Comment.Math.Univ.Carolin. 49,1 (2008)53–65

Small sets and hypercyclic vectors

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Abstract. We study the "smallness" of the set of non-hypercyclic vectors for some classical hypercyclic operators.

Keywords: hypercyclic operators, porous sets, Haar-null sets

Classification: 47A16, 28A05

1. Introduction

Let X be a separable Fréchet space. A continuous linear operator T on X is said to be *hypercyclic* if there exists some vector $x \in X$ whose T-orbit $O_T(x) := \{T^n(x);\, n \in \mathbb{N}\}$ is dense in X. Such a vector is said to be hypercyclic for T, and the set of hypercyclic vectors is denoted by $HC(T)$. We refer to [6] for background on hypercyclicity.

It is easy to see that if $T \in \mathcal{L}(X)$ is hypercyclic, then $HC(T)$ is a dense G_δ subset of X. Thus, $X \setminus HC(T)$ is always "small" in the sense of Baire category, provided $HC(T) \neq \emptyset$. However, there exist many other natural notions of smallness in infinite-dimensional analysis. In this note, we consider two of them: σ-porosity, and Haar-negligibility.

In a metric space (E, d), a set A is said to be *porous* if the following property holds true: for each point $a \in A$, there exists some positive constant λ and a sequence of positive numbers (r_n) tending to 0 such that, for each $n \in \mathbb{N}$, one can find $x \in B(a, r_n)$ with $B(x, \lambda r_n) \cap A = \emptyset$. The set A is said to be *σ-porous* if it can be covered by countably many porous sets. Porosity was introduced by E.P. Dolženko in 1967 ([4]), and extensively studied since then; see [9] and [10] for more details.

In a Polish abelian group G, a universally measurable set A is said to be *Haar-null* if there exists some Borel probability measure μ on X such that $\mu(A+x) = 0$ for all $x \in G$. This notion was discovered by J.P.R. Christensen in 1972 ([3]), and it has received much attention in the last few years; see e.g. [7].

In this note, we study the smallness of the set of non-hypercyclic vectors for weighted backward shifts on $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ ($1 \leq p < \infty$) and operators commuting with translations on the space of entire functions $\mathcal{H}(\mathbb{C})$. We first recall the definitions.

Let $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ $(1 \le p < \infty)$, and let us denote by $(e_i)_{i\in\mathbb{N}}$ the canonical basis of X. If $\mathbf{w} = (w_i)_{i\ge 1}$ is any bounded sequence of positive numbers, then the weighted backward shift on X associated to $\mathbf{w}$ is the operator $B_{\mathbf{w}} : X \to X$ defined by $B_{\mathbf{w}}(e_0) = 0$ and $B_{\mathbf{w}}(e_i) = w_i e_{i-1}$, $i \ge 1$. By a result of H. Salas ([8]), $B_{\mathbf{w}}$ is hypercyclic if and only if

$$\limsup_{n\to\infty} \prod_{i=1}^{n} w_i = \infty.$$

For example, the operator $T = 2B$ is hypercyclic, where B is the usual, unweighted backward shift on X. This is a classical result of S. Rolewicz.

If $\lambda \in \mathbb{C}$, then the translation-by-λ operator on $\mathcal{H}(\mathbb{C})$ is the operator τ_λ defined by $\tau_\lambda f(z) = f(z+\lambda)$. By a classical result of G.D. Birkhoff, τ_λ is hypercyclic whenever $\lambda \ne 0$. More generally, it was proved by G. Godefroy and J.H. Shapiro ([5]) that if $T \in \mathcal{L}(H(\mathbb{C}))$ is not a scalar multiple of the identity and commutes with all translation operators, then T is hypercyclic. A typical example is the derivation operator D: this is another classical result, due to S. McLane.

The first two sections of the paper concern σ-porosity, which had already been considered in [1]. We show that if a weighted backward shift T has at least one orbit staying away from 0, then the set of non-hypercyclic vectors for T is not σ-porous; this improves the first main result in [1]. On the other hand, we give a criterion for an operator to be "σ-porous hypercyclic", which can be applied to an interesting shift constructed by D. Preiss (unpublished), and to translation operators on $\mathcal{H}(\mathbb{C})$ for a certain class of metrics; here, of course, an operator T is said to be σ-porous hypercyclic if the set of non-hypercyclic vectors for T is σ-porous. The final section concerns Haar-negligibility. We show that weighted backward shifts with large weights are not "Haar-null hypercyclic", and we get the same conclusion for a class of operators on $\mathcal{H}(\mathbb{C})$ commuting with translations.

2. Weighted shifts which are not σ-porous hypercyclic

In this section, we exhibit a class of non σ-porous subsets in Banach spaces, and we apply the result to show that quite a lot of weighted backward shifts on $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ are not σ-porous hypercyclic.

In what follows, X is a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and we assume that X has an unconditional basis $(e_i)_{i\in\mathbb{N}}$. We denote by (e_i^*) the associated sequence of coordinate functionals. We will consider subsets of X of the form

$$F_{[L]}^{\mathcal{A}} = \{x \in X;\, \forall\, n \in \mathbb{N}\ L_n(x) \in \mathcal{A}\},$$

where $\mathcal{A}$ is a subset of $\mathbb{K}^{\mathbb{N}}$ and $[L] = (L_n)$ is a sequence of continuous linear maps from X into $\mathbb{K}^{\mathbb{N}}$. We view a sequence $[L] = (L_n) \subset \mathcal{L}(X, \mathbb{K}^{\mathbb{N}})$ as an infinite matrix (L_{nj}) with entries in X^*, and we denote by $\mathfrak{L}$ the family of all such matrices.

We will say that a matrix $[L]' = (L'_n) \in \mathfrak{L}$ is an *admissible modification* of a matrix $[L]$ if $L'_{nj} = \alpha_{nj} L_{nj}$ for all $(n,j) \in \mathbb{N}^2$, where (α_{nj}) is a bounded sequence of scalars.

Finally, we say that a set $\mathcal{A} \subset \mathbb{K}^{\mathbb{N}}$ is *monotone* if, whenever $(u_j) \in \mathcal{A}$ and $|v_j| \geq |u_j|$ for all $j \in \mathbb{N}$, it follows that $(v_j) \in \mathcal{A}$.

Theorem 2.1. *Let $\mathcal{A}$ be a monotone subset of $\mathbb{K}^{\mathbb{N}}$, and let $[L] \in \mathfrak{L}$. Assume that $F^{\mathcal{A}}_{[L]} \neq \emptyset$, and that the following properties hold:*

- *$L_{nj} \in \bigcup_i \mathbb{K} e^*_i$ for all pairs $(n,j) \in \mathbb{N}^2$;*
- *$F^{\mathcal{A}}_{[L]'}$ is closed in X for each admissible modification $[L]'$ of $[L]$.*

Then $F^{\mathcal{A}}_{[L]}$ is not σ-porous.

From this, we get the following improvement of the first main result of [1].

Corollary 2.2. *Let T be a weighted backward shift on $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $1 \leq p < \infty$. If there exists some point $x \in X$ such that $\inf_n \|T^n(x)\| > 0$, then T is not σ-porous hypercyclic.*

Proof: It is enough to show that the set

$$F = \{x \in X;\, \forall\, n \in \mathbb{N}\ \|T^n(x)\| \geq 1\}$$

is not σ-porous. Now, the set F is nonempty and has the form $F^{\mathcal{A}}_{[L]}$, with $L_{nj}(x) = \langle e^*_j, T^n(x)\rangle$ and $\mathcal{A} = \{(u_j) \in \mathbb{K}^{\mathbb{N}};\ \|\sum_j u_j\, e_j\| \geq 1\}$, where we have put $\|\sum_j u_j\, e_j\| = \infty$ if (u_j) does not define an element of X.

Since $L_{nj} = T^{*n}(e^*_j)$ and T is a shift, we have $L_{nj} \in \mathbb{K} e^*_{n+j}$ for each $(n,j) \in \mathbb{N}^2$. Moreover, if $[L]' = (L'_n)$ is an admissible modification of $[L]$, so that $L'_{nj} = \alpha_{nj} L_{nj}$ for some bounded sequence of scalars (α_{nj}), then each $L'_n : X \to \mathbb{K}^{\mathbb{N}}$ defines a bounded operator $T'_n : X \to X$, namely $T'_n(\sum_j x_j e_j) = T^n(\sum_j \alpha_{nj} x_j e_j)$. Therefore, $F^{\mathcal{A}}_{[L]'} = \{x \in X;\, \forall\, n \in \mathbb{N}\ \|T'_n(x)\| \geq 1\}$ is closed in X. Thus, we may apply 2.1. □

To prove 2.1, we will use a version of *Foran's Lemma* (see [9, Lemma 4.3]), which is essentially the only known tool to check that some sets are not σ-porous.

A subset A of a metric space (E,d) is said to be λ-*porous* ($\lambda > 0$) at some point $a \in E$ if there exists a sequence of positive numbers (r_n) tending to 0 such that, for each $n \in \mathbb{N}$, one can find $x \in B(a, r_n)$ with $B(x, \lambda r_n) \cap A = \emptyset$. The set A is said to be λ-porous if it is λ-porous at each point $a \in A$. It is proved in [10] that given $\lambda \in (0, \frac{1}{2})$, any σ-porous set $A \subset E$ can in fact be covered by countably many λ-porous sets. From this and Lemma 4.3 in [9] applied to the porosity relation V defined by $V(x,A) \Leftrightarrow A$ is λ-porous at x, one gets the following result.

Lemma 2.3. *Let (E,d) be a complete metric space, and let $\lambda \in (0, \frac{1}{2})$. Let also $\mathcal{F}$ be a family of nonempty closed subsets of E. Assume $\mathcal{F}$ has the following property: for each set $F \in \mathcal{F}$ and each open set V such that $V \cap F \neq \emptyset$, one can find $F' \in \mathcal{F}$ such that $F' \subset F$, $F' \cap V \neq \emptyset$ and F is λ-porous at no point of F'. Then no set $F \in \mathcal{F}$ is σ-porous.*

Proof of Theorem 2.1: Let us denote by $[L]_0$ the matrix given in the hypotheses of Theorem 2.1, and by $\mathfrak{L}_0$ the family of all matrices $[L] \in \mathfrak{L}$ which are admissible modifications of $[L]_0$ and satisfy $F^{\mathcal{A}}_{[L]} \neq \emptyset$. For notational simplicity, we will drop the superscript $\mathcal{A}$ in $F^{\mathcal{A}}_{[L]}$ and simply write $F_{[L]}$.

If $L \in \mathfrak{L}_0$, we can choose a map $(n,j) \mapsto \langle n,j\rangle$ from $\mathbb{N}^2$ into $\mathbb{N}$ such that $L_{nj} \in \mathbb{K}e^*_{\langle n,j\rangle}$. We do not indicate explicitly that the map $\langle\, ,\,\rangle$ depends on the matrix $[L]$, but this will cause no confusion.

Finally, for each set $J \subset \mathbb{N}$, we denote by π_J the canonical projection from X onto $\overline{\text{span}}\,\{e_i;\, i \in J\}$. This projection is well-defined by unconditionality of the basis (e_i).

Let $L \in \mathfrak{L}_0$. For each triple $\mathbf{p} = (\varepsilon, K, I)$, where $\varepsilon > 0$, $K > 1$ and I is a finite subset of $\mathbb{N}$, we define a new matrix $[L^{\mathbf{p}}]$ in the following way:

$$L^{\mathbf{p}}_{nj} = \begin{cases} (1+\varepsilon)^{-1} L_{nj} & \text{if } \langle n,j\rangle \in I; \\ K^{-1}\, L_{nj} & \text{if } \langle n,j\rangle \notin I. \end{cases}$$

Then $[L^{\mathbf{p}}]$ is an admissible modification of $[L]$, and hence an admissible modification of the matrix $[L]_0$ we started with. Moreover, if $x \in F_{[L]}$, then $y := (1+\varepsilon)\pi_I(x) + K\pi_{\mathbb{N}\setminus I}(x)$ satisfies $L^{\mathbf{p}}_{nj}(y) = L_{nj}(x)$ for all $(n,j) \in \mathbb{N}^2$, whence $y \in F_{[L^{\mathbf{p}}]}$. Thus $[F^{\mathbf{p}}] \neq \emptyset$, so that $[L^{\mathbf{p}}] \in \mathfrak{L}_0$. Notice also that $F_{[L^{\mathbf{p}}]} \subset F_{[L]}$ by the monotonicity property of $\mathcal{A}$.

Claim 1. *Let $[L] \in \mathfrak{L}_0$, and let $K > 1$ be given. If $V \subset X$ is an open set such that $F_{[L]} \cap V \neq \emptyset$, then one can find ε, I such that $F_{[L^{\mathbf{p}}]} \cap V \neq \emptyset$, where $\mathbf{p} = (\varepsilon, K, I)$.*

Proof: Here, the basis (e_i) needs not be unconditional because we consider projections on finite or co-finite sets only. Choose a point $x \in V \cap F_{[L]}$ and $r > 0$ such that $B(x,r) \subset V$. Since (e_i) is a basis for X, one can choose a finite set $I \subset \mathbb{N}$ such that $\|\pi_{\mathbb{N}\setminus I}(x)\|$ is very small, and then $\varepsilon > 0$ such that $y := (1+\varepsilon)\pi_I(x) + K\pi_{\mathbb{N}\setminus I}(x)$ satisfies $\|y - x\| < r$. This point y shows that $F_{[L^{\mathbf{p}}]} \cap V \neq \emptyset$. □

Claim 2. *Let $[L] \in \mathfrak{L}_0$ and $\lambda \in (0,1)$. If $K > 0$ is large enough, then, for each $\mathbf{p} = (\varepsilon, K, I)$, the set $F_{[L]}$ is λ-porous at no point of $F_{[L^{\mathbf{p}}]}$.*

Proof: Let $\alpha \in (0,1)$ to be chosen later. If $x \in X$, $\varepsilon > 0$ and a finite set $I \subset \mathbb{N}$ are given, one can find $\delta = \delta(x, \varepsilon, I) > 0$ such that if $y \in X$ satisfies $\|y - x\| < \delta$, then

$$|\langle e_i^*, y\rangle| \geq (1+\varepsilon)^{-1} |\langle e_i^*, x\rangle| \quad \text{for all} \quad i \in I.$$

Since (e_i) is unconditional, one may associate to each such point y another point $\tilde{y} \in X$ such that

$$\langle e_i^*, \tilde{y}\rangle = \begin{cases} \langle e_i^*, y\rangle & \text{if} \quad i \in I \quad \text{or} \quad |\langle e_i^*, y\rangle| > \frac{\alpha}{2} |\langle e_i^*, x\rangle|; \\ \alpha\langle e_i^*, x\rangle + (1-\alpha)\langle e_i^*, y\rangle & \text{otherwise.} \end{cases}$$

Then $|\langle e_i^*, \tilde{y}\rangle| \geq (1+\varepsilon)^{-1}|\langle e_i^*, x\rangle|$ if $i \in I$, and $|\langle e_i^*, \tilde{y}\rangle| \geq \frac{\alpha}{2}|\langle e_i^*, x\rangle|$ if $i \notin I$, by the triangle inequality. Thus, if $K \geq 2\alpha^{-1}$, we get that for any $\mathbf{p} = (\varepsilon, K, I)$ and each point $y \in B(x, \delta)$, the following implication holds:

$$x \in F_{[L\mathbf{p}]} \Rightarrow \tilde{y} \in F_{[L]}.$$

Moreover, we also have $|\langle e_i^*, \tilde{y} - y\rangle| \leq \alpha\, |\langle e_i^*, x - y\rangle|$ for all $i \in I$, so that

$$\|\tilde{y} - y\| \leq C\alpha \|x - y\|,$$

where C is the unconditionality constant of the basis (e_i). If we now choose $\alpha < C^{-1}\lambda$, we conclude that if $K \geq 2\alpha^{-1}$, then $F_{[L]}$ is λ-porous at no point $x \in F_{[L\mathbf{p}]}$ when $\mathbf{p}$ has the form (ε, K, I). □

It follows from the above claims that the family $\mathcal{F} := (F_{[L]})_{L \in \mathfrak{L}_0}$ satisfies the hypotheses of Lemma 2.3. Thus, no set $F_{[L]}$ is σ-porous ($[L] \in \mathcal{L}_0$), and the proof of 2.1 is complete. □

3. A criterion for σ-porosity

It is well-known that an operator $T \in \mathcal{L}(X)$ is hypercyclic if and only if it is *topologically transitive*, which means that for each pair (U, V) of nonempty open subsets of X, one can find $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ (see [GE]). In this section, we show that if an operator is "topologically transitive with estimate", then it is σ-porous hypercyclic. The following easy lemma will be needed.

Lemma 3.1. *Let (E, d) be a metric space, and let $A \subset E$. Assume there exist $\delta_0 > 0$, a dense set $D \subset E$ and some constant $c > 0$ such that: for all $u \in D$ and every $\delta \in (0, \delta_0)$, one can find $x \in E$ such that $d(x, u) < \delta$ and $B(x, c\delta) \cap A = \emptyset$. Then A is porous.*

Proof: Let a be any point of E. For any $\delta \in (0, \delta_0)$, one can find $u \in D$ with $d(u, a) < \frac{\delta}{2}$, and then $x \in E$ with $d(x, u) < \frac{\delta}{2}$ and $B(x, c\frac{\delta}{2}) \cap A = \emptyset$. Then we have $x \in B(a, \delta)$ and $B(x, \frac{c}{2}\,\delta) \cap A = \emptyset$, which shows that A is $\frac{c}{2}$-porous at each point $a \in E$ (actually *very porous* in the sense of [9]). $\square$

Since this involves no additional complication, we formulate the announced criterion for an arbitrary sequence of continuous maps $T_n : E \to E$ from a metric space (E, d) into itself. The sequence $\mathbf{T} = (T_n)_{n\in\mathbb{N}}$ is said to be *universal* if there is some $x \in E$ such that the set $\{T_n(x);\, n \in \mathbb{N}\}$ is dense in E, and the set of universal points for $\mathbf{T}$ is denoted by $\mathrm{Univ}(\mathbf{T})$.

If $T : (E, d) \to (E, d)$ is a continuous map, then, for each $r > 0$, we denote by $\omega^{-1}(T, r)$ the largest number $\delta \in [0, 1]$ such that $d(x, y) < \delta \Rightarrow d(T(x), T(y)) < r$.

Theorem 3.2. *Let (E, d) be a separable metric space, and let $\mathbf{T} = (T_n)$ be a sequence of continuous maps, $T_n : E \to E$. Assume there exist a dense set $\mathcal{D} \subset E$ and for each pair $(v, r) \in \mathcal{D} \times (0, 1)$, a dense set $\mathcal{D}_{v,r} \subset E$ and positive real constants $\delta_{v,r}$, $c_{v,r}$ such that the following holds true. For each $u \in \mathcal{D}_{v,r}$ and every $\delta \in (0, \delta_{v,r})$, one can find $x \in E$ and $n \in \mathbb{N}$ such that*

(a) *$d(x, u) < \delta$ and $d(T_n(x), v) < r$;*
(b) *$\omega^{-1}(T_n, r) \geq c_{v,r}\delta$.*

Then $X \setminus \mathrm{Univ}(\mathbf{T})$ is σ-porous.

Proof: It is enough to show that for each pair $(v, r) \in \mathcal{D} \times (0, 1)$, the set

$$A_{v,r} := \{x \in E;\ \forall\, n \in \mathbb{N}\ d(T^n(x), v) \geq 2r\}$$

is porous. Indeed, $E \setminus \mathrm{Univ}(\mathbf{T})$ is a countable union of such sets $A_{v,r}$, by separability of E. Now, it follows from the triangle inequality that for each $u \in \mathcal{D}_{v,r}$ and every $\delta \in (0, \delta_{v,r})$, one can find $x \in B(u, \delta)$ and $n \in \mathbb{N}$ such that $d(T_n(y), v) < 2r$ for all $y \in B(x, c_{r,v}\delta)$, hence $B(x, c_{v,r}\delta) \cap A_{v,r} = \emptyset$. Since $\mathcal{D}_{v,r}$ is dense in X, this shows that $A_{v,r}$ is porous by 3.1. $\square$

In the linear setting, we get from 3.2 the following "universality criterion with estimate", which is a natural variant of the well-known Hypercyclicity Criterion (see [6]).

Corollary 3.3. *Let X be a separable Banach space, and let $\mathbf{T} = (T_n)$ be a sequence of continuous linear operators on X. Assume there exist a dense set $\mathcal{D}^* \subset X$ and for each pair $(v, r) \in \mathcal{D}^* \times (0, 1)$, a dense set $\mathcal{D}^*_{v,r} \subset X$ and positive real constants $\delta^*_{v,r}$, $C_{v,r}$ such that the following holds true. For each $u \in \mathcal{D}^*_{v,r}$ and every $\delta \in (0, \delta^*_{v,r})$, one can find $n \in \mathbb{N}$ and $z \in X$ such that*

(a1) *$\|T_n(u)\| < r$;*
(a2) *$\|z\| < \delta$ and $\|T_n(z) - v\| < r$;*
(b) *$\|T_n\| \leq \frac{C_{v,r}}{\delta}$.*

Then $X \setminus \mathrm{Univ}(\mathbf{T})$ is σ-porous.

PROOF: One can apply 3.2 with $\mathcal{D}_{v,r} := \mathcal{D}^*_{v,r/2}$, $\delta_{v,r} := \delta^*_{v,r/2}$ and $c_{v,r} := \frac{r}{C_{v,r/2}}$. Given $u \in \mathcal{D}^*_{v,r/2}$ and $\delta \in (0, \delta^*_{v,r/2})$, choose n and z according to 3.3 and set $x := u + z$. □

Remark. It follows from the above proofs that, in 3.2 as well as in 3.3, the set $X \setminus \mathrm{Univ}(\mathbf{T})$ can in fact be covered by countably many closed sets which are σ-very porous in the sense of [9].

We now give two illustrations of 3.2. The first one is a recent unpublished result of D. Preiss, and the second one is a generalization of the second main result in [1]. We would like to thank D. Preiss for allowing us to include his example in this paper.

Example 1 (Preiss). *There exist weighted backward shifts on $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ $(1 \le p < \infty)$ which are σ-porous hypercyclic.*

PROOF: We apply Corollary 3.3 with $\mathcal{D}^* = c_{00}$, the space of all finitely supported vectors $x \in X$, and $\mathcal{D}^*_{v,r} = c_{00}$ for all $(v, r) \in c_{00} \times (0, 1)$. Let $T_{\mathbf{w}}$ be a backward shift on X associated to some bounded sequence of positive numbers $\mathbf{w} = (w_k)_{k\ge1}$, and let us see what properties of $\mathbf{w}$ are needed. For $p \le q \in \mathbb{N}$, we set $w_{p,q} = \prod_{p\le k\le q} w_k$ (where we have put $w_0 = 0$). Finally, we denote by $(e_i)_{i\in\mathbb{N}}$ the canonical basis of X.

Let $v = \sum_i v(i)e_i \in c_{00}$ be supported on some interval $[0, p)$. If n is any positive integer, then the vector

$$z_n := \sum_{i=0}^{p-1} \frac{v(i)}{w_{1+i,n+i}}\, e_{n+i}$$

satisfies $T^n_{\mathbf{w}}(z_n) = v$, and we have $\|z_n\| \le \|v\| \max\{(w_{1+i,n+i})^{-1};\ 0 \le i < p\}$. Moreover, if $u \in c_{00}$, then $T^n_{\mathbf{w}}(u) = 0$ if n is large enough. Finally, we have $\|T^n_{\mathbf{w}}\| = \sup\{w_{1+i,n+i};\ i \in \mathbb{N}\}$. Thus, we see that $T_{\mathbf{w}}$ will be σ-porous hypercyclic provided for each positive integer p the following holds for suitable constants M_p and C_p: for every $M \ge M_p$, one can find infinitely many integers n satisfying

(a) $w_{1+i,n+i} > M$ for all $i \in \{0; \dots; p-1\}$;
(b) $w_{1+i,n+i} \le C_p\, M$ for all $i \in \mathbb{N}$.

A weight sequence $\mathbf{w}$ with that property can be obtained as follows. Let us denote by $\mathbb{N}^*$ the set of all positive integers, and let $(p_j, r_j)_{j\ge1}$ be a sequence in $\mathbb{N}^* \times (0, \infty)$ to be specified later. Then one can construct a sequence $\mathbf{w} = (w_k)_{k\ge1} \subset (0, 2)$ and an increasing sequence of positive integers $(n_j)_{j\ge1}$ such that the following properties hold for each j:

(i) $w_{1,k} = r_j$ for all $k \in [n_j\,, n_j + p_j)$;
(ii) $w_{1+i,n_j+i} \le 2$ for all $i \ge p_j$.

To do this, first choose n_1 so that one can find $w_1, \ldots, w_{n_1} \in (0,2)$ with $w_{1,n_1} = r_1$. Set $w_k = 1$ for $k \in (n_1\,, n_1 + p_1)$ in order to have (i) for $j = 1$. Then choose $w_{n_1+p_1}, \ldots, w_{2n_1-1+p_1} \in (0,1)$ small enough to ensure $w_{1+i,n_1+i} \leq 2$ for all $i \in [p_1, n_1+p_1)$. At this point, choose $\varepsilon_1 > 0$ such that $(1+\varepsilon_1)^{n_1} \leq 2$, and find n_2 large enough to ensure that one can construct $w_{2n_1+p_1}, \ldots, w_{n_2} \in (0, 1+\varepsilon_1)$ with $w_{1,n_2} = r_2$. Then (ii) is satisfied for $j = 1$ and all $i \in [p_1\,, n_2 - n_1]$. Repeating the procedure, one gets the sequences (n_j) and (w_k).
Now assume that the sequence (p_j, r_j) enumerates $\mathbb{N}^* \times \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of all positive rational numbers. Fixing p and setting $\mathbf{N}_p := \{n_j;\, p_j = p\}$, we show that (a) and (b) hold with suitable constants M_p and C_p and infinitely many $n \in \mathbf{N}_p$. If $n = n_j \in \mathbf{N}_p$, then, writing $w_{1+i,n+i} = \frac{r_j}{w_{1,i}}$ if $i < p$, we see that property (i) and (ii) give

$$\begin{cases} a_p r_j \leq w_{1+i,n+i} \leq b_p r_j & \text{for all } \ i \in \{0; \ldots; p-1\}, \\ w_{1+i,n+i} \leq 2 & \text{for all } \ i \geq p, \end{cases}$$

where a_p, b_p depend only on p. Thus, (a) and (b) are satisfied for a given M provided $\frac{M}{a_p} < r_j \leq \frac{C_p M}{b_p}$ and $C_p M \geq 2$. Choosing $C_p > \frac{b_p}{a_p}$, this holds for infinitely many j's whenever $M \geq M_p := \frac{2}{C_p}$. $\square$

Our second illustration concerns translation operators on $\mathcal{H}(\mathbb{C})$. Since porosity makes sense only when a metric is given, we first have to choose some compatible metric on $\mathcal{H}(\mathbb{C})$. There are various reasonable ways of doing so. For example, to each sequence of positive numbers $\overline{\varepsilon} = (\varepsilon_n)$ such that $\sum_0^\infty \varepsilon_n < \infty$, one may associate the metric $d_{\overline{\varepsilon}}$ defined by

$$d_{\overline{\varepsilon}}(f,g) = \sum_0^\infty \varepsilon_n \min(1, \|f-g\|_{K_n}),$$

where K_n is the disk $\overline{D}(0,n)$ and $\|f\|_K = \sup\{|f(z)|;\, z \in K\}$. With that kind of metrics, we have the following result.

Example 2. *Let $\overline{\varepsilon} = (\varepsilon_n)$ be a summable sequence of positive numbers, and assume there exists some constant $c > 0$ such that $\sum_{k>n} \varepsilon_k \geq c\,\varepsilon_n$ for all $n \in \mathbb{N}$. If $T \neq \mathrm{id}$ is a translation operator on $\mathcal{H}(\mathbb{C})$, then T is σ-porous hypercyclic with respect to the metric $d_{\overline{\varepsilon}}$.*

Proof: The operator T is defined by $Tf(s) = f(s+\alpha)$, where $\alpha \in \mathbb{C} \setminus \{0\}$. We check that the hypotheses of Theorem 3.2 are satisfied with $\mathcal{D} = \mathcal{H}(\mathbb{C})$ and $\mathcal{D}_{v,r} = \mathcal{H}(\mathbb{C})$, $\delta_{v,r} = 1$ for all $(v,r) \in \mathcal{H}(\mathbb{C}) \times (0,1)$. Thus, we have to find some suitable constants $c_{v,r}$. Let us fix a pair $(v,r) \in \mathcal{H}(\mathbb{C}) \times (0,1)$, together with $u \in \mathcal{H}(\mathbb{C})$.

For simplicity, we will write d instead of $d_{\overline{\varepsilon}}$ and $\|\cdot\|_n$ instead of $\|\cdot\|_{K_n}$. We fix a positive integer $p \geq |\alpha|$ and $\eta > 0$ such that

$$\|f-g\|_p < \eta \Rightarrow d(f,g) < r.$$

Finally, we assume without loss of generality that $\sum_0^\infty \varepsilon_n = 1$.

Let $\delta \in (0,1)$, and set $N := \min\{n \in \mathbb{N};\ \sum_{k>n} \varepsilon_k < \frac{\delta}{2}\}$. To ensure property (a) in 3.2, it is enough to find some function $x \in \mathcal{H}(\mathbb{C})$ and some integer n such that $\|x-u\|_N < \frac{\delta}{2}$ and $\|T^n(x)-v\|_p < \eta$. In other words, we require $|x(s)-u(s)| < \frac{\delta}{2}$ on K_N and $|x(s)-v(s-n\alpha)| < \eta$ on $n\alpha + K_p$. Now, the two disks K_N and $n\alpha + K_p$ are disjoint whenever $n|\alpha| > N+p$, and in that case Runge's Theorem provides an $x \in \mathcal{H}(\mathbb{C})$ satisfying the required properties. Let n be the smallest integer satisfying $n|\alpha| > N+p$. Then (a) is satisfied with n and some $x \in \mathcal{H}(\mathbb{C})$, so it only remains to show that (b) holds for some suitable constant $c_{v,r}$.

By the choice of n and since $p \geq |\alpha|$, we have $n|\alpha| \leq N+2p$ and hence $\|T^n(f)-T^n(g)\|_p \leq \|f-g\|_{N+3p}$ for all $f,g \in \mathcal{H}(\mathbb{C})$. By the choice of p and η, it is therefore enough to find some constant c, which may depend on v, r, p, η but must be independent of δ (and hence of N) such that

$$d(f,g) < c\delta \Rightarrow \|f-g\|_{N+3p} < \eta. \tag{1}$$

By assumption on $\overline{\varepsilon}$, there exists some constant c_p such that

$$\sum_{k\geq N+3p} \varepsilon_k \geq c_p \sum_{k\geq N} \varepsilon_k \geq c_p \frac{\delta}{2}\,,$$

where the second inequality comes from the choice of N. By definition of the metric $d_{\overline{\varepsilon}}$, it follows that for any $f,g \in \mathcal{H}(\mathbb{C})$, we have

$$\frac{c_p}{2}\,\delta\, \min(1, \|f-g\|_{N+3p}) \leq d(f,g).$$

Therefore, (1) will be satisfied provided $c < \frac{c_p}{2}\min(1,\eta)$. This concludes the proof. □

Remark 1. We do not know what happens if the sequence (ε_n) tends very quickly to 0. We do not know either what can be said about the derivation operator D, another classical example of hypercyclic operator on $\mathcal{H}(\mathbb{C})$. In view of 2.1 and since D is a weighted backward shift with increasing weights, it seems reasonable in that case to "conjecture" that, at least for a certain class of metrics $d_{\overline{\varepsilon}}$, the operator D is not σ-porous hypercyclic.

Remark 2. Let X be a separable Fréchet space whose topology is generated by an increasing sequence of semi-norm $(\rho_n)_{n\in\mathbb{N}}$, and define a metric d on X by

$$d(x,y) = \sum_0^\infty \varepsilon_n \min(1, \rho_n(x-y)),$$

where (ε_n) is as in Example 2. Then one proves in exactly the same way that an operator $T \in \mathcal{L}(X)$ is σ-porous hypercyclic provided it has the following property: given $(u, v) \in X \times X$ and $(N, p) \in \mathbb{N} \times \mathbb{N}$, one can find for each $\varepsilon \in (0, 1)$ a point $x \in X$ and an integer n such that

- $\rho_N(x - u) < \varepsilon$ and $\rho_p(T^n(x) - v) < \varepsilon$;
- $\rho_p(T^n(z)) \leq A_p\, \rho_{N+B_p}(z)$ for all $z \in X$, where $A_p > 0$ and $B_p \in \mathbb{N}$ depend only on p.

A similar property, called *Runge transitivity*, is introduced in [2].

4. Haar-negligibility

In this section, we give some examples of hypercyclic operators which are *not* Haar-null hypercyclic. The main tool will be the following well-known and simple lemma (see [3]).

Lemma 4.1. *Let G be a Polish abelian group, and let A be a universally measurable subset of G. If A contains a translate of each compact set $K \subset G$, then A is not Haar-null.*

Our first result will be applied below to weighted shifts. Let us say that a sequence $(f_i)_{i\in\mathbb{N}}$ in a Banach space X is *semi-basic* if there exists some finite constant C such that for all finitely supported sequences of scalars $(\lambda_i)_{i\in\mathbb{N}}$ and each $p \in \mathbb{N}$, we have

$$|\lambda_p|\, \|f_p\| \leq C\, \|\sum_i \lambda_i f_i\|.$$

Proposition 4.2. *Let X be a Banach space with a Schauder basis $(e_i)_{i\in\mathbb{N}}$, and let $T \in \mathcal{L}(X)$. For each integer $n \geq 1$, set $\theta_n := \limsup_{i\to\infty} \frac{\|T^n(e_i)\|}{\|e_i\|}$. Assume the following properties hold true.*

(a) *All sequences $(T^n(e_i))_{i\in\mathbb{N}}$ are semi-basic, with uniformly bounded constants.*
(b) *For each increasing sequence of natural numbers $(p_n)_{n\geq 1}$, the series $\sum \frac{1}{\theta_n} e_{p_n}$ is convergent.*

Then $X \setminus HC(T)$ is not Haar-null.

Proof: Replacing e_i by $\frac{e_i}{\|e_i\|}$, we may assume that the Schauder basis (e_i) is normalized. It is enough to show that the set

$$F = \{x;\, \forall\, n \in \mathbb{N}\ \|T^n(x)\| \geq 1\}$$

is not Haar-null. We show that F contains a translate of each compact subset of X. If $K \subset X$ is compact, then the sequence of coordinate functionals (e_i^*)

tends to 0 uniformly on K, because $\inf_i \|e_i\| > 0$. Writing x_i instead of $\langle e_i^*, x\rangle$, it follows that one can choose an increasing sequence of integers $(p_n)_{n\geq 1}$ such that

$$\begin{cases} \forall\, x \in K \quad |x_{p_n}| \leq \frac{1}{\theta_n} \\ \|T^n(e_{p_n})\| \geq \frac{1}{2}\theta_n. \end{cases}$$

Now, put $z = \sum_1^\infty \frac{2}{\theta_n} e_{p_n}$. For all $x \in K$ and all $n \geq 1$, we have

$$\begin{aligned} \|T^n(x+z)\| &= \left\| \sum_{i=0}^{\infty} (z_i + x_i)\, T^n(e_i) \right\| \\ &\geq C^{-1}\, |z_{p_n} + x_{p_n}|\, \|T^n(e_{p_n})\| \\ &\geq \frac{\theta_n}{2C} \left(\frac{2}{\theta_n} - \frac{1}{\theta_n} \right) = \frac{1}{2C}, \end{aligned}$$

where C is a constant independent of n and K. It follows that $K + z \subset (2C)^{-1} F$. Since K is an arbitrary compact subset of X, this concludes the proof. □

From 4.2, we immediately get the following result, which says that weighted shifts with "large" weights are not Haar-null hypercyclic.

Corollary 4.3. *Let T be a weighted backward shift on $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ ($1 \leq p < \infty$), with weight sequence $(w_n)_{n\geq 1}$. For each integer $n \geq 1$, set $\theta_n := \limsup_{i\to\infty} \theta_{ni}$, where $\theta_{ni} = \prod_{i-n<j\leq i} w_j$. If the sequence $(1/\theta_n)_{n\geq 1}$ defines an element of X (i.e. if the series $\sum \frac{1}{\theta_{i+1}} e_i$ is convergent in X), then T is not Haar-null hypercyclic. This holds in particular if* $\inf_n w_n > 1$.

Remark. The hypothesis in 4.3 is stronger than the corresponding one in 2.2. Indeed, choosing some increasing sequence of integers $(i_n)_{n\geq 1}$ with $\theta_{ni_n} \geq \frac{1}{2}\theta_n$ for all n and setting $x := \sum_1^\infty \frac{1}{\theta_{ni_n}} e_{i_n}$, we have $\|T^n(x)\| \geq 1$ for each positive integer n.

We now turn to operators on $\mathcal{H}(\mathbb{C})$ which commute with translations. By a result of Godefroy and Shapiro ([5]), these are exactly the operators of the form $T = \Phi(D)$, where D is the derivation operator and Φ is an entire function of exponential type. Moreover, we recall that such an operator is always hypercyclic, unless it is a multiple of the identity ([5]).

In what follows, we denote by $c_k(f)$, $k \in \mathbb{N}$, the Taylor coefficients of a function $f \in \mathcal{H}(\mathbb{C})$. Let $\mathfrak{C}$ be the class of all functions $\Phi : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:

- Φ is an entire function of exponential type;
- for all $k, n \in \mathbb{N}$, one can write $c_k(\Phi^n) = a_k\, b_n\, p_{nk}$, where $p_{nk} \geq 0$.

Clearly, the family $\mathfrak{E}$ contains all entire functions of exponential type with non-negative coefficients, and all exponential functions $e^{\alpha z}$, $\alpha \in \mathbb{C}$. More generally, it is easily checked that $\mathfrak{E}$ contains all entire functions of exponential type Φ such that $c_k(\Phi) \in \alpha^k \mathbb{R}^+$, for all $k \in \mathbb{N}$ and some fixed complex number α.

Proposition 4.4. *Let T be an operator on $X = \mathcal{H}(\mathbb{C})$ of the form $T = \Phi(D)$, where D is the derivation operator and $\Phi \in \mathfrak{E}$. Assume there exists at least one T-orbit whose closure does not contain 0. Then $X \setminus HC(T)$ is not Haar-null.*

In particular, we get

Corollary 4.5. *If T is the derivation operator or a translation operator on $\mathcal{H}(\mathbb{C})$, then T is not Haar-null hypercyclic.*

Proof: In both cases, the operator T has the required form, and there exists a function $f \in \mathcal{H}(\mathbb{C})$ whose orbit stays away from 0: if T is the derivation operator, one may take $f(z) = e^z$, and if T is a translation operator $f = \mathbf{1}$. □

The proof of 4.4 relies on the following lemma.

Lemma 4.6. *Let T be as in 4.4. Then there exists a sequence $(a_n) \subset \mathbb{C}$ such that the following property holds true: for each compact set $\mathcal{K} \subset X$, one can find a single function $\varphi \in X$ such that*

(i) $\forall n \in \mathbb{N}\ T^n\varphi(0) \in \mathbb{R}^+ a_n$;
(ii) $\forall f \in \mathcal{K}\ \forall n \in \mathbb{N}\ |T^n f(0)| \leq |T^n \varphi(0)|$.

Proof: Write $c_k(\Phi^n) = a_n\, b_k\, p_{nk}$, with $p_{nk} \geq 0$. We show that (a_n) does the job. Let $\mathcal{K}$ be a compact subset of $\mathcal{H}(\mathbb{C})$, and for each $k \in \mathbb{N}$, put

$$c_k = \sup\{|c_k(f)|;\ f \in \mathcal{K}\}.$$

By Cauchy's inequalities, we have $\lim_{k\to\infty} c_k^{1/k} = 0$. Thus, there exists an entire function φ such that $b_k\, c_k(\varphi) = |b_k|\, c_k$ for all $k \in \mathbb{N}$. Since $T^n f(0) = [\Phi^n(D) f](0) = a_n \sum_k p_{nk}\, k!\, b_k\, c_k(f)$ for each $n \in \mathbb{N}$ and all $f \in X$, this function φ clearly works. □

Proof of 4.4: We fix a sequence (a_n) satisfying the conclusion of the previous lemma. By assumption, there exist some function $f_0 \in X$ and some neighbourhood $\mathcal{U}$ of 0 in X such that $T^n f_0 \notin \mathcal{U}$ for all $n \in \mathbb{N}$. We may assume that $\mathcal{U}$ has the form $\{u \in X;\ \sup_{K_0} |u(z)| < \varepsilon_0\}$ for some compact set $K_0 \subset \mathbb{C}$ and some $\varepsilon_0 > 0$; and replacing f_0 by f_0/ε_0, we may assume that $\varepsilon_0 = 1$. Thus, we have at hand some compact set $K_0 \subset \mathbb{C}$ and some $f_0 \in X$ such that $\sup\{|T^n f_0(z)|;\ z \in K_0\} \geq 1$ for all $n \in \mathbb{N}$. Since T commutes with all translation operators, this means that $\sup\{|T^n f(0)|;\ f \in \mathcal{K}_0\} \geq 1$ for all n, where $\mathcal{K}_0 = \{\tau_z f_0;\ z \in K_0\}$. Since $\mathcal{K}_0$ is a compact subset of X, one can apply Lemma 4.6 to get $\varphi \in X$ such that

$$\forall n \in \mathbb{N}\ \ T^n\varphi(0) \in \mathbb{R}^+ a_n \ \text{ and } \ |T^n\varphi(0)| \geq 1.$$

Now, let $\mathcal{K}$ be any compact subset of X. By Lemma 4.6, one can find $\psi \in X$ such that $T^n\psi(0) \in \mathbb{R}^+ a_n$ and $|T^n f(0)| \leq |T^n\psi(0)|$, for all $f \in \mathcal{K}$ and each $n \in \mathbb{N}$. Putting $h = \varphi + \psi$, we have $|T^n(h)(0)| = |T^n\varphi(0)| + |T^n\psi(0)| \geq 1 + |T^n\psi(0)|$ for each $n \in \mathbb{N}$, hence $|T^n(f+h)(0)| \geq 1$ for all $f \in \mathcal{K}$ and each $n \in \mathbb{N}$. In particular, it follows that $\mathcal{K} + h \subset X \setminus HC(T)$. Thus, we have proved that $X \setminus HC(T)$ contains a translate of each compact subset of X. $\square$

From the above propositions, the following questions obviously come to mind.

- Does there exist a weighted backward shift on $\ell^2(\mathbb{N})$ which is Haar-null hypercyclic?
- Does there exist a nontrivial operator on $\mathcal{H}(\mathbb{C})$ commuting with translations which is Haar-null hypercyclic?

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(*Received* March 15, 2007, *revised* October 25, 2007)