AB-compacta

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Abstract. We define a compactum X to be AB-compact if the *cofinality* of the character $\chi(x, Y)$ is countable whenever $x \in Y$ and $Y \subset X$. It is a natural open question if every AB-compactum is necessarily first countable.

We strengthen several results from [Arhangel'skii and Buzyakova, *Convergence in compacta and linear Lindelöfness*, Comment. Math. Univ. Carolin. **39** (1998), no. 1, 159–166] by proving the following results.

(1) Every AB-compactum is countably tight.
(2) If $\mathfrak{p} = \mathfrak{c}$ then every AB-compactum is Frèchet-Urysohn.
(3) If $\mathfrak{c} < \aleph_\omega$ then every AB-compactum is first countable.
(4) The cardinality of any AB-compactum is at most $2^{<\mathfrak{c}}$.

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1. Introduction

As usual, a compactum is an infinite compact Hausdorff space. Let us say that $x \in X$ is a K-point of X if for no *uncountable* regular cardinal ϱ is there a ϱ-sequence in $X \setminus \{x\}$ converging to x. Clearly, any point of first countability of X is a K-point of X. On the other hand, Kunen constructed in ZFC compacta with K-points that are *not* points of first countability. In fact, the methods of [6] and [7] can be used to provide K-points in compacta of character λ for any singular cardinal λ of countable cofinality. This result of Kunen answered a problem that had been first raised in [2] and then in [1].

It is well-known that if x is a non-isolated point of a compactum X of character $\chi(x, X) = \kappa(\geq \omega)$ then there is a κ-sequence of points of X distinct from x that converges to x. So, if $x \in X$ is a K-point then for every closed subspace $F \subset X$ with $x \in F$ we must have $\mathrm{cf}(\chi(x, F)) \leq \omega$. In fact, since $x \in Y \subset X$ implies $\chi(x, Y) = \chi(x, \overline{Y})$, we even have $\mathrm{cf}(\chi(x, Y)) \leq \omega$ for all subspaces Y of X containing x. Now, we say that x is an AB-point of X if it satisfies this

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latter condition. Moreover, an AB-compactum is one in which every point is an AB-point.

Again, it is obvious that first countable compacta are AB and the natural question if the converse is also true can be raised. Arhangel'skii and Buzyakova have shown in [1] that under CH the answer to this question is affirmative if all points of X are K-points, i.e. for K-compacta. Contrasting this with Kunen's above result shows that the situation with the "global" question is quite different from the "local" one. In [1] (see also [8]) some weaker questions concerning K-compacta were also raised (but using different terminology): Are they of cardinality $\leq \mathfrak{c}$? Are they sequential? The aim of this note is to give some partial answers to all of these questions that are significantly stronger than those given in [1].

Our results are stronger than those of Arhangel'skii and Buzyakova on one hand because they apply to the class all AB-compacta that is (potentially) wider than that of K-compacta. Moreover, the assumptions of our results are strictly weaker than the ones used in [1]. But before turning to our results let us formulate the following intriguing questions that we could not answer.

Problem 1. *Is there an AB-point of some compactum that is not a K-point? Is there an AB-compactum that is not a K-compactum?*

Note that if, as we conjecture, all AB-compacta are first countable then the answer to the second question is negative.

Finally, let us note that the AB-property makes sense for arbitrary topological spaces but it is easy to find (non-compact) ones that are not first countable. Indeed, consider first Kunen's example of a compactum X having a K-point x of character $\lambda > \omega$. Then the space $\widetilde{X}$ obtained from X by isolating all points of $X \backslash \{x\}$ is clearly a (non-compact) AB-space with $\chi(x, \widetilde{X}) = \lambda$.

2. The results

We start with a result that, on one hand, may be considered as a step towards verifying our above conjecture and, on the other hand, will play a great role in establishing our other results.

Theorem 2. *Every AB-compactum is countably tight.*

Proof: Assume that X is a compactum with $t(X) > \omega$. Then by the main result of [5] there is a free sequence of length ω_1 in X, say $Y = \{x_\alpha : \alpha < \omega_1\}$, converging to some point $x \in X$. But then we have

$$\overline{Y} \backslash \{x\} = \bigcup \{ \overline{\{x_\beta : \beta < \alpha\}} : \alpha < \omega_1 \},$$

hence $\chi(x, \overline{Y}) = \omega_1$, which shows that X is not AB-compact. □

As we mentioned in the introduction, it was explicitly asked in [1] if K-compacta are sequential; moreover an affirmative answer was given to this question under MA. Our next result is a strengthening of this. We note that the assumption $\mathfrak{p} = \mathfrak{c}$ is equivalent to MA (σ-centered) and so is strictly weaker than MA.

Theorem 3. *If $\mathfrak{p} = \mathfrak{c}$ holds then every AB-compactum X is Frèchet-Urysohn.*

PROOF: Assume that $x \in \overline{A}$ with $A \subset X$. Then, by Theorem 2, there is a countable set $S \subset A$ with $x \in \overline{S}$. But then $\chi(x, \overline{S}) \leq \mathfrak{c}$ and so the AB-property and $\operatorname{cf}(\mathfrak{c}) > \omega$ imply even $\chi(x, \overline{S}) < \mathfrak{c}$. However, in this case $\mathfrak{p} = \mathfrak{c}$ is known to imply that x is the limit of an ω-sequence of points from $S \setminus \{x\}$. □

It is well-known, see e.g. Corollary 3.4 of [3], that first countability of any compactum X is decided by its subspaces of cardinality ω_1. More precisely: If all subspaces of X of size ω_1 of X are first countable then so is X. We shall need below a strengthening of this that we now turn to.

In fact, we first present a reflection result concerning the pseudo-character of a point in a T_3-space. This may be of some independent interest.

Theorem 4. *Let X be a T_3-space and κ be an uncountable regular cardinal such that there is no free sequence of length κ in X. If $p \in X$ has pseudocharacter $\psi(p, X) \geq \kappa$ then either there is a discrete subspace $D \subset X$ with $|D| < \kappa$ such that $\psi(p, \overline{D}) \geq \kappa$ or there is a discrete $E \subset X$ with $|E| = \kappa$ that converges to p.*

PROOF: By transfinite recursion on $\alpha < \kappa$, we define a sequence of points $x_\alpha \in X \setminus \{p\}$ and a decreasing sequence of closed $G_{<\kappa}$-sets H_α containing p with the properties $x_\alpha \in H_\alpha$ and $\overline{D_\alpha} \cap H_\alpha \subset \{p\}$, where $D_\alpha = \{x_\beta : \beta < \alpha\}$. (A $G_{<\kappa}$-set H is one that is the intersection of fewer than κ open sets, i.e. satisfies $\psi(H, X) < \kappa$.)

Now assume that $\alpha < \kappa$ and both x_β and H_β have been appropriately defined for all $\beta < \alpha$. Note that then D_α is a free sequence in the subspace $X \setminus \{p\}$ and hence is discrete. So if we have $p \in \overline{D_\alpha}$ and $\psi(p, \overline{D_\alpha}) \geq \kappa$ then we are done and we simply stop the recursion.

Otherwise we have either $p \notin \overline{D_\alpha}$ or $\psi(p, \overline{D_\alpha}) < \kappa$ and in either case we may obviously find a closed $G_{<\kappa}$-set H_α satisfying

$$p \in H_\alpha \subset \cap\{H_\beta : \beta < \alpha\} \text{ and } \overline{D_\alpha} \cap H_\alpha \subset \{p\}.$$

(We have to use here both the regularity of the cardinal κ and the regularity of the space X.) Note that then $H_\alpha \neq \{p\}$ since we have $\psi(p, X) \geq \kappa$, consequently a point $x_\alpha \in H_\alpha \setminus \{p\}$ can be chosen.

So if we do not stop for any $\alpha < \kappa$ then we obtain in this way the κ-sequence $E = \{x_\alpha : \alpha < \kappa\}$, which is a free sequence in $X \setminus \{p\}$ and therefore discrete. It remains to show that E converges to p. Assume, arguing indirectly, that for some open neighbourhood U of p we have $|E \setminus U| = \kappa$. Clearly, then $E \setminus U$, as a subsequence of E, would be a free sequence in X, contradicting our assumption. □

Note that in a compactum the character and the pseudocharacter of any point coincide. Therefore, in this case we get from Theorem 4 the following corollary.

Corollary 5. *Assume that some point p of a compactum X has character at least κ where κ is an uncountable regular cardinal. If there is no convergent* discrete *κ-sequence in X then we have $\chi(p, \overline{D}) \geq \kappa$ for some* discrete *$D \subset X$ with $|D| < \kappa$.*

Proof: By the main result of [5], if there is no convergent discrete (even free) κ-sequence in X then there is no free κ-sequence in X, either. Consequently, we may apply Theorem 4 and from there only the first alternative may hold. □

Let us say that a space is *simple* iff it has exactly one non-isolated point. The following is then an immediate consequence of Corollary 5.

Corollary 6. *If κ is an uncountable regular cardinal and all simple subspaces of size $\leq \kappa$ of a compactum X have character $< \kappa$ then all points of X have character $< \kappa$. In particular, if all simple subspaces of size $\leq \omega_1$ of a compactum X are first countable then so is X.*

Our next result shows that for an AB-compactum first countability is even decided by its countable simple subspaces.

Theorem 7. *Let X be an AB-compactum in which the closure of any countable discrete subspace is first countable, or equivalently: all countable simple subspaces of X are first countable. Then X is first countable, as well.*

Proof: By Corollary 6 it suffices to show that every simple subspace Y of X with $|Y| = \omega_1$ is first countable. Consider an ω_1-type enumeration $\{x_\alpha : \alpha < \omega_1\}$ of Y where x_0 is the unique non-isolated point of Y. Then, putting $Y_\alpha = \{x_\beta : \beta < \alpha\}$, for every $0 < \alpha < \omega_1$ we have by our assumption that

$$\chi(x_0, Y_\alpha) = \chi(x_0, \overline{Y_\alpha}) \leq \omega.$$

Note that, as X is countably tight, we also have $\overline{Y} = \bigcup_{\alpha<\omega_1} \overline{Y_\alpha}$, consequently

$$\psi(x_0, \overline{Y}) = \chi(x_0, \overline{Y}) \leq \omega_1.$$

But the AB-property then implies $\chi(x_0, \overline{Y}) = \chi(x_0, Y) = \omega$. □

Recall that a compactum is called weakly ω-monolithic if the closure of any countable discrete subspace in it is second countable. So it is immediate from Theorem 7 that every weakly ω-monolithic AB-compactum is first countable. For the (potentially smaller) class of K-compacta this weaker result was proved in [1].

Trivially, every point of a countable space has character $\leq \mathfrak{c}$, so another immediate corollary of Theorem 7 is the following.

Corollary 8. *If $\mathfrak{c} < \aleph_\omega$ then every AB-compactum is first countable.*

Perhaps the main result of this paper is the following which, analogously to Theorem 7, states that countable subspaces (or more precisely their closures) control the size of any AB-compactum.

Theorem 9. *For an arbitrary AB-compactum X we have*

$$|X| \leq \big(\sup\{|\overline{S}| : S \in [X]^\omega\}\big)^\omega.$$

Proof: Set

$$\mu = \big(\sup\{|\overline{S}| : S \in [X]^\omega\}\big)^\omega$$

and assume, arguing indirectly, that $|X| > \mu$. Since X is countably tight and $\mu = \mu^\omega$ implies $\mu^+ = (\mu^+)^\omega$, it is easy to show that for any set $A \subset X$ with $|A| = \mu^+$ we have $|\overline{A}| = \mu^+$ as well. Consequently, we may assume without any loss of generality that $|X| = \mu^+$.

Now, consider an ω-closed elementary submodel $\mathcal{M}$ of a suitably chosen universe $H(\lambda)$ with $|\mathcal{M}| = \mu$, where the regular cardinal λ is sufficiently large, moreover we have both $X \in \mathcal{M}$ and $\mu \subset \mathcal{M}$.

We claim that then $X \cap \mathcal{M}$ is closed in X. Indeed, for every countable set $S \in [X \cap \mathcal{M}]^\omega$ we have $S \in \mathcal{M}$ as $\mathcal{M}$ is ω-closed and then $\overline{S} \in \mathcal{M}$ as well. But we have $|\overline{S}| \leq \mu$ and therefore $\overline{S} \subset \mathcal{M}$.

Next, observe that for every point $x \in X$ we trivially have $\chi(x, X) \leq \mu^+$ and so by the AB-property even $\chi(x, X) < \mu$, for both μ and μ^+ have uncountable cofinality. Since $\mu \subset \mathcal{M}$, this in turn implies that for every point $x \in X \cap \mathcal{M}$ there is a neighbourhood base $\mathcal{V}_x \in \mathcal{M}$ such that $|\mathcal{V}_x| < \mu$ and hence $\mathcal{V}_x \subset \mathcal{M}$.

From this it is a standard procedure to show that $X \subset \mathcal{M}$ that is clearly a contradiction. Indeed, assume that $y \in X \setminus \mathcal{M}$. Then for each point $x \in X \cap \mathcal{M}$ there is a neighbourhood $V_x \in \mathcal{V}_x \subset \mathcal{M}$ with $y \notin V_x$. As $X \cap \mathcal{M}$ is compact, there are finitely many points $x_1, \ldots, x_n \in X \cap \mathcal{M}$ such that $X \cap \mathcal{M} \subset V_{x_1} \cup \cdots \cup V_{x_n}$, contradicting that

$$y \notin V_{x_1} \cup \cdots \cup V_{x_n}.$$

□

Now, if X is a separable AB-compactum then $w(X) \leq \mathfrak{c}$ and consequently $\chi(X) \leq \mathfrak{c}$. But $\mathrm{cf}(\mathfrak{c}) > \omega$ implies that we actually have $\chi(x, X) < \mathfrak{c}$ for all $x \in X$. It follows then that $|X| \leq 2^{<\mathfrak{c}} = \sum_{\kappa<\mathfrak{c}} 2^\kappa$, see e.g. 2.5 of [4]. Note also that $(2^{<\mathfrak{c}})^\omega = 2^{<\mathfrak{c}}$, as can be easily checked.

As a consequence we obtain the following result that is remarkable for in [1] no upper bound was given even for the size of K-compacta.

Corollary 10. *Every AB-compactum has cardinality at most* $2^{<\mathfrak{c}}$. *This, of course is optimal if* $2^{<\mathfrak{c}} = \mathfrak{c}$, *in particular if* $\mathfrak{t} = \mathfrak{c}$ *holds* (*which is a consequence e.g. of MA*).

Another immediate consequence of Theorem 9 that is worth noting reads as follows.

Corollary 11. *If every separable subspace of an AB-compactum* X *has cardinality at most* $\mathfrak{c}$ *then* $|X| \leq \mathfrak{c}$ *as well.*

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