More on ordinals in topological groups

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Abstract. Let τ be an uncountable regular cardinal and G a T_1 topological group. We prove the following statements: (1) If τ is homeomorphic to a closed subspace of G, G is Abelian, and the order of every non-neutral element of G is greater than 5 then $\tau \times \tau$ embeds in G as a closed subspace. (2) If G is Abelian, algebraically generated by $\tau \subset G$, and the order of every element does not exceed 3 then $\tau \times \tau$ is not embeddable in G. (3) There exists an Abelian topological group H such that ω_1 is homeomorphic to a closed subspace of H and $\{t^2 : t \in T\}$ is not closed in H whenever $T \subset H$ is homeomorphic to ω_1. Some other results are obtained.

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1. Introduction

We continue to study topological groups that contain a topological copy of an ordinal. In [BUZ] it is proved that if a torsion free Abelian topological group G contains a copy of an uncountable regular cardinal τ then G contains topological copies of $\prod_{i\in N} \tau$ for every N. Here and throughout the paper we reserve the letter N to denote a positive integer. It is natural to ask if one can embed $\prod_{i\in N} \tau$ in G as a closed subspace given that τ is closed in G. In this paper we answer this question in affirmative for $N = 2$ (Theorem 2.7). Technical difficulties in constructing such an embedding are justified by Example 4.1 of a topological group with pathologies described in the abstract. In [BUZ] it is shown that if an Abelian group G is algebraically generated by $\tau + 1$, where τ is an uncountable regular cardinal and the order of every element of G does not exceed $2^N - 1$, then $\prod_{i\in N}(\tau + 1)$ is not embeddable in G. It is natural to expect that this statement holds if we replace $\tau + 1$ by τ. In Theorem 3.3 we show that it is so for $N = 2$. To achieve this goal we give an algebraic description of subsets homeomorphic to $\tau \times \tau$ in a topological group G algebraically generated by τ (Lemma 3.1).

We will use $\star$ sign for the binary operation in an abstract topological group G. For classical groups we use generally accepted signs. The neutral element of G is denoted by e_G. The order of $g \in G$ is denoted by $o(g)$. If the order of every element of G does not exceed N we write $o(G) \le N$. The equality $o(G) = N$ means that $o(G) \le N$ and the order of at least one element equals N. A group G is *torsion free* if the order of every non-neutral element of G is infinite.

An ordinal τ, when treated as a topological space, is endowed with the topology of linear order. We agree that α^n denotes $\alpha \star \alpha \star \ldots \star \alpha$, where α is an element of a group G. The topological second power of the space τ will be denoted by $\tau \times \tau$. The topological N-th power of the space τ will be denoted by $\prod_{i\in N} \tau$.

In notation and terminology we will follow [ENG] and [PON]. All spaces are assumed to satisfy T_1. It is a classical theorem of Kolmogorov and Pontryagin that a T_1 topological group is Tychonov (see, for example, [PON]).

Let G be a topological group and $\langle x_\alpha : \alpha < \tau\rangle$ a sequence of elements of G, where τ is an infinite regular cardinal. We say that $\langle x_\alpha : \alpha < \tau\rangle$ converges to $x \in G$ if for any open neighborhood U of x there exists an ordinal $\alpha_U < \tau$ such that $x_\alpha \in U$ whenever $\alpha > \alpha_U$. For short, any sequence of type $\langle x_\alpha : x_\alpha = \lambda, \alpha < \tau\rangle$ will be written as $\langle \lambda : \alpha < \tau\rangle$. We will use the following facts quite often.

Fact 1. *Let G be a topological group and τ an infinite regular cardinal. If $\langle x_\alpha : \alpha < \tau\rangle$, $\langle y_\alpha : \alpha < \tau\rangle$ converge in G to x and y, respectively, then $\langle x_\alpha \star y_\alpha : \alpha < \tau\rangle$ converges to $x \star y$ and $\langle x_\alpha^{-1} : \alpha < \tau\rangle$ to x^{-1}.*

In the next statement and throughout the paper we will use the fact that if cofinality of τ is uncountable then $\beta(\prod_{i\in N} \tau)$ is naturally homeomorphic to $\prod_{i\in N}(\tau + 1)$.

Fact 2. *Let G be a topological group that contains an uncountable regular cardinal τ as a subspace. If $\langle \alpha^n : \alpha < \tau\rangle$ does not converge in G then $\{\alpha^n : \alpha < \tau\}$ is closed in G.*

PROOF: Observe that $\{\alpha^n : \alpha < \tau\}$ is a continuous image of the subset $S = \{\langle \alpha : i \in n\rangle : \alpha < \tau\}$ of $\prod_{i\in n} \tau$ under f defined by $f(\langle \alpha_i : i \in n\rangle) = \alpha_0 \star \ldots \star \alpha_{n-1}$. If $f(S)$ is not closed in G then $\tilde f(\langle \tau : i \in n\rangle) \in G$, where $\tilde f$ is the continuous extension of f over $\prod_{i\in n}(\tau + 1)$. Since $\langle\langle \alpha : i \in n\rangle : \alpha < \tau\rangle$ converges to $\langle \tau : i \in n\rangle$ and $\tilde f$ is continuous, $\langle \alpha^n : \alpha < \tau\rangle$ converges to $\tilde f(\langle \tau : i \in n\rangle)$. □

2. Topological square of an ordinal in a topological group

Suppose G is an Abelian torsion free group and $\omega_1 \subset G$. It is proved in [BUZ] that $\omega_1 \times \omega_1$ embeds in G. Namely, it is shown that there exists an unbounded closed subset T of ω_1 such that $S = \{\alpha^2 \star \beta : \alpha, \beta \in T\}$ is homeomorphic to $T \times T$ under the correspondence $\alpha^2 \star \beta \leftrightarrow \langle \alpha, \beta\rangle$. It is natural to ask whether $\omega_1 \times \omega_1$ embeds in G as a closed subspace provided ω_1 is closed in G. It turns out that the set S need not be closed in G even if T is. More precisely, the diagonal subset $D = \{\alpha^2 \star \alpha : \alpha \in T\}$ or a horizontal $H_\beta = \{\alpha^2 \star \beta : \alpha \in T\}$ may not be closed. That is, it may happen that ω_1 is closed in G while $\{\alpha^2 : \alpha < \omega_1\}$ or $\{\alpha^3 : \alpha < \omega_1\}$ is not. This pathology is represented in Example 4.1.

In this section we overcome the described difficulties. The main result of this section (Theorem 2.7) implies that if G is an Abelian torsion free topological group and an uncountable regular cardinal τ is a closed subspace of G then $\tau \times \tau$ is homeomorphic to a closed subspace of G. To achieve this goal we find a set $S = \{s_{\langle\alpha,\beta\rangle} : \langle\alpha,\beta\rangle \in \tau\times\tau\}$ that is closed in G and homeomorphic to $\tau\times\tau$ under the correspondence $s_{\langle\alpha,\beta\rangle} \leftrightarrow \langle\alpha,\beta\rangle$. Lemma 3.1 of the next section shows that if such an S exists then the elements of the upper triangle $S_U = \{s_{\langle\alpha,\beta\rangle} : \beta \geq \alpha\}$ of S must be described by a well-defined simple formula $f_U(\alpha,\beta)$ such that $s_{\langle\alpha,\beta\rangle} = f_U(\alpha,\beta)$. The same is true for the elements of the lower triangle. Thus, the desired correspondence must be given by a pair of formulas. Luckily, we do not have to consider all the pairs. A winner is always among the following three:

$$f(\alpha,\beta) = \begin{cases} \beta^2 \star \alpha & \text{if } \alpha \geq \beta \\ \alpha \star \beta^2 & \text{if } \beta \geq \alpha \end{cases}$$

$$g(\alpha,\beta) = \begin{cases} \beta^2 \star \alpha^3 & \text{if } \alpha \geq \beta \\ \alpha^4 \star \beta & \text{if } \beta \geq \alpha \end{cases}$$

$$h(\alpha,\beta) = \begin{cases} \beta^2 \star \alpha^2 & \text{if } \alpha \geq \beta \\ \alpha^3 \star \beta & \text{if } \beta \geq \alpha \end{cases}$$

However, none of these formulas (nor any other formula) is universal.

Lemma 2.1. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than N, and $\tau \subset G$ is an uncountable regular cardinal. Let A, B be distinct natural numbers less than N. Then for any $\lambda \in \tau$ there exists $\lambda^* \in (\lambda,\tau)$ such that the following hold:*

1. $\lambda^A \star y^{N-A} \neq z^B \star w^{N-B}$,
2. $\lambda^A \star y^{N-A} \neq \lambda^B \star w^{N-B}$,
3. $\lambda^A \star \lambda^{N-A} \neq z^B \star w^{N-B}$

for any $y, z, w \in [\lambda^*,\tau)$.

Proof: Assume that no such λ^* exists. Assume that the requirement 1 is unattainable. Then there exist strictly increasing sequences $\langle y_n\rangle_n$, $\langle z_n\rangle_n$, and $\langle w_n\rangle_n$ of elements of (λ,τ) such that

- P1. $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = p$;
- P2. $\lambda^A \star y_n^{N-A} = z_n^B \star w_n^{N-B}$.

By P1, P2, and the continuity of $\star$, we have $\lambda^A \star p^{N-A} = p^B \star p^{N-B}$. Therefore, $\lambda^A = p^A$. By commutativity, $(\lambda \star p^{-1})^A = e_G$. This contradicts the facts that $\lambda < p$ and the order of every non-neutral element is greater than N, which is greater than A.

Assume now that the requirement 2 is not attainable. By the same argument we arrive at the equality $\lambda^{A-B} = p^{A-B}$. Since A, B are distinct we may assume that $A > B$. Then $0 < A - B < N$. Since $o(g) > N > A - B$ for every non-neutral element $g \in G$, we conclude that $p = \lambda$, contradicting $p > \lambda$.

The case when 3 fails is similar to the first case. $\square$

Lemma 2.2. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than N, and $\tau \subset G$ is an uncountable regular cardinal. Let A, B be distinct natural numbers less than N. Then there exists a τ-sized closed subset T of τ such that*

$$\{\alpha^A \star \beta^{N-A} : \alpha < \beta \ \text{ and } \ \alpha, \beta \in T\} \cap \{\beta^B \star \alpha^{N-B} : \alpha > \beta \ \text{ and } \ \alpha, \beta \in T\} = \emptyset.$$

Proof: We will construct $T = \{\lambda_\alpha : \alpha < \tau\}$ inductively.

Assume that for all $\beta < \alpha < \tau$, we have defined λ_β and λ^*_β that meet the following conditions:

1. $\lambda^*_\gamma < \lambda^*_\beta < \tau$ for $\gamma < \beta$;
2. $\lambda_\beta = \sup\{\lambda^*_\gamma : \gamma < \beta\}$ for $\beta > 0$;
3. $\lambda_\beta < \lambda^*_\beta$;
4. $\lambda^A_\beta \star y^{N-A} \neq z^B \star w^{N-B}$ for any $y, z, w \in [\lambda^*_\beta, \tau)$;
5. $x^A \star y^{N-A} \neq \lambda^B_\beta \star w^{N-B}$ for any $x, y, w \in [\lambda^*_\beta, \tau)$;
6. $\lambda^A_\beta \star y^{N-A} \neq \lambda^B_\beta \star w^{N-B}$ for any $y, w \in [\lambda^*_\beta, \tau)$.

If $\alpha = 0$ put $\lambda_\alpha = 0$. If $\alpha > 0$ put $\lambda_\alpha = \sup\{\lambda^*_\beta : \beta < \alpha\}$. Let λ^*_α be an ordinal strictly between λ_α and τ that meets requirements 4–6. Such an ordinal exists by 1 and 2 of Lemma 2.1. By 2, T is closed in τ. By 3 and regularity of τ, T is τ-sized. To finish the proof we need to show that $x^A \star y^{N-A} \neq z^B \star w^{N-B}$ whenever $x, y, z, w \in T$, $x < y$ and $z < w$.

Case $[x < z]$. Then $y, z, w \geq x^*$. Now apply 4.

Case $[x > z]$. Apply 5.

Case $[x = z]$. Apply 6.

$\square$

Lemma 2.3. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than N, and $\tau \subset G$ is an uncountable regular cardinal. Let A be a natural number less than N. Then there exists a τ-sized closed subset T of τ such that $\star$ is one-to-one on $\{\langle \alpha^A, \beta^{N-A}\rangle : \alpha \leq \beta$ and $\alpha, \beta \in T\}$.*

Proof: We will construct $T = \{\lambda_\alpha : \alpha < \tau\}$ inductively.

Assume that for all $\beta < \alpha < \tau$, we have defined λ_β and λ^*_β that meet the following conditions:

1. $\lambda^*_\gamma < \lambda^*_\beta < \tau$ for $\gamma < \beta$;
2. $\lambda_\beta = \sup\{\lambda^*_\gamma : \gamma < \beta\}$ for $\beta > 0$;
3. $\lambda_\beta < \lambda^*_\beta$;
4. $\lambda^A_\beta \star y^{N-A} \neq z^A \star w^{N-A}$ for any $y, z, w \in [\lambda^*_\beta, \tau)$;
5. $\lambda^A_\beta \star \lambda^{N-A}_\beta \neq z^A \star w^{N-A}$ for any $z, w \in [\lambda^*_\beta, \tau)$.

If $\alpha = 0$ put $\lambda_\alpha = 0$. If $\alpha > 0$ put $\lambda_\alpha = \sup\{\lambda^*_\beta : \beta < \alpha\}$. Let λ^*_α be an ordinal strictly between λ_α and τ that meets requirements 4 and 5. Such an ordinal exists by 1 and 3 of Lemma 2.1. By 2, T is closed in τ. By 3 and regularity of τ, T is τ-sized. To finish the proof we need to show that $x^A \star y^{N-A} \neq z^A \star w^{N-A}$ for any distinct $\langle x, y\rangle$ and $\langle z, w\rangle$ in $T \times T$, where $x \leq y$ and $z \leq w$.

Case $[x < z$ and $x \neq y]$. Then $y, z, w \geq x^*$. Now apply 4.

Case $[x < z$ and $x = y]$. Apply 5.

Case $[z < x$ and $z \neq w]$. Apply 4.

Case $[z < x$ and $z = w]$. Apply 5.

Case $[x = z]$. Apply the fact that multiplication by a scalar is a homeomorphism. □

Lemma 2.4. *Suppose G is an Abelian topological group, the order of every non-neutral element of G is greater than N, $\tau \subset G$ is an uncountable regular cardinal, and A, B are distinct natural numbers less than N. Then there exists a τ-sized closed subset T of τ such that $\star$ is one-to-one on the following sets*

1. $\{\langle\alpha^A, \beta^{N-A}\rangle : \alpha < \beta; \alpha, \beta \in T\} \cup \{\langle\alpha^{N-B}, \beta^B\rangle : \alpha > \beta; \alpha, \beta \in T\}$;
2. $\{\langle\alpha^A, \beta^{N-A}\rangle : \alpha \leq \beta$, *and* $\alpha, \beta \in T\}$;
3. $\{\langle\alpha^{N-B}, \beta^B\rangle : \alpha \geq \beta$, *and* $\alpha, \beta \in T\}$.

Proof: By Lemma 2.3 there exists a τ-sized T_1 closed in τ such that $\star$ is one-to-one on set 2. By Lemma 2.3 there exists a τ-sized $T_2 \subset T_1$ closed in τ such that $\star$ is one-to-one on set 3. By Lemma 2.2, there exists a τ-sized $T \subset T_2$ closed in τ such that $\star$ is one-to-one on the set 1. Clearly T is as desired. □

Lemma 2.5. *Let G be an Abelian topological group and $\tau \subset G$ an uncountable regular cardinal. Suppose τ is closed in G and $\{\alpha^2 : \alpha < \tau\}$ is not. Then $\langle\alpha^{2n+1} : \alpha < \tau\rangle$ is not convergent in G for any $n \in \omega$.*

Proof: Assume the contrary and let n be the smallest number for which the conclusion does not hold. Let p be the limit of the transfinite sequence $\langle \alpha^{2n+1} : \alpha < \tau \rangle$. Since τ is closed in G, we conclude that $n > 0$. Since $\{\alpha^2 : \alpha < \tau\}$ is not closed, the sequence $\langle \alpha^2 : \alpha < \tau \rangle$ converges to some element $q \in G$. Therefore, $\langle (\alpha^{-1})^2 : \alpha < \tau \rangle$ converges to q^{-1}. Then the sequence $\langle \alpha^{2n+1} \star (\alpha^{-1})^2 : \alpha < \tau \rangle$ must converge to $p \star q^{-1}$. Every term of the sequence is simplified to $\alpha^{2(n-1)+1}$. Therefore $\langle \alpha^{2(n-1)+1} : \alpha < \tau \rangle$ converges in G, contradicting the choice of n. $\square$

Lemma 2.6. *Let G be an Abelian topological group and $\tau \subset G$ an uncountable regular cardinal. Suppose τ is closed in G and $\{\alpha^3 : \alpha < \tau\}$ is not. Then neither $\langle \alpha^2 : \alpha < \tau \rangle$ nor $\langle \alpha^4 : \alpha < \tau \rangle$ is convergent in G.*

Proof: Since $\{\alpha^3 : \alpha < \tau\}$ is not closed, the sequence $\langle \alpha^3 : \alpha < \tau \rangle$ converges to some $p \in G$ (see Fact 2). Assume $\langle \alpha^2 : \alpha < \tau \rangle$ converges to $q \in G$. Then $\langle \alpha^2 \star \alpha^{-3} : \alpha < \tau \rangle$ converges to $q \star p^{-1}$. Therefore, $\{\alpha^{-1} : \alpha < \tau\}$ is not closed in G, a contradiction.

The proof that the sequence of powers of 4 is not convergent is similar. $\square$

Theorem 2.7. *Let G be an Abelian topological group with the order of every non-neutral element greater than 5. If a regular cardinal τ embeds in G as a closed subspace, then so does $\tau \times \tau$.*

Proof: Since the topological square of ω is homeomorphic to ω, the conclusion is clear for ω. Thus, we may assume that τ is uncountable. We may assume that τ is a closed subspace of G. Let N be a natural number less than or equal to 5 and greater than 2. Let A, B be distinct natural numbers less than N. Put $L = \{\langle \alpha, \beta \rangle : \beta \le \alpha < \tau\}$ and $U = \{\langle \alpha, \beta \rangle : \alpha \le \beta < \tau\}$. Define $f : \tau \times \tau \to G$ as follows:

$$f(\alpha, \beta) = \begin{cases} \beta^B \star \alpha^{N-B} & \text{if } \alpha \ge \beta \\ \alpha^A \star \beta^{N-A} & \text{if } \beta \ge \alpha \end{cases}$$

Since both formulas of f give the same value when $\alpha = \beta$, f is well-defined.

Let us show that f is continuous. It suffices to show that $f|_U$ and $f|_L$ are continuous. Let ϕ be the restriction of $\star$ to $S_{N-B} \times S_B$, where $S_B = \{\beta^B : \beta < \tau\}$ and $S_{N-B} = \{\alpha^{N-B} : \alpha < \tau\}$. Define $\mu : \tau \times \tau \to S_{N-B} \times S_B$ by letting $\mu(\alpha, \beta) = \langle \alpha^{N-B}, \beta^B \rangle$. Then $f|_L = \phi \circ \mu|_L$. Similarly, $f|_U$ is the composition of two continuous functions, and is, therefore, continuous.

By Lemma 2.4, we may assume that f is one-to-one on L, U, and the co-diagonal $\{\langle \alpha, \beta \rangle : \alpha \neq \beta \textit{ and } \alpha, \beta \in \tau\}$. Since any two elements of $\tau \times \tau$ lie both in the co-diagonal, or both in L, or both in U, we may assume that f is one-to-one. We only need to show that under a correct choice of A,B, and N, f is a homeomorphism and $f(\tau \times \tau)$ is closed in G.

Case I. Assume that $\{\alpha^2 : \alpha < \tau\}$ is not closed. Put $B = 2$, $A = 4$, and $N = 5$. Suppose that either f is not a homeomorphism or $f(\tau \times \tau)$ is not closed in G. Since f is one-to-one there exists $\lambda \leq \tau$ such that $\tilde{f}(\lambda, \tau)$ (or $\tilde{f}(\tau, \lambda)$) is in G, where $\tilde{f}$ is the continuous extension of f over $(\tau + 1) \times (\tau + 1)$ (recall that $\beta(\tau \times \tau) \cong (\tau + 1) \times (\tau + 1)$). This means that one of the following sets has a complete accumulation point in G:

1. horizontal sets: $H_\beta = \beta^2 \star \{\alpha^3 : \beta < \alpha < \tau\}$, where β is fixed;
2. vertical sets: $V_\alpha = \alpha^4 \star \{\beta : \alpha < \beta < \tau\}$, where α is fixed;
3. diagonal set: $D = \{\alpha^5 : \alpha < \tau\}$.

Horizontal and diagonal sets are closed by Lemma 2.5 and Fact 2. The vertical sets are closed since τ is closed in G.

Case II. Assume that $\{\alpha^3 : \alpha < \tau\}$ is not closed. Put $B = 2$, $A = 3$, and $N = 4$. As in Case I, we need to show that the following sets are closed:

1. horizontal sets: $H_\beta = \beta^2 \star \{\alpha^2 : \beta < \alpha < \tau\}$, where β is fixed;
2. vertical sets: $V_\alpha = \alpha^3 \star \{\beta : \alpha < \beta < \tau\}$, where α is fixed;
3. diagonal set: $D = \{\alpha^4 : \alpha < \tau\}$.

Horizontal and diagonal sets are closed by Lemma 2.6 and Fact 2. The vertical sets are closed since τ is closed in G.

Case III. Assume that $\{\alpha^2 : \alpha < \tau\}$ and $\{\alpha^3 : \alpha < \tau\}$ are closed. Put $B = 2$, $A = 1$, and $N = 3$. As in Case I, we need to show that the following sets are closed:

1. horizontal sets: $H_\beta = \beta^2 \star \{\alpha : \beta < \alpha < \tau\}$, where β is fixed;
2. vertical sets: $V_\alpha = \alpha \star \{\beta^2 : \alpha < \beta < \tau\}$, where α is fixed;
3. diagonal set: $D = \{\alpha^3 : \alpha < \tau\}$.

The sets are closed by this case's assumptions.

Since one of these cases takes place and each case leads to the required conclusion, we are done. □

In [BUZ] it is proved that if G is an Abelian topological group, the order of every non-neutral element of G is greater than 3, and a regular cardinal τ embeds in G, then $\tau \times \tau$ embeds in G as well. Therefore, it is natural to ask if "5" can be replaced by "3" in the hypothesis of Theorem 2.7. Another motivation for this replacement is given by Theorem 3.2 of the next section. It is easy to see that the requirement "$o(g) > 3$ *for every* $g \in G \setminus \{e_G\}$" implies that the order of every non-neutral element of G is greater than or equal to 5. Indeed, if G had an element of order 4, then it would have had an element of order 2, contradicting the assumption. Thus, we need to show that the conclusion of Theorem 2.7 holds if the order of every non-neutral element is greater than or equal to 5. This goal is partly achieved by Theorem 2.9.

Lemma 2.8. *Let G be a topological group and $\tau \subset G$ an uncountable regular cardinal. If τ is closed in G and $o(G) = 5$ then $\{\alpha^2 : \alpha < \tau\}$ and $\{\alpha^3 : \alpha < \tau\}$ are closed in G.*

PROOF: Since the order of every element must divide the maximum order, which is 5, $o(g) = 5$ for every non-neutral element. Now apply a known fact that in an Abelian group with $o(G) = 5$, taking a square of each element or taking a cube of each element defines a topological isomorphism of G onto itself. □

Theorem 2.9. *Let G be an Abelian topological group, $o(G) = 5$, and τ an uncountable regular cardinal. If τ embeds in G as a closed subspace, then so does $\tau \times \tau$.*

PROOF: Define $f : \tau \times \tau \to G$ as follows:

$$f(\alpha, \beta) = \begin{cases} \beta^2 \star \alpha & \text{if } \alpha \geq \beta \\ \alpha \star \beta^2 & \text{if } \beta \geq \alpha \end{cases}$$

By Lemma 2.4, where $N = 3, A = 1$, and $B = 2$, we may assume that f is one-to-one. By the argument of Theorem 2.7, f is continuous. We need to show that f is homeomorphism and $f(\tau \times \tau)$ is closed in G. As in Case III of Theorem 2.7, we need to show that the following sets are closed:

1. horizontal sets: $H_\beta = \beta^2 \star \{\alpha : \beta < \alpha < \tau\}$, where β is fixed;
2. vertical sets: $V_\alpha = \alpha \star \{\beta^2 : \alpha < \beta < \tau\}$, where α is fixed;
3. diagonal set: $D = \{\alpha^3 : \alpha < \tau\}$.

The sets are closed by Lemma 2.8. □

Remark. Note that the order requirements in Theorems 2.7 and 2.9 need not be placed on all elements of the group. It is enough to require that there exists a closed unbounded subset T of τ such that the order of every non-neutral element of $\mathrm{Env}(T)$ is greater than 5 or $o(\mathrm{Env}(T)) = 5$.

3. Topological groups of order 3 algebraically generated by τ

The main result of this section (Theorem 3.3) will partly justify the order requirement in the hypothesis of Theorem 2.7. For our discussion we need the following result from [BUZ].

Lemma ([Lemma 3.2, BUZ]). *Let G be an Abelian topological group algebraically generated by $(\tau + 1) \subset G$ (or by $\tau \subset G$), where τ is an uncountable regular cardinal. Suppose $X = \{x_\alpha : \alpha < \tau\} \subset G$ is homeomorphic to τ. Then there exist a natural number K_X, an unbounded closed subset I_X of τ, and $g_X \in G$ such that $x_\alpha = g_X \star \alpha^{K_X}$ for every $\alpha \in I_X$.*

If, additionally, $o(G) = N$, then K_X can be chosen strictly between 0 *and* N.

The next lemma describes an algebraic structure of subspaces of an Abelian group G that are homeomorphic to $\tau \times \tau$ if G is algebraically generated by τ. In words, the lemma states that if S is a subset of such a group G and is homeomorphic to $\tau \times \tau$, then there exists $R \subset S$ homeomorphic to $\tau \times \tau$ such that the elements of the lower (or upper) triangle of R are the values of a fixed power function defined on $\tau \times \tau$.

Lemma 3.1. *Let G be an Abelian topological group algebraically generated by an uncountable regular cardinal τ. Let $\{s_{\langle\alpha,\beta\rangle} : \langle\alpha,\beta\rangle \in \tau\times\tau\}$ be homeomorphic to $\tau\times\tau$ under $s_{\langle\alpha,\beta\rangle} \leftrightarrow \langle\alpha,\beta\rangle$. Then there exist an unbounded closed subset T of τ, natural numbers N and K, an element $t^* \in G$, and an unbounded closed subset T_β of τ for every $\beta \in T$ such that $s_{\langle\alpha,\beta\rangle} = t^* \star \beta^K \star \alpha^N$ for every $\beta \in T$ and $\alpha \in T_\beta$.*
If, in addition, $o(G) = M$, then K and N can be chosen strictly between 0 *and M.*

Proof: Fix $\beta \in \tau$. By Lemma 3.2 of [BUZ], there exist a natural number N_β, a constant $p_\beta \in G$, and a τ-sized P_β closed in τ such that $s_{\langle\alpha,\beta\rangle} = p_\beta \star \alpha^{N_\beta}$ for all $\alpha \in P_\beta$.

For every $\beta \in \tau$ select non-negative integers A_β and B_β and sequences $\langle x_\beta(i) : i < A_\beta\rangle$ and $\langle y_\beta(j) : j < B_\beta\rangle$ such that

P1. $p_\beta = x_\beta(0) \star \ldots \star x_\beta(A_\beta - 1) \star y_\beta(0) \star \ldots \star y_\beta(B_\beta - 1)$;

P2. $x_\beta(i) < \beta$ for all $i < A_\beta$;

P3. $y_\beta(j) \geq \beta$ for all $j < B_\beta$.

Such sequences exist because τ generates algebraically G. There exist non-negative integers A and B, a natural number N, and a stationary $S_1 \subset \tau$ such that $\langle A_\beta, B_\beta, N_\beta\rangle = \langle A, B, N\rangle$ for all $\beta \in S_1$.

Property P2 defines A-many regressive functions on stationary S_1. Applying the Pressing Down Lemma (see, for example, [KUN]), we can find stationary $S_2 \subset S_1$ such that $x_\beta(i) = x_\gamma(i)$ for all $\beta, \gamma \in S_2$ and $i < A$. In other words, there exists a constant $t^* \in G$ such that

P4. $s_{\langle\alpha,\beta\rangle} = t^* \star y_\beta(0) \star \ldots \star y_\beta(B-1) \star \alpha^N$ for any $\beta \in S_2$ and $\alpha \in P_\beta$.

Claim. *For every $\gamma < \tau$ there exist $\beta > \gamma$ and an unbounded closed subset T_β of τ such that $s_{\langle\alpha,\beta\rangle} = t^* \star \beta^B \star \alpha^N$ for all $\alpha \in T_\beta$.*

To prove the claim select $\langle\beta_n\rangle_n$ a strictly increasing sequence of elements of S_2 greater than γ such that

P5. $\beta_{n+1} > \max\{y_{\beta_n}(i) : i < B\}$.

Such a sequence exists because S_2 is unbounded in τ. Put $\beta = \lim_{n\to\infty} \beta_n$ and $T_\beta = \bigcap_n P_{\beta_n}$. Clearly, T_β is closed and unbounded in τ. Pick any $\alpha \in T_\beta$. Since $s_{\langle\alpha,\beta\rangle} \leftrightarrow \langle\alpha,\beta\rangle$ is a homeomorphism, $s_{\langle\alpha,\beta\rangle} = \lim_{n\to\infty} s_{\langle\alpha,\beta_n\rangle}$. By P4, $s_{\langle\alpha,\beta\rangle} = \lim_{n\to\infty} t^* \star y_{\beta_n}(0) \star \ldots \star y_{\beta_n}(B-1) \star \alpha^N$. By P5 and P3, $s_{\langle\alpha,\beta\rangle} = t^* \star \beta^B \star \alpha^N$. The claim is proved.

Observe that in the proof of the claim, instead of $\langle\beta_n\rangle_{n\in\omega}$ we could use a transfinite sequence $\langle\beta_i\rangle_{i\in\lambda}$, where $\lambda < \tau$ is a regular cardinal. By Claim, there exists an unbounded subset S_3 of τ such that for any $\beta \in S_3$ there exists an unbounded closed subset T_β of τ such that $s_{\langle\alpha,\beta\rangle} = t^* \star \beta^B \star \alpha^N$ for all $\alpha \in T_\beta$. Put T to be the closure of S_3 in τ. By the same argument as in Claim, for any $\beta \in T$ there exists an unbounded closed subset T_β of τ such that $s_{\langle\alpha,\beta\rangle} = t^* \star \beta^B \star \alpha^N$ for all $\alpha \in T_\beta$.

Put $K = B$. To show that $K \neq 0$, assume the contrary. Pick any distinct $\beta, \gamma \in T$. Since T_β and T_γ are closed in τ and unbounded, there exists $\alpha \in T_\beta \cap T_\gamma$. Then $s_{\langle\alpha,\beta\rangle} = t^* \star \alpha^N$ and $s_{\langle\alpha,\gamma\rangle} = t^* \star \alpha^N$, contradicting to the facts that $\beta \neq \gamma$ and $N \neq 0$.

Finally, if $o(G) = M$, then $o(g)$ divides M for all $g \in G$. Therefore, by [Lemma 3.2, BUZ], N can be chosen strictly between 0 and M. Also, when B is introduced for the first time it can be chosen less than M and at the end of the proof put $K = B$. $\square$

Lemma 3.2. *Let G be a topological group algebraically generated by an uncountable regular cardinal τ. If $o(G) = 2$ then $\tau \times \tau$ is not embeddable in G.*

Proof: First note that $o(G) = 2$ implies that G is Abelian. Assume the conclusion of the lemma is false and let $S = \{s_{\langle\alpha,\beta\rangle} : \langle\alpha,\beta\rangle \in \tau \times \tau\} \subset G$ be homeomorphic to $\tau \times \tau$ under $\langle\alpha,\beta\rangle \leftrightarrow s_{\langle\alpha,\beta\rangle}$.

By Lemma 3.1, there exist an unbounded closed subset T of τ, an element $t^* \in G$, natural numbers $K, N < 2$ (that is, $K = N = 1$), and an unbounded closed subset T_β of τ for every $\beta \in T$ such that $s_{\langle\alpha,\beta\rangle} = t^* \star \beta^1 \star \alpha^1$ for all $\beta \in T$ and $\alpha \in T_\beta$.

Select sequences $\langle\alpha_n\rangle_n$ and $\langle\beta_n\rangle_n$ with the following properties:

1. $\alpha_n > \beta_n$, $\beta_n \in T$, $\alpha_n \in T_{\beta_n}$;
2. $\beta_{n+1} > \alpha_n$.

Such sequences exist because T and T_β are unbounded in τ. By 2, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lambda < \tau$. By 1, we have $s_{\langle\alpha_n,\beta_n\rangle} = t^* \star \beta_n \star \alpha_n$. By continuity of $\star$, we have $\lim_{n\to\infty} s_{\langle\alpha_n,\beta_n\rangle} = t^* \star \lambda^2 = s_{\langle\lambda,\lambda\rangle}$. Since $o(G) = 2$, we have $s_{\langle\lambda,\lambda\rangle} = t^*$. Since λ can be as large as we wish, we conclude that almost all diagonal elements $s_{\langle\lambda,\lambda\rangle}$ are equal to each other, a contradiction. $\square$

Theorem 3.3. *Let G be an Abelian topological group algebraically generated by an uncountable regular cardinal τ. If $o(G) \leq 3$ then $\tau \times \tau$ is not embeddable in G.*

Proof: By Lemma 3.2, we may assume that $o(G) = 3$. Assume the conclusion of the theorem is not true and let $S = \{s_{\langle\alpha,\beta\rangle} : \langle\alpha,\beta\rangle \in \tau \times \tau\}$ be homeomorphic to $\tau \times \tau$ under $\langle\alpha,\beta\rangle \leftrightarrow s_{\langle\alpha,\beta\rangle}$.

By Lemma 3.1, there exist an unbounded closed subset T of τ, an element $t^* \in G$, natural numbers $K, N < 3$, and an unbounded closed subset T_β of τ for every $\beta \in T$ such that $s_{\langle\alpha,\beta\rangle} = t^* \star \beta^K \star \alpha^N$ for any $\beta \in T$ and $\alpha \in T_\beta$.

By Lemma 3.1, there exist an unbounded closed subset P of τ, an element $p^* \in G$, natural numbers $L, M < 3$, and an unbounded closed subset P_α of τ for every $\alpha \in P$ such that $s_{\langle\alpha,\beta\rangle} = p^* \star \alpha^L \star \beta^M$ for any $\alpha \in P$ and $\beta \in P_\alpha$.

Claim 1. *$p^* = t^*$ and $K + N \equiv L + M \pmod 3$.*

To prove the claim select sequences $\langle\alpha_{2n}, \beta_{2n}\rangle$ and $\langle\alpha_{2n+1}, \beta_{2n+1}\rangle$ such that

1. $\alpha_{2n} > \beta_{2n}$, $\beta_{2n} \in T$, $\alpha_{2n} \in T_{\beta_{2n}}$;
2. $\beta_{2n+1} > \alpha_{2n+1}$, $\alpha_{2n+1} \in P$, $\beta_{2n+1} \in P_{\alpha_{2n+1}}$;
3. $\max\{\alpha_n, \beta_n\} < \min\{\alpha_{n+1}, \beta_{n+1}\}$.

By 3, $\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} \beta_{2n} = \lim_{n\to\infty} \alpha_{2n+1} = \lim_{n\to\infty} \beta_{2n+1} = \lambda < \tau$. By 1 and 2, we have $s_{\langle\alpha_{2n},\beta_{2n}\rangle} = t^* \star \beta_{2n}^K \star \alpha_{2n}^N$ and $s_{\langle\alpha_{2n+1},\beta_{2n+1}\rangle} = p^* \star \alpha_{2n+1}^L \star \beta_{2n+1}^M$. By continuity of $\star$, we have $\lim_{n\to\infty} s_{\langle\alpha_{2n},\beta_{2n}\rangle} = t^* \star \lambda^{K+N} = s_{\langle\lambda,\lambda\rangle}$ and $\lim_{n\to\infty} s_{\langle\alpha_{2n+1},\beta_{2n+1}\rangle} = p^* \star \lambda^{L+M} = s_{\langle\lambda,\lambda\rangle}$. Put $D = K + N - L - M$. Then $\lambda^D = p^* \star (t^*)^{-1}$. By the same argument, we can find $\mu > \lambda$ such that $\mu^D = p^* \star (t^*)^{-1}$. Since $\mu^D = \lambda^D$, we conclude that $D \equiv 0 \pmod 3$.
To show that $p^* = t^*$, plug in 0 for D in $\mu^D = p^* \star (t^*)^{-1}$ and get $p^* = t^*$. The claim is proved.

Claim 2. $N \not\equiv M \pmod 3$.
To prove the claim, assume the contrary. Put $c = p^* = t^*$. Pick λ in $T \cap P$ and $\mu \in T_\lambda \cap P_\lambda$ such that $\mu > \lambda$. Such λ and μ exist because $T, P, T_\lambda, P_\lambda$ are unbounded closed subsets of τ. Then $s_{\langle\mu,\lambda\rangle} = c \star \lambda^K \star \mu^N$ and $s_{\langle\lambda,\mu\rangle} = c \star \lambda^L \star \mu^M$. If $N \equiv M \pmod 3$, then, by Claim 1, $K \equiv L \pmod 3$. Therefore, $s_{\langle\mu,\lambda\rangle} = s_{\langle\lambda,\mu\rangle}$, a contradiction. The claim is proved.

Since $N \neq M$, $0 < N < 3$, and $0 < M < 3$, we may assume that $N = 1$ and $M = 2$. Since $K + N \equiv L + M \pmod 3$, $0 < K < 3$, and $0 < L < 3$, we have $L = 1$ and $K = 2$. Using the same argument as in Claim 1, we can find distinct $\lambda, \mu < \tau$ such that $s_{\langle\lambda,\lambda\rangle} = c \star \lambda^{K+N} = c \star \lambda^3$ and $s_{\langle\mu,\mu\rangle} = c \star \mu^{K+N} = c \star \mu^3$.

Since the order of every non-neutral element is 3, $s_{\langle\lambda,\lambda\rangle} = s_{\langle\mu,\mu\rangle}$, a contradiction. □

4. Other results on groups algebraically generated by an ordinal

In the beginning of Section 2 we discussed difficulties that occur in finding a closed copy of $\omega_1 \times \omega_1$ in an Abelian topological group G containing a closed copy of ω_1. The mentioned difficulties lie in the fact that $\{\alpha^2 : \alpha < \omega_1\}$ or $\{\alpha^3 : \alpha < \omega_1\}$ need not be closed in G even if ω_1 is. The following example and its obvious modifications show that each of the three formulas discussed at the beginning of Section 2 is not universal.

Example 4.1. *There exists a torsion free Abelian topological group G such that*

1. *ω_1 is homeomorphic to a closed subspace of G;*
2. *if X is closed in G and homeomorphic to ω_1 then $\{g^2 : g \in X\}$ is not closed in G.*

Proof: For every $\alpha \in \omega_1 + 1$, let $\phi_\alpha \in C_p(C_p(\omega_1 + 1))$ be the evaluation map, that is, $\phi_\alpha(f) = f(\alpha)$ for all $f \in C_p(\omega_1 + 1)$. Put $H = \{\phi_\alpha : \alpha < \omega_1 + 1\}$. Here are facts we need:

A: H and $\omega_1 + 1$ are homeomorphic under the correspondence $\phi_\alpha \leftrightarrow \alpha$ ([ARH]);

B: $\phi_{\omega_1} \notin \mathrm{Env}(\{\phi_\alpha : \alpha < \omega_1\})$. To prove this exclusion, fix any $\alpha_1, \dots, \alpha_n < \omega_1$. There exists $f \in C_p(\omega_1 + 1)$ such that $f(\omega_1) = 1$ and $f(\alpha_i) = 0$ for all $i = 1, \dots, n$. Then for any constants $c_1, \dots, c_n$, we have $(c_1\phi_{\alpha_1} + \dots + c_n\phi_{\alpha_n})(f) = 0$ while $\phi_{\omega_1}(f) = 1$.

C: $C_p(C_p(\omega_1 + 1))$ is Abelian with respect to the pointwise function addition.

Let G consist of all words of $\mathrm{Env}(H)$ that contain ϕ_{ω_1} with even coefficients only (coefficients are computed after collecting like terms).

By Fact B, G is a group. Since $\phi_{\omega_1} \notin G$, the set $H \setminus \{\phi_{\omega_1}\}$ is a closed subspace of G homeomorphic to ω_1 (see Fact A).

Since $2\phi_{\omega_1} \in G$, the set $\{2\phi_\alpha : \alpha < \omega_1\}$ is not closed in G. This is a particular case of the second property of our example.

To prove the second property, fix $F = \{f_\alpha : \alpha < \omega_1\}$ a closed subspace of G homeomorphic to ω_1 under $f_\alpha \leftrightarrow \alpha$. Recall that $G \subset \mathrm{Env}(H)$. Applying Lemma 3.2 of [BUZ], we find an unbounded closed subset T of ω_1, a natural number N, and a function $c \in \mathrm{Env}(H)$ such that $f_\alpha = c + N\phi_\alpha$ for all $\alpha \in T$. Then $\{2f_\alpha : \alpha \in T\} = \{2c + 2N\phi_\alpha : \alpha < \omega_1\}$ has $2c + 2N\phi_{\omega_1}$ as the complete accumulation point in G which is not in $\{2f_\alpha : \alpha \in T\}$. □

In the rest of this work we show what degree of compactness can exist in groups algebraically generated by countably compact ordinals. For our next theorem we need the following folklore lemma about continuous images of scattered (= "every subspace has an isolated point") compacta.

Lemma 4.2 (folklore). *Let X and Y be Hausdorff topological spaces. If X is a scattered compactum and $f : X \to Y$ is a continuous surjection, then Y has an isolated point.*

Proof: Let C be a closed subspace of X such that $f|_C$ is irreducible and $f(C) = Y$. Such a C exists because X is compact. Since C is scattered, there exists $x \in C$ isolated in C. Since $f|_C$ is irreducible and C is compact, $f(x)$ is isolated in $Z = f(C)$. □

Theorem 4.3. *Let G be a topological group algebraically generated by an ordinal τ of uncountable cofinality. Then G is not pseudocompact.*

Proof: Assume G is pseudocompact. By Comfort-Ross ([C&R]) theorem, βG is a topological group and G is its subgroup. We may assume that $\tau + 1$ is the closure of τ in βG. Let $H = \mathrm{Env}(\tau + 1)$ and let H_n be the set of all elements of βG that can be represented as a combination of n elements of $(\tau + 1)$. It is clear that H_n is a continuous image of $\prod_{i\in n}(\tau + 1)$. Since G is algebraically generated by τ, we have $G \subset \bigcup_n H_n = H$. We have two cases to consider.

First, assume that $\bigcup_n H_n$ is compact (that is, equal to βG). Then by the Baire category theorem one of the summands (say H_5) has a non-empty interior in βG. Let U be a non-empty subset of βG such that $\mathrm{Cl}_{\beta G}(U) \subset H_5$. Since H_5 is a continuous image of $\prod_{i\in 5}(\tau+1)$ and finite powers of ordinal spaces are scattered, by Lemma 4.2, there exists a point $x \in \mathrm{Cl}_{\beta G}(U)$ that is isolated in $\mathrm{Cl}_{\beta G}(U)$. Then x belongs to U. Therefore, x is isolated in βG. Since βG is a compact group with an isolated point, βG is finite, which contradicts the inclusion $\tau \subset G$.

Now assume that H is not compact. Since $H = \bigcup_n H_n$ is σ-compact and not compact, $\beta G \setminus H \subset \beta G \setminus G$ is a non-empty G_δ-set in βG that lies in $\beta G \setminus G$, contradicting pseudocompactness of G.

Since the only two possible cases lead to contradictions, we conclude that G is not pseudocompact. □

Example 4.4. *There exists a σ-compact topological group G algebraically generated by ω_1.*

Proof: For every $\alpha \leq \omega_1$, let $\mathbf{x}_\alpha \in 2^{\omega_1}$ be defined as follows: $\mathbf{x}_\alpha(\beta) = 0$ for all $\beta < \alpha$ and $\mathbf{x}_\alpha(\beta) = 1$ for all $\alpha \leq \beta < \omega_1$. It is clear that $H = \{\mathbf{x}_\alpha : \alpha < \omega_1\}$ is a topological copy of ω_1. The space $H^* = \{\mathbf{x}_\alpha : \alpha \leq \omega_1\}$ is a topological copy of $\omega_1 + 1$. Let $G = \mathrm{Env}(H)$. Thus, G is algebraically generated by a topological copy of ω_1. Since $2\mathbf{x}_0 = \mathbf{x}_{\omega_1}$, $G = \mathrm{Env}(H^*)$. Since H^* is compact, G is σ-compact. □

In conclusion, let us state several problems that are naturally suggested by the results of the paper.

Question 4.5. *Is commutativity important in Theorem 2.7 or/and Theorem 3.3?*

Question 4.6. *Let G be an Abelian topological group that contains a closed copy of an uncountable regular cardinal τ. Suppose G is torsion free. Does $\prod_{i\in N}\tau$ embed in G as a closed subspace for every natural number N?*

In [BUZ] it is proved that if a topological group G contains closed copies of τ and $\tau+1$, where τ is an uncountable regular cardinal then $\tau\times(\tau+1)$ embeds in G as a closed subspace. This result prompts the following question.

Question 4.7. *Let X and βX embed in a topological group G as closed subspaces. Suppose that X is not paracompact. Is it true that G is not normal?*

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