On a characterization of normal and countably paracompact spaces via set-avoiding selections

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Abstract. We give a characterization of normal and countably paracompact spaces via continuous set-avoiding selections.

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1. Introduction

Throughout this note all spaces are assumed to be T_1-spaces and $\mathbb{B}$ stands for a Banach space. We often denote by $(\mathbb{B}, \rho)$ the Banach space $\mathbb{B}$ with the metric ρ generated by the given norm on $\mathbb{B}$. In order to state our main result let us set up some of the terminology. If Y is a space, then 2^Y denotes the set of all non-empty subsets of Y. For a space $\mathbb{B}$ we denote

$$E(\mathbb{B}) = \{A \in 2^{\mathbb{B}} : A \text{ is convex and } \dim A < \infty\}.$$

Let $\phi : X \to 2^Y$ be a map. Then ϕ is *lower semi-continuous* (LSC for short) if $\{x \in X : \phi(x) \cap U \neq \emptyset\}$ is open in X for every open U in Y. A map $f : X \to Y$ is called a *selection* for ϕ if $f(x) \in \phi(x)$ for every $x \in X$. An open ball with a radius $\varepsilon > 0$ and a center x in a given metric space will be denoted by $B(x, \varepsilon)$.

All other definitions and concepts are introduced at the end of this section. Now, we formulate our main theorem.

Theorem 1. *For a T_1-space X the following are equivalent.*

(i) *X is normal and countably paracompact.*
(ii) *For every separable Banach space $\mathbb{B}$ and for every LSC map $\phi : X \to E(\mathbb{B})$ such that $\dim \phi(x) = \dim \phi(y)$ for every $x, y \in X$ there exists a continuous selection f for ϕ such that $f(x) \in [\phi(x)]^\circ$.*
(iii) *For every separable Banach space $\mathbb{B}$ and for every LSC map $\phi : X \to E(\mathbb{B})$ such that $\dim \phi(x) = \dim \phi(y)$ for every $x, y \in X$ there exists a continuous selection f for ϕ such that for every $x \in X$ there exist a neighborhood V_x of x and an $\varepsilon_x > 0$ with $B(f(y), \varepsilon_x) \cap \phi(y) \subset [\phi(y)]^\circ$ for every $y \in V_x$.*

In the case when $\mathbb{B}$ is the real line the equivalence (i)$\Leftrightarrow$(ii) in the above theorem was virtually shown by Dowker [4] and Katětov [6]. By [7, Theorem 3.1$'''$], we have that (ii) with the dimension requirement being removed is equivalent for X to be perfectly normal. The following example also shows that we cannot remove the dimension requirement for ϕ in the above theorem.

Example 1. Let X be normal and countably paracompact which is not perfectly normal and let $F \subset X$ be closed but not a G_δ-set in X. Define a LSC map $\phi : X \to 2^{[0,1]}$ as follows:

$$\begin{aligned} \phi(x) &= \{0\} \quad \text{if } \ x \in F, \quad \text{and} \\ \phi(x) &= (0,1) \ \ \text{if } \ x \notin F. \end{aligned}$$

Clearly, $\dim \phi(x) \neq \dim \phi(y)$ if $x \in F$ and $y \in X \setminus F$. On the other hand, ϕ does not have a continuous selection since otherwise, we would have that F is a G_δ-set in X.

Let A be a subset of $\mathbb{B}$. Let $\operatorname{aff} A$ denote the affine hull of A, $\langle A \rangle$ the convex hull and $\overline{A}$ the closure of A in $\mathbb{B}$. ∂A means the relative boundary of A, that is, the boundary with respect to $\operatorname{aff} A$, and we define $A^\circ = A \setminus \partial A$. Recall that if A is convex in $\mathbb{R}^n$, $n \in \mathbb{N}$, then $A^\circ \neq \emptyset$. A set $\{a_0, a_1, \ldots, a_k\}$ in $\mathbb{B}$ is called *a geometrically independent set* (see, for example, [1], [8]) if the set $\{a_1 - a_0, a_2 - a_0, \ldots, a_k - a_0\}$ is a linearly independent set.

If $\delta > 0$ and $A \subset (\mathbb{B}, \rho)$ then $B(A, \delta)$ is the open ball around A with radius δ, i.e.

$$B(A, \delta) = \{x : \rho(x, A) < \delta\}.$$

If $x \in (\mathbb{B}, \rho)$ then we always write $B(x, \delta)$ instead of $B(\{x\}, \delta)$.

Further, if A, C are nonempty compact subsets of a normed linear space $(Y, \|\cdot\|)$ then $d_{\mathrm{H}}(A, C)$ stands for the Hausdorff metric between A and C (see [5] for more details).

We conclude this section with one more definition. For a map $\phi : X \to 2^Y$ we define the following condition.

(*) for every $x \in X$ there is a neighborhood U_x of x such that $\dim \phi(x) \geq \dim \phi(y)$ for every $y \in U_x$.

We see later on that Theorem 1 remains valid if we replace the dimension requirement for ϕ everywhere in the theorem with the requirement for ϕ to satisfy (*).

The paper is arranged as follows. In the next section we establish some lemmas that we need in the sequel. In Section 3 we prove our main results.

2. Some lemmas

In this section we state and prove several lemmas that are needed for proving our main results. Let us point out that some of the lemmas are of independent interest.

The proof of the next lemma directly follows from [3, Lemma 2]. For reader's convenience we give the proof.

Lemma 2. *Let $k \in \mathbb{N}$, and let $\{a_1, \dots, a_k\}$ be a linearly (geometrically) independent set in a normed linear space $(Y, \|\cdot\|)$. Then there is a $\delta > 0$ such that every subset $\{b_1, \dots, b_k\}$ of Y with $\|a_i - b_i\| < \delta$ for every $1 \le i \le k$ is also a linearly (geometrically) independent set.*

Proof: Let

$$M = \left\{(t_1, \dots, t_k) \in \mathbb{R}^k : \left\|\sum_{i=1}^k t_i a_i\right\| \le 2\right\}.$$

Since $\{a_1, \dots, a_k\}$ is a linearly independent set we have that M is compact. Indeed, let us consider a map $f : \mathbb{R}^k \to \operatorname{span}\{a_1, \dots, a_k\}$, defined by the formula

$$f((t_1, \dots, t_k)) = \sum_{i=1}^k t_i a_i \quad \text{for} \quad (t_1, \dots, t_k) \in \mathbb{R}^k.$$

Then f is bijective because $\{a_1, \dots, a_k\}$ is a linearly independent set. In addition, f is a continuous linear function between two finite dimensional Banach spaces and hence f is a linear homeomorphism (see, for example, [8, Corollary 1.1.14]). Therefore, M is compact as an inverse image of a compact set under a homeomorphism. Thus, we can take an α to be an upper bound for the set $\{\sum_{i=1}^k |t_i| \colon (t_1, \dots, t_k) \in M\}$. Further, we can choose a δ such that $0 < \delta < \frac{1}{2\alpha}$. Let $\{b_1, \dots, b_k\} \subset Y$ with $\|a_i - b_i\| < \delta$ for every $1 \le i \le k$. For $(t_1, \dots, t_k) \in M$ we have that

$$\left\|\sum_{i=1}^k t_i(a_i - b_i)\right\| \le \sum_{i=1}^k |t_i| \|(a_i - b_i)\| < \sum_{i=1}^k |t_i| \delta \le \delta\alpha \le \frac{1}{2}\,.$$

Define

$$D = \left\{(t_1, \dots, t_k) \in \mathbb{R}^k : \left\|\sum_{i=1}^k t_i b_i\right\| \le 1\right\}.$$

Now, we are going to show that $D \subset M$. Indeed, let $(t_1, \dots, t_k) \in D$. Set $a' = \sum_{i=1}^k t_i b_i$ and notice that $\|a'\| \le 1$. Consider $a'' = \sum_{i=1}^k t_i a_i$. We need to show that $\|a''\| \leq 2$. Clearly, we may assume that $\|a''\| \neq 0$, so we consider $a''/\|a''\|$. Obviously, $(t_1, \dots, t_k)/\|a''\| \in M$ and therefore,

$$\|a' - a''\|/\|a''\| = \left\| \sum_{i=1}^{k} \frac{t_i}{\|a''\|}(a_i - b_i) \right\| \leq \frac{1}{2}.$$

So, $\|a''\| - \|a'\| \leq \|a' - a''\| \leq \|a''\|/2$ and therefore, $\|a''\| \leq 2\|a'\| \leq 2$. Hence $(t_1, \dots, t_k) \in M$.

Next, being a subset of the compact set M, D is bounded. That implies that $\{b_1, \dots, b_k\}$ is a linearly independent set. Indeed, if we assume that $\{b_1, \dots, b_k\}$ is a linearly dependent set then there is $(t_1, \dots, t_k) \in \mathbb{R}^k$, not all entries equal to zero, such that $\sum_{i=1}^{k} t_i b_i = 0$. Thus, $(\beta t_1, \dots, \beta t_k) \in D$ for every real β. That means that D is unbounded — a contradiction. Consequently, $\{b_1, \dots, b_k\}$ is a linearly independent set. The result for the geometrically independent sets follows easily from that of linearly independent sets. That completes the proof. □

Lemma 3. *Let $A = \{a_1, \dots, a_k\}$ and $C = \{c_1, \dots, c_k\}$ be two sets of points in a normed linear space $(Y, \|\cdot\|)$ such that $\|a_i - c_i\| < \delta$ for every $1 \leq i \leq k$. Then $d_{\mathrm{H}}(\langle A \rangle, \langle C \rangle) < \delta$.*

Proof: Let $m = \max\{\|a_i - c_i\| \colon 1 \leq i \leq k\}$ and notice that $m < \delta$. Then, by the assumptions, we have that $C \subset \langle A \rangle + B(\mathbf{0}, \delta)$. Next, since $\langle A \rangle + B(\mathbf{0}, \delta)$ is convex we get that $\langle C \rangle \subset \langle A \rangle + B(\mathbf{0}, \delta)$. Hence $\langle C \rangle \subset B(\langle A \rangle, \delta)$. Finally, by symmetry we have that $\langle A \rangle \subset B(\langle C \rangle, \delta)$ and that completes the proof. □

Lemma 4. *Let X be a topological space and $(Y, \|\cdot\|)$ be a normed linear space. Let $\phi : X \to E(\mathbb{B})$ be LSC satisfying (*). Then $P_n = \{x \in X \colon \dim \phi(x) = n - 1\}$ is clopen for every $n \in \mathbb{N}$.*

Proof: Clearly, we have that $X = \bigcup_{n=1}^{\infty} P_n$ with $P_i \cap P_j = \emptyset$ if $i \neq j$. We are going to show that each P_n is open. Let $n \in \mathbb{N}$ and $x \in P_n$. Since ϕ satisfies (*) it suffices to show that there is a neighborhood W_x of x such that $\dim \phi(y) \geq \dim \phi(x)$ for every $y \in W_x$. Indeed, if $n = 1$ then it is a triviality. So, we assume that $n \geq 2$. We find a geometrically independent set $\{a_1, \dots, a_n\} \subset \phi(x)$. By Lemma 2, we find a $\delta > 0$ such that if $\|b_i - a_i\| < \delta$ for each $1 \leq i \leq n$ then the subset $\{b_1, \dots, b_n\}$ of Y is also a geometrically independent set. For every $1 \leq i \leq n$ let

$$U_i = \{y \in X : \phi(y) \cap B(a_i, \delta) \neq \emptyset\}.$$

Since ϕ is LSC we have that each U_i is open in X. Set $W_x = \bigcap_{i=1}^{n} U_i$. Take $y \in W_x$. Observe that $\phi(y) \cap B(a_i, \delta) \neq \emptyset$ for every $1 \leq i \leq n$. By the choice of δ we get that $\phi(y)$ contains at least n geometrically independent vectors and hence $\dim \phi(y) \geq \dim \phi(x) = n - 1$. Consequently, each P_n is open. Furthermore, each P_n is closed since the complement of each P_n is open as a union of open sets. That completes the proof. □

Remark 1. Observe that, in the proof of Lemma 4, we show that if $\phi : X \to E(\mathbb{B})$ is LSC then for every $x \in X$ there is a neighborhood U of x such that $\dim \phi(y) \geq \dim \phi(x)$ for every $y \in U$.

Lemma 5. *Let X be a topological space and $(\mathbb{B}, \rho)$ be a Banach space. Let $\phi : X \to E(\mathbb{B})$ be LSC such that $\dim \phi(x) = \dim \phi(y)$ for every $x, y \in X$. Then for every $x \in X$ there are a neighborhood W_x of x, a $\mu_x > 0$ and a LSC map $\psi_x : W_x \to 2^{\mathbb{B}}$ such that each $\psi_x(y)$ is convex and compact and $B(\psi_x(y), \mu_x) \cap \phi(y) \subset [\phi(y)]^\circ$ for every $y \in W_x$.*

Proof: Let $x \in X$. If $\dim \phi(x) = 0$ then the lemma is trivially satisfied by setting $W_x = X$, $\psi_x = \phi$ and taking μ_x to be any positive number. So, we may assume that $n - 1 = \dim \phi(x) > 0$.

We find a geometrically independent set $\{a_1, \ldots, a_n\} \subset [\phi(x)]^\circ$ and set

$$F = \langle \{a_1, \ldots, a_n\} \rangle.$$

Pick $p \in F^\circ$. By Lemma 2, we can find a δ such that if $\|b_i - a_i\| < \delta$ then $\{b_1, \ldots, b_n\}$ is also a geometrically independent set. Let

$$\begin{aligned} &\delta^* = \rho(p, \partial F), \\ &\mu_x = \min\{\frac{1}{3}\delta^*, \delta\}, \quad \text{and} \\ &U_i = \{y \in X : \phi(y) \cap B(a_i, \mu_x) \neq \emptyset\}. \end{aligned}$$

Notice that every U_i is open in X since ϕ is LSC. Set

$$W_x = \bigcap_{i=1}^{n} U_i.$$

Further, we are going to construct a LSC map $\psi_x : W_x \to 2^{\mathbb{B}}$ such that each image is compact and convex and

$$\psi_x(x) = \{p\} \quad \text{and}$$
$$B(\psi_x(y), \mu_x) \cap \phi(y) \subset [\phi(y)]^\circ \quad \text{for every} \quad y \in W_x.$$

Let $y \in W_x$ be arbitrary. Take $\{b_1, \ldots, b_n\}$ such that $b_i \in \phi(y)$ and $\rho(a_i, b_i) < \mu_x$ for every $1 \leq i \leq n$. Next, observe that $\{b_1, \ldots, b_n\}$ is a geometrically independent set. Set

$$G = \langle \{b_1, \ldots, b_n\} \rangle \quad \text{and} \quad G_y = B(p, \mu_x) \cap \phi(y).$$

Now, by Lemma 3, we have that

$$\rho(p, G) < \mu_x \leq \frac{1}{3}\delta^*.$$

In addition, we have the following claim.

Claim 1. $\rho(p, \partial G) \geq \frac{2}{3}\delta^*$.

Proof: Let $z \in \partial G$. Then $z = \sum_{i=1}^n t_i b_i$, where the right-hand side is a convex combination of the b_i's with at least one coefficient zero. Consider $z^* = \sum_{i=1}^n t_i a_i$. Then we have that $z^* \in \partial F$ since at least one of the t_i's is zero and $\{a_1, \dots, a_n\}$ is a geometrically independent set. Moreover, $\rho(z, z^*) = \|\sum_{i=1}^n t_i(b_i - a_i)\| \leq \sum_{i=1}^n t_i\|b_i - a_i\| < \sum_{i=1}^n t_i \mu_x = \mu_x$. Consequently, $\rho(p, z) \geq \rho(p, z^*) - \rho(z, z^*) > \delta^* - \mu_x \geq \frac{2}{3}\delta^*$. □

Furthermore, we prove the following claim.

Claim 2. $B(\overline{G_y}, \mu_x) \cap \phi(y) \subset G^\circ \subset [\phi(y)]^\circ$.

Proof: Obviously, we have that $\dim \phi(y) = n - 1$ and $\dim G = n - 1$. Consequently, $\operatorname{aff} G = \operatorname{aff} \phi(y)$ and $G^\circ \subset [\phi(y)]^\circ$ since both G and $\phi(y)$ are convex and finite-dimensional. Next, having in mind that $\rho(p, G) < \mu_x$ and $B(\overline{G_y}, \mu_x) \subset B(p, 2\mu_x) \subset B(p, \frac{2}{3}\delta^*)$ we have that

$$\emptyset \neq G_y \cap G \subset \overline{G_y} \subset B(\overline{G_y}, \mu_x) \cap \operatorname{aff} \phi(y) \subset$$
$$\subset B(\overline{G_y}, \mu_x) \cap \operatorname{aff} G \subset B(p, \frac{2}{3}\delta^*) \cap \operatorname{aff} G.$$

Further, applying Claim 1, we obtain that

$$B(p, \frac{2}{3}\delta^*) \cap \operatorname{aff} G \subset G^\circ \subset [\phi(y)]^\circ.$$

Hence the claim results. □

Now, define $\psi_x : W_x \to 2^{\mathbb{B}}$, as follows: $\psi_x(x) = \{p\}$ and $\psi_x(y) = \overline{G_y}$ for $y \in W_x \setminus \{x\}$. Let us verify that ψ_x is LSC. Indeed, let U be open in $\mathbb{B}$ and let $O = \{z \in W_x : U \cap \psi_x(z) \neq \emptyset\}$. We need to show that O is open. Observe that, by the definition of ψ_x, we have that

$$O = \{z \in W_x : U \cap (\overline{B(p, \mu_x) \cap \phi(z)}) \neq \emptyset\}$$
$$= \{z \in W_x : U \cap B(p, \mu_x) \cap \phi(z) \neq \emptyset\}.$$

Now, O is open since ϕ is LSC and $U \cap B(p, \mu_x)$ is open in $\mathbb{B}$.

All other required properties of ψ_x are obvious. We are done. □

Remark 2. Considering Lemma 5, let us suppose that $(\mathbb{B}, \rho)$ is separable. Then, without loss of generality, we can assume that the set $\mathcal{Q} = \{W_x : x \in X\}$ is countable. Indeed, if $\dim \phi(x) = 0$ then we can take $\mathcal{Q} = \{X\}$. If $n - 1 = \dim \phi(x) > 0$ then we consider a countable base $\mathcal{O}$ for $\mathbb{B}$. Then, observe that, for every $1 \leq i \leq n$, $B(a_i, \mu_x)$ can be replaced by $O_{x,i} \in \mathcal{O}$ such that $a_i \in O_{x,i} \subset B(a_i, \mu_x)$. Further, we redefine U_i's, i.e. $U_i = \{y \in X : \phi(y) \cap O_{x,i} \neq \emptyset\}$. Thus, W_x is completely determined by $(O_{x,1}, \dots, O_{x,n})$. Since the set of all finite subcollections of $\mathcal{O}$ is countable we may assume that $\mathcal{Q}$ is countable as well.

3. Main results

First, we are going to prove Theorem 1.

Proof of Theorem 1: (i)⇒(iii). For every $x \in X$ we can find a neighborhood W_x of x, a positive number μ_x and a LSC map $\psi_x : W_x \to 2^{\mathbb{B}}$ as in Lemma 5. Next, by Remark 2, without loss of generality, we can assume that the set $\mathcal{Q} = \{W_x : x \in X\}$ is countable. Further, for each $W \in \mathcal{Q}$ we fix an $x_W \in W$ such that $W = W_{x_W}$. Then, we can arrange the countable set $\{(W, \mu_{x_W}, \psi_{x_W}) : W \in \mathcal{Q}\}$ in a sequence $\{(W_i, \mu_i, \psi_i)\}_{i=1}^{\infty}$. Since X is normal and countably paracompact we can find an open locally finite refinement (O_i) of (W_i) such that $X = \bigcup_{n=1}^{\infty} O_i$ and $\overline{O_i} \subset W_i$ for each $i \in \mathbb{N}$. Moreover, by Michael [7, Theorem 3.1$''$] (or, by [7, Theorem 3.1$'$]), we find a continuous selection $f_{\overline{O}_i}$ for $\psi_i \restriction \overline{O_i}$. Further, for $x \in X$ we define

$$A_x = \{k \in \mathbb{N} : x \in O_k\} \quad \text{and} \quad \varepsilon_x^* = \min\{\mu_k : k \in A_x\}.$$

Now, we define a function $\theta : X \to 2^{\mathbb{B}}$ as follows:

$$\theta(x) = \langle \{f_{\overline{O}_k}(x) : k \in A_x\} \rangle \quad \text{for} \quad x \in X.$$

Let us verify that θ is a LSC function. Let $x \in X$. Consider $B(p, \delta)$ with $\delta > 0$ and $p \in \theta(x)$. For every $k \in A_x$ we find a neighborhood U_k of x in O_k such that $f_{\overline{O}_k}(y) \in B(f_{\overline{O}_k}(x), \delta)$ for every $y \in U_k$. Set $U = \bigcap_{k \in A_x} U_k$. Clearly, U is a neighborhood of x in X. Observe that, for each $y \in U$ and $k \in A_x$ we have that $B(f_{\overline{O}_k}(x), \delta) \cap \theta(y) \neq \emptyset$. Now, by Lemma 3 and by the definition of θ, we have that $B(p, \delta) \cap \theta(y) \neq \emptyset$ for every $y \in U$. Hence θ is LSC.

Further, obviously, each image $\theta(x)$ is compact and is a subset of $[\phi(x)]^\circ$. Moreover, for each $x \in X$, we have that

$$B(\theta(x), \varepsilon_x^*) \cap \phi(x) \subset [\phi(x)]^\circ.$$

Now, again by [7, Theorem 3.1$''$], we can find a continuous selection f for θ. Let $x \in X$ be arbitrary. Find a neighborhood V_x of x such that the set $\{k : O_k \cap V_x \neq \emptyset\}$ is finite. Set

$$C_x = \{k : V_x \cap O_k \neq \emptyset\} \quad \text{and} \quad \varepsilon_x = \min\{\mu_k : k \in C_x\}.$$

Let us prove that V_x, ε_x and f are as required. Indeed, pick an $y \in V_x$. Then $A_y \subset C_x$ and therefore $\varepsilon_y^* \geq \varepsilon_x$. Thus,

$$B(\theta(y), \varepsilon_x) \cap \phi(y) \subset B(\theta(y), \varepsilon_y^*) \cap \phi(y) \subset [\phi(y)]^\circ.$$

Now, since $f(y) \in \theta(y)$, we get that $B(f(y), \varepsilon_x) \cap \phi(y) \subset [\phi(y)]^\circ$. Hence the implication (i)⇒(ii) results.

The implication (iii)$\Rightarrow$(ii) is obvious; we need to show that (ii)$\Rightarrow$(i). To show that X is normal we use a very standard idea. Let F_1, F_2 be closed in X with empty intersection. Define $\phi : X \to E([0,1])$ as follows: $\phi(x) = [0, \frac{1}{3}]$ for $x \in F_1$, $\phi(x) = [\frac{2}{3}, 1]$ for $x \in F_2$ and $\phi(x) = [0,1]$ for $x \in X \setminus (F_1 \cup F_2)$. Then, clearly, ϕ is LSC. By the hypotheses, we can find a continuous selection $f : X \to [0,1]$ for ϕ. Then the open sets $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$, that have no common points, contain F_1 and F_2 respectively. The fact, that X is countably paracompact, follows from the proof of [2, Theorem 4] ((2)$\Rightarrow$(1)). That completes the proof of the theorem. □

Now, we are in a position to state and prove the following variation of Theorem 1.

Theorem 6. *For a T_1-space X the following are equivalent.*

(i) *X is normal and countably paracompact.*

(ii) *For every separable Banach space $\mathbb{B}$ and for every LSC map $\phi : X \to E(\mathbb{B})$ satisfying (*) there exists a continuous selection f for ϕ such that for every $x \in X$ there exist a neighborhood V_x of x and an $\varepsilon_x > 0$ with $B(f(y), \varepsilon_x) \cap \phi(y) \subset [\phi(y)]^\circ$ for every $y \in V_x$.*

Proof: It follows directly from Lemma 4 and Theorem 1. □

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