a space X are related to properties of some or all remainders of X. A famous classical result in this direction is the following theorem of M. Henriksen and J. Isbell [9]:

Theorem 1.1. *A space X is of countable type if and only if the remainder in any* (*in some*) *compactification of X is Lindelöf* [9].

In fact, in this article we are mostly concerned with the case of remainders of topological groups, and it continues the line of research in [5], [6].

A space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ which has a countable base of open neighbourhoods in X. All metrizable spaces and all locally compact spaces, as well as all Čech-complete spaces, are of countable type [1]. It follows from Henriksen's and Isbell's Theorem 1.1 that every remainder of a metrizable space is Lindelöf and hence, is paracompact.

When *every* remainder of a Tychonoff space X has a certain property $\mathcal{P}$ we say, following [9], that X *has property $\mathcal{P}$ at infinity*.

One should expect that the properties of a space in general should be quite different from the properties of its remainders. This is unlike the situation with the whole compactifications: the best thing we can expect to happen is that some properties of a space would pass to the compactification. On the contrary, in the study of relationship between properties of a space and of its remainders in compactifications we should look for "orthogonal" (very much unlike) properties of a space and remainder to be brought in some kind of duality, as in Henriksen-Isbell's Theorem.

In this article we consider what kind of remainders can have a topological group, and we discover a basic fact in this direction: every remainder of a topological group must belong to at least one of two rather narrow classes of spaces the intersection of which is the class of compacta (Theorem 2.4). We also establish some corollaries of this main result one of which answers several questions asked ten years ago in [3]: a remainder Y of a topological group is paracompact if and only if Y is Lindelöf. This result implies that if a remainder of a topological group G is paracompact then G is a p-space. A few open problems are formulated.

Recall that *paracompact p-spaces* [1] are preimages of metrizable spaces under perfect mappings. *A Lindelöf p-space* is a preimage of a separable metrizable space under a perfect mapping.

For the definition of a p-space see [1], where it was shown that every p-space is of countable type, and that every metrizable space is a p-space.

Clearly, every separable metrizable space has a separable metrizable remainder. Here is a parallel result from [5]:

Theorem 1.2. *If X is a Lindelöf p-space, then every remainder of X is a Lindelöf p-space.*

Theorem 1.2 is not a straightforward result; its proof in [5] is based on a deep theorem of V.V. Filippov on preservation of the class of paracompact p-spaces by perfect mappings [8]. Unfortunately, Theorem 1.2 cannot be generalized to paracompact p-spaces, and there are metrizable spaces that do not have a metrizable remainder [5].

2. A Dichotomy Theorem

Recall that a *π-base* of a space X at a subset F of X is a family γ of non-empty open subsets of X such that every open neighbourhood of F contains at least one element of γ.

A *strong π-base* of a space X at a subset F of X is an infinite family γ of non-empty open subsets of X such that every open neighbourhood of F contains all but finitely many elements of γ. Clearly, every infinite subfamily of a strong π-base is again a strong π-base; therefore, a strong π-base can be always assumed to be countable. Obviously, every strong π-base is a π-base.

Lemma 2.1. *Suppose that X is a nowhere locally compact space, and that bX is a compactification of X. Then the following two conditions are equivalent:*

1) *the remainder $Y = bX \setminus X$ is not pseudocompact;*
2) *there exists a non-empty compact subspace F of X which has a strong countable π-base in X.*

Proof: Note that both X and Y are dense in bX. Suppose that Y is not pseudocompact. Then there exists a countable family $\eta = \{U_n : n \in \omega\}$ of open non-empty subsets U_n of bX such that η has no accumulation point in Y. Then the set F of all accumulation points for η in bX is a non-empty compact subset of bX contained in X.

Put $V_n = U_n \cap X$. Clearly, each V_n is a non-empty open subset of X.

Claim 1: The family $\xi = \{V_n : n \in \omega\}$ is a strong π-base of the space X at the set F.

Take any open neighbourhood V of F in X. Then $V = U \cap X$, for some open subset U of bX. Since F is compact, and $F \subset U$, we can find an open neighbourhood W of F in bX such that the closure $\overline{W}$ of W in bX is contained in U. We claim that $U_k \subset \overline{W}$, for all but finitely many $k \in \omega$. Assume the contrary, and put $G_n = U_n \setminus \overline{W}$. Then G_n is a non-empty open subset of bX, for infinitely many $n \in \omega$. Thus, by compactness of bX some $z \in bX$ must be an accumulation point for the family $\mu = \{G_n : n \in \omega\}$ in bX. However, on one hand, $z \in bX \setminus \overline{W}$ and, on the other hand, $z \in F \subset W$, a contradiction. It follows that G_n is empty for all but finitely many values of n. Therefore, $U_n \subset \overline{W} \subset U$ for all but finitely many values of n. It follows that all, except finitely many, sets $V_n = U_n \cap X$ are contained in the set $V = X \cap U$. Claim 1 is verified.

Let us prove that 2) implies 1). Suppose that F is a non-empty compact subspace of X with a strong π-base γ. For each $V \in \gamma$ fix an open subset U_V of bX such that $U_V \cap X = V$, and put $G_V = U_V \cap Y$. Clearly, $\{G_V : V \in \gamma\}$ is an infinite family of non-empty open sets in Y without accumulation points in Y. Hence, Y is not pseudocompact. □

Lemma 2.2. *Suppose that G is a topological group, and that F is a non-empty compact subspace of G such that G has a countable π-base at F. Then G is a paracompact p-space.*

Proof: Fix a countable π-base η of G at F. Put $B = FF^{-1}$ and $\xi = \{VV^{-1} : V \in \eta\}$. Then B is compact, since G is a topological group. It follows, by a standard compactness argument making use of the continuity of multiplication in G, that for each open neighbourhood OB of B there are open neighbourhoods O_1 and O_2 of F and F^{-1}, respectively, such that $O_1O_2 \subset OB$. Hence, OB contains some element of ξ, that is, we have established the following fact:

Fact 1: ξ is a countable π-base of G at B.

Observe that the neutral element e of G belongs to every element of ξ. Using this fact and Fact 1, we can easily construct a sequence $\nu = \{W_n : n \in \omega\}$ of open neighbourhoods of e in G such that ν is a π-base of G at B, and $\overline{W_{n+1}} \subset W_n$, for each $n \in \omega$. Put $P = \cap\nu$. Clearly, P is a closed subset of B. Hence, P is compact.

Claim 2: ν is a countable base of open neighbourhoods of P in G.

Indeed, take any open neighbourhood OP of P in G, and put $M = B \setminus OP$. Then M is compact, and $P \cap M = \emptyset$. It follows from the definition of P and compactness of M that $\overline{W_k} \cap M = \emptyset$, for some $k \in \omega$. Put $L = \overline{W_k} \setminus OP$. Clearly, L is a closed subset of G disjoint from B. Therefore $G \setminus L$ is an open neighbourhood of B, and there exists $m > k$ such that $W_m \subset G \setminus L$. Since $W_m \subset W_k$, we have $W_m \subset OP$. Claim 2 is verified.

However, every topological group that contains a non-empty compact subspace with a countable base of open neighbourhoods is a paracompact p-space (see [11] and [2] for a discussion of this result). Hence, G is a paracompact p-space. □

Lemma 2.1 immediately implies the next statement:

Corollary 2.3. *Suppose that X is a nowhere locally compact space. Then every remainder of X is pseudocompact if and only if some remainder of X is pseudocompact.*

Now we can prove the main result.

Theorem 2.4. *For any topological group G, any remainder of G in a compactification bG of G is either pseudocompact or Lindelöf.*

PROOF: Assume that the subspace $Y = bG \setminus G$ is not pseudocompact. It follows that G is nowhere locally compact. Then, by Lemma 2.1, there exists a non-empty compact subspace F of G with a countable π-base in G. Now it follows from Lemma 2.2 that G is a paracompact p-space. Then the remainder $bG \setminus G$ is Lindelöf according to the following theorem (see Theorem 4.1 in [5]) which nicely complements Theorem 2.4. □

Theorem 2.5. *A topological group G has a Lindelöf remainder in a compactification if and only if G is a paracompact p-space (and then all remainders of G are Lindelöf).*

Theorem 2.4 permits to improve Theorem 2.5. Our next result shows that several classical restrictions on remainders of topological groups are in fact equivalent.

Theorem 2.6. *Suppose that G is a topological group, and $Y = bG \setminus G$ is a remainder of G in a compactification bG of G. Then the following conditions are equivalent:*

1) *Y is Dieudonné complete;*

2) *Y is paracompact;*
3) *every remainder of G in a compactification is Lindelöf;*
4) *G is a paracompact p-space.*

PROOF: Suppose that 1) holds. Since every pseudocompact Dieudonné complete space is compact, it follows from Theorem 2.4 that Y is Lindelöf. Then G is a paracompact p-space and every remainder of G is Lindelöf, by Theorem 2.5. $\square$

Notice that Theorem 2.4 and Theorem 2.5 imply the following characterization of topological groups all remainders of which are pseudocompact.

Corollary 2.7. *Any (some) remainder of a topological group G is pseudocompact if and only if G is not a paracompact p-space.*

Corollary 2.8. *For any dense subspace X of an arbitrary locally pseudocompact non-locally compact topological group G, any remainder of X is pseudocompact.*

PROOF: Assume the contrary. Then, by Lemma 2.1, the space X contains a non-empty compact subspace F with a countable π-base in X. Since X is dense in G, it follows that$F$ has a countable π-base in G. Then, by Lemma 2.2, G is a paracompact p-space. This implies that G is locally compact, since G is locally pseudocompact. Thus, we have arrived at a contradiction. $\square$

To prove one more result of the same kind, we need the following lemma:

Lemma 2.9. *The free topological group $F(X)$ of a non-discrete space X cannot contain a non-empty compact subset F with a countable base $\mathcal{B}$ of open neighbourhoods.*

PROOF: Assume the contrary. Let Y_n be the set of words in $F(X)$ of the length $\leq n$. Then Y_n is a nowhere dense closed subset of $F(X)$ (see [10], [2]). Let $\mathcal{B} = \{V_n : n \in \omega\}$, where $V_{n+1} \subset V_n$. Since V_n is open, we can fix $a_n \in V_n \cap (F(X) \setminus Y_n)$ for each $n \in \omega$. Put $B = F \cup \{a_n : n \in \omega\}$. Clearly, B is a compact subset of $F(X)$ and $B \setminus Y_n$ is non-empty, for each $n \in \omega$. However, since B is compact, the set B must be contained in some Y_n (see [2], [10]), a contradiction. $\square$

Theorem 2.10. *If $G = F(X)$ is the free topological group of an arbitrary space X, then every remainder of G is pseudocompact.*

PROOF: This follows from Theorem 2.4 and Lemma 2.9, since no such G is a paracompact p-space. $\square$

Corollary 2.11. *If G is a submetrizable non-metrizable topological group, then each remainder of G is pseudocompact.*

PROOF: By Theorem 2.4, it is enough to show that no remainder of G is Lindelöf. Assume the contrary. Then, by Theorem 2.5, G is a paracompact p-space. However, every submetrizable paracompact p-space is metrizable (see [1]). Hence, G is metrizable, a contradiction. $\square$

After Theorems 2.4 and 2.5 it is natural to ask the following two general questions:

Problem 2.12. *Which pseudocompact spaces can serve as remainders of topological groups in compactifications?*

Problem 2.13. *Which Lindelöf spaces can serve as remainders of topological groups in compactifications?*

In the main case of non-locally compact topological groups we can reformulate these questions as follows.

Problem 2.14. *When a pseudocompact space X has a remainder homeomorphic to a topological group?*

Problem 2.15. *When a Lindelöf space X has a remainder homeomorphic to a topological group?*

At present, we are far from a complete answer to these questions. However, we present below some related examples.

Using the criteria obtained above, we can easily identify many spaces that cannot serve as remainders of topological groups.

Example 2.16. 1) Let Z be the product of an infinite family of non-separable metrizable spaces. Then no remainder of Z is homeomorphic to a topological group.

Indeed, Z is not Lindelöf and is not pseudocompact. Therefore, Z cannot be a remainder of a topological group.

2) Let Y be the product of an uncountable family of non-compact metrizable spaces. Then no remainder of Y is homeomorphic to a topological group.

Indeed, Y is Dieudonné complete but is not Lindelöf.

3) Let X be a nowhere locally compact pseudocompact space with a G_δ-diagonal. Then no remainder of X is homeomorphic to a topological group. Indeed, otherwise this group is separable and metrizable, by a theorem in [6]. However, according to Theorem 1.2, then X must be Lindelöf and therefore, compact, a contradiction.

Obviously, the class of topological groups with Lindelöf remainders is hereditary with respect to closed subgroups. On the contrary, it is easy to see that the class of topological groups with pseudocompact remainders does not have this property. However, the following very general stability theorem holds.

Theorem 2.17. *If γ is a family of topological spaces such that at least one element X of γ is a non-locally compact topological group with a pseudocompact remainder P, then every remainder of the topological product of γ is pseudocompact.*

Proof: Let H be the topological product of γ. Clearly, the space H is nowhere locally compact, since X is nowhere locally compact. Hence, by Corollary 2.3, it is enough to show that some remainder of H is pseudocompact. For each $Y \in \gamma$ we fix a compactification bY, and denote by Z the product space $P \times \Pi\{bY : Y \in \gamma \setminus \{X\}\}$. Then Z is pseudocompact and Z is dense in the compactification $B = \Pi\{bY : Y \in \gamma\}$ of the product space $H = \Pi\{Y : Y \in \gamma\}$. Clearly, Z is contained in the remainder $B \setminus H$. It follows that Z is dense in $B \setminus H$. Hence, $B \setminus H$ is pseudocompact. □

The next result shows that the class of pseudocompact spaces which have a remainder homeomorphic to a topological group is very large and cannot be characterized by a topological property inherited by closed subspaces.

Theorem 2.18. *Every space Y can be represented as a closed subspace of a pseudocompact space Z some remainder of which is a topological group.*

Proof: Clearly, Y can be represented as a nowhere dense subspace of some compact space B. Put $X = B \setminus Y$, and let $F(B)$ be the free topological group of B. Further, let G be the topological subgroup of $F(B)$ algebraically generated by X. Obviously, X is dense in B which implies that G is dense in $F(B)$. Observe that $G \cap Y = \emptyset$. Hence, G is not closed in $F(B)$. Since $F(B)$ and G are topological groups, it follows that G is not open in $F(B)$. Therefore, G is not locally compact.

Take any compactification H of $F(B)$, and put $Z = H \setminus G$. Clearly, G is dense in H. Since G is nowhere locally compact, it follows that Z is also dense in H. Thus, H is a compactification of Z, and topological group G is a remainder of Z. Clearly, $G \cap B = X$, we have: $Z \cap B = Y$. Observe, that B is compact and hence, is closed in H. It follows that Y is closed in Z.

Claim: G is not a paracompact p-space.

Indeed, assume the contrary. Then G contains a non-empty compact subspace F with a countable base of neighbourhoods in G. Since G is dense in $F(B)$, it follows that F has a countable base of open neighbourhoods in $F(B)$ as well. However, this contradicts Lemma 2.9. The Claim is proved. It follows now from Theorems 2.4 and 2.5 that the remainder Z of topological group G is pseudocompact.

□

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