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On the lattices of quasivarieties of differential groupoids

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Abstract. The main result of Romanowska A., Roszkowska B., *On some groupoid modes*, Demonstratio Math. **20** (1987), no. 1–2, 277–290, provides us with an explicit description of the lattice of varieties of differential groupoids. In the present article, we show that this variety is $\mathcal{Q}$-universal, which means that there is no convenient explicit description for the lattice of quasivarieties of differential groupoids. We also find an example of a subvariety of differential groupoids with a finite number of subquasivarieties.

Keywords: mode, differential groupoid, lattice of subquasivarieties, $\mathcal{Q}$-universal quasivariety

Classification: 08C15, 20N02

Introduction

A *differential groupoid* is a structure with one fundamental binary operation satisfying the identities

(I) $$x \cdot x = x,$$

(E) $$(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t),$$

(D) $$x \cdot (x \cdot y) = x.$$

Let **Dm** denote the variety of differential groupoids.

Many authors use the term *medial* groupoid instead of *entropic*, i.e., satisfying (E), see [3]. Differential groupoids were studied in [5]–[7], where they were called LIR-*groupoids* (*left normal, idempotent, and reductive groupoids*) and a different basis for identities was used. The term *differential groupoid* appeared in [8]. For more information, the reader is referred to the monograph [9].

For $i \geqslant 0$ and $n > 0$, let $\mathbf{D}_{i,n}$ denote the subvariety of **Dm** defined by the identity

(1) $$xy^{i+n} = xy^i,$$

where $xy^k = (\dots((x \cdot \underbrace{y) \cdot y)\dots) \cdot y}_{k \text{ times}}$. The structure of the lattice $\mathrm{L}_{\mathrm{v}}(\mathbf{Dm})$ of subvarieties of **Dm** is described by [6, Theorem 5.3], cf. also [9, Theorem 8.4.14].

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Proposition 1. *Let $\mathbb{N}_c$ denote the lattice of natural numbers with the usual order and let $\mathbb{N}_d$ denote the lattice of positive integers ordered by the divisibility relation.*

Proper subvarieties of **Dm** *form a lattice which is isomorphic to the direct product $\mathbb{N}_c \times \mathbb{N}_d$. Moreover, a pair (i, n) corresponds to the variety $\mathbf{D}_{i,n}$.*

A quasivariety **K** of groupoids is said to be $\mathcal{Q}$-*universal* if, for every quasivariety $\mathbf{K}'$ of structures of finite type, the lattice $\mathsf{L_q}(\mathbf{K}')$ of subquasivarieties of $\mathbf{K}'$ is a homomorphic image of some sublattice of the lattice $\mathsf{L_q}(\mathbf{K})$ of subquasivarieties of **K**. For every $\mathcal{Q}$-universal quasivariety **K**, the lattice $\mathsf{L_q}(\mathbf{K})$ is highly complicated. Namely, $|\mathsf{L_q}(\mathbf{K})| = 2^\omega$; moreover, this lattice satisfies no nontrivial lattice identity and contains a sublattice that is isomorphic to the ideal lattice of a free ω-generated lattice.

In Section 1, we prove that the variety **Dm** is $\mathcal{Q}$-universal. This shows that there is no convenient description for the lattice $\mathsf{L_q}(\mathbf{Dm})$. The following question naturally arises: Which proper subvarieties of differential groupoids are $\mathcal{Q}$-universal? In Section 2, we show that $\mathbf{D}_{1,1}$ is not $\mathcal{Q}$-universal.

1. The variety Dm is $\mathcal{Q}$-universal

We use the standard notation for class operators. Namely, **Q** stands for taking the least quasivariety containing a given class, while $\mathbf{P_s}$, **S**, and **H** stand for formation of subdirect products, subgroupoids, and homomorphic images, respectively. For every class operator **O** and classes **X** and **K**, we denote by $(\mathbf{O}\cap\mathbf{K})(\mathbf{X})$ the class $\mathbf{O}(\mathbf{X}) \cap \mathbf{K}$.

Our proof is based on the following sufficient condition for $\mathcal{Q}$-universality (cf. [2, Theorem 5.4.26]).

Proposition 2. *A quasivariety* **K** *of groupoids is $\mathcal{Q}$-universal if there exist a subclass* **B** *of* **K** *and a family $(\mathcal{A}_i)_{i<\omega}$ of finite groupoids in* **B** *such that the following conditions are satisfied.*

- (Q1) *For every $n < \omega$ and* **B**-*congruences θ and θ' on $\mathcal{A}_n$, if $\mathcal{A}_n/\theta'$ is embeddable into $\mathcal{A}_n/\theta$ then either $\theta = \theta'$ or $\mathcal{A}_n/\theta'$ is a trivial groupoid.*
- (Q2) *For every $n < \omega$, the meet semilattice L_n of* **B**-*congruences on $\mathcal{A}_n$ is a subsemilattice of the meet semilattice of congruences on $\mathcal{A}_n$. Moreover, the meet semilattice of subsets of an n-element set is embeddable into L_n.*
- (Q3) *If $m \neq n$ then the class $\mathbf{A}_n \cap \mathbf{S}(\mathbf{A}_m)$, where $\mathbf{A}_n = \mathbf{H}(\mathcal{A}_n) \cap \mathbf{B}$, consists of trivial groupoids only.*
- (Q4) *For every $\mathbf{X} \subseteq \mathbf{K}$ and $n < \omega$, we have*

$$\mathbf{Q}(\mathbf{X}) \cap \mathbf{A}_n = (\mathbf{P_s} \cap \mathbf{A}_n)(\mathbf{S} \cap \mathbf{A}_n)(\mathbf{X}).$$

For more information on $\mathcal{Q}$-universal quasivarieties, the reader is referred to [1, Section 5].

Recall that a groupoid G is called a *left zero band* if G satisfies the identity $x \cdot y = x$, i.e., if $G \in \mathbf{D}_{0,1}$. We say that a groupoid G is an **Lz**-**Lz**-*sum* (of left zero bands G_i over a left zero band I) *satisfying the left normal law* if there exists a partition $G = \bigcup_{i \in I} G_i$ and, for every pair $(i,j) \in I^2$, there exists a map $h_{ij} : G_i \to G_i$ such that the following conditions are satisfied:

(i) h_{ii} is the identity map for every $i \in I$,
(ii) $h_{ij}(h_{ik}(x)) = h_{ik}(h_{ij}(x))$ for all $i,j,k \in I$ and $x \in G_i$,
(iii) $a_i \cdot a_j = h_{ij}(a_i)$ for all $i,j \in I$, $a_i \in G_i$, and $a_j \in G_j$.

The structure of differential groupoids was completely described in [6, Section 2], cf. also [4, 5, 7]. Namely, we have $G \in \mathbf{Dm}$ if and only if G is an **Lz**-**Lz**-sum satisfying the left normal law.

Let $\mathcal{C}_0$ denote the trivial groupoid whose universe is $\{\infty\}$. For every $n > 0$, let $\mathcal{C}_n$ denote the **Lz**-**Lz**-sum of $G_1 = \{0, 1, \dots, n-1\}$ and $G_2 = \{\infty\}$, where $h_{12}(k) \equiv k+1 \pmod{n}$ and h_{21} is the identity map. We have $\mathcal{C}_n \in \mathbf{Dm}$ for each $n \geqslant 0$.

We describe congruences on the constructed groupoids. Let m divide n. For every $k < n$, let r_k denote the remainder in the division of k by m. It is easy to see that the map defined by the rule

$$\infty \mapsto \infty, \quad k \mapsto r_k$$

is a homomorphism from $\mathcal{C}_n$ onto $\mathcal{C}_m$. Let θ_m denote the kernel of this homomorphism.

Lemma 3. *Let $n > 0$ and let θ be a congruence on $\mathcal{C}_n$. Then either $\mathcal{C}_n/\theta$ is a trivial groupoid or $\theta = \theta_m$ for some divisor m of n.*

Proof: If $(\infty, k) \in \theta$, where $0 \leqslant k < n$, then, as in [4, p. 378], we find that $\mathcal{C}_n/\theta$ is a trivial groupoid. If $(\infty, k) \notin \theta$ for all k with $0 \leqslant k < n$ then $\theta \leqslant \theta_1$. By [7, Propositions 2.2 and 2.5], we conclude that the restriction of θ to G_1 is a congruence on a cyclic abelian group of order n. Hence, $\theta = \theta_m$ for some m dividing n. $\square$

Let $\mathbf{B}$ denote the subclass of $\mathbf{Dm}$ consisting of trivial groupoids and differential groupoids that are not left zero bands. We have $\mathcal{C}_n \in \mathbf{B}$ if and only if $n \neq 1$.

Let $\mathbb{P}$ denote the set of prime numbers. Consider a partition $\mathbb{P} = \bigcup_{i<\omega} P_i$ with $|P_i| = i + 1$ for all $i < \omega$ and $P_i \cap P_j = \emptyset$ for all $i \neq j$. Let $k_i = \prod_{p \in P_i} p$. Put $\mathcal{A}_i = \mathcal{C}_{k_i}$ for $i < \omega$.

Theorem 4. *The class $\mathbf{B}$ and the family $(\mathcal{A}_i)_{i<\omega}$ satisfy conditions* (Q1)–(Q4) *of Proposition* 2. *Hence,* $\mathbf{Dm}$ *is a $\mathcal{Q}$-universal quasivariety.*

Proof: We have $(\mathcal{A}_i)_{i<\omega} \subseteq \mathbf{B}$. It is easy to see that, for $i, j < \omega$, the groupoid $\mathcal{C}_i$ is embeddable into the groupoid $\mathcal{C}_j$ if and only if $i = j$. By Lemma 3, this immediately implies (Q1) and (Q3). Since L_i is obtained from the meet semilattice of congruences on $\mathcal{A}_i$ by removing the congruence θ_1, we also obtain (Q2).

We prove (Q4). Let $\mathbf{X} \subseteq \mathbf{Dm}$ and let $n < \omega$. The inclusion $\mathbf{Q}(\mathbf{X}) \cap \mathbf{A}_n \supseteq (\mathbf{P_s} \cap \mathbf{A}_n)(\mathbf{S} \cap \mathbf{A}_n)(\mathbf{X})$ is obvious.

Consider a nontrivial groupoid $\mathcal{B} \in \mathbf{Q}(\mathbf{X}) \cap \mathbf{A}_n$. By [2, Corollary 2.3.4], we have $\mathbf{Q}(\mathbf{X}) = \mathbf{SP_uP}(\mathbf{X})$, where $\mathbf{P}$ and $\mathbf{P_u}$ are the class operators for formation of direct products and ultraproducts. Hence, there exists a family $(\mathcal{B}_i)_{i\in I}$ of groupoids and an ultrafilter U over I such that $\mathcal{B}$ is a subgroupoid of the ultraproduct $\prod_{i\in I} \mathcal{B}_i/U$. Moreover, each $\mathcal{B}_i$ is the direct product of a family $(\mathcal{B}_{ij})_{j\in I_i}$ of groupoids in $\mathbf{X}$.

Since $\mathcal{B}$ is a homomorphic image of the finite groupoid $\mathcal{A}_n$, we conclude that $\mathcal{B}$ is a finite groupoid too. There exists a first-order sentence φ such that, for every groupoid $\mathcal{X}$, the following two conditions are equivalent: (a) $\mathcal{X}$ satisfies φ; (b) $\mathcal{B}$ is embeddable into $\mathcal{X}$. In particular, $\prod_{i\in I} \mathcal{B}_i/U$ satisfies φ. By the Łoś Theorem, there exists an $i \in I$ such that $\mathcal{B}_i$ satisfies φ. Hence, there exists an embedding $\alpha : \mathcal{B} \to \mathcal{B}_i$.

Let $\pi_j : \prod_{j\in I_i} \mathcal{B}_{ij} \to \mathcal{B}_{ij}$ be the jth projection map. Denote by ψ_j the composition $\pi_j \circ \alpha$ of homomorphisms. For every $j \in I_i$, let $\mathcal{G}_j$ be the homomorphic image of $\mathcal{B}$ with respect to ψ_j. Then $\mathcal{G}_j$ is a subgroupoid of $\mathcal{B}_{ij}$ and a homomorphic image of $\mathcal{A}_n$.

We show that $\mathcal{B}$ is a subdirect product of the family $(\mathcal{G}_j)_{j\in I_i}$, i.e., if $x, y \in B$ and $x \neq y$ then there exists a $j \in I_i$ such that $\psi_j(x) \neq \psi_j(y)$ (or, which is equivalent, $\bigcap_{j\in I_i} \ker\psi_j$ is the equality relation Δ_B on B). Indeed, since α is an embedding, we have $\alpha(x) \neq \alpha(y)$. Since each π_j, $j \in I_i$, is a projection, we have $\psi_j(x) = \pi_j(\alpha(x)) \neq \pi_j(\alpha(y)) = \psi_j(y)$ for at least one $j \in I_i$.

Let $J = \{j \in I_i : \mathcal{G}_j \notin \mathbf{D}_{0,1}\}$. If $J = \emptyset$ then $\mathcal{B}$ is a left zero band, a contradiction. By Lemma 3, we have $\ker\psi_j \subseteq \ker\psi_k$ for all $j \in J$ and $k \in I_i \setminus J$. Hence $\bigcap_{j\in J} \ker\psi_j = \bigcap_{j\in I_i} \ker\psi_j = \Delta_B$. Therefore, $\mathcal{B}$ is a subdirect product of the family $(\mathcal{G}_j)_{j\in J} \subseteq \mathbf{B}$. Consequently, $\mathcal{B} \in (\mathbf{P_s} \cap \mathbf{A}_n)(\mathbf{S} \cap \mathbf{A}_n)(\mathbf{X})$. $\square$

2. The variety $\mathbf{D}_{1,1}$ is not $\mathcal{Q}$-universal

In this section, we find subdirectly irreducible groupoids in $\mathbf{D}_{1,1}$ and show that the lattice $\mathrm{L_q}(\mathbf{D}_{1,1})$ is finite.

For $i = n = 1$, identity (1) has the following form:

$$xy^2 = xy. \tag{1'}$$

Define a relation $\leqslant$ on G as follows:

$$a \leqslant b \iff b = ax_1 \ldots x_n \text{ for some } x_1, \ldots, x_n \in G,$$

where, $ax_1 \ldots x_n = (\ldots((a \cdot x_1) \cdot x_2) \ldots \cdot x_n)$. Using the left normal law

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \tag{L}$$

(see [9, Proposition 5.6.2]) and ($1'$), it is easy to check that the relation $\leqslant$ is a partial order on G and

$$x \leqslant y \ \text{ implies } \ xz \leqslant yz \tag{2}$$

for all $x, y, z \in G$.

Assume that G is a finite groupoid. Let M denote the set of maximal elements with respect to the order $\leqslant$ and, for every $m \in M$, let G_m denote the order ideal generated by m (or the *orbit* of m). It is easy to see that $m_1 \neq m_2$ implies that $G_{m_1} \cap G_{m_2} = \emptyset$.

As in [9, p. 537] (cf. also [5]), let β denote the congruence on G defined as follows:

$$(a, b) \in \beta \iff a, b \in G_m \ \text{ for some } \ m \in M.$$

Then G is an **Lz**-**Lz**-sum of its β-orbits.

Let $\mathcal{G}_0$ denote the two-element left zero band with the universe $\{0, 1\}$. Let $\mathcal{G}_1$ denote the **Lz**-**Lz**-sum of β-orbits $\{0, 1\}$ and $\{2\}$, where $0 < 1$, i.e., $0 \cdot 2 = 1$ and $x \cdot y = x$ if the pair (x, y) is different from $(0, 2)$.

Theorem 5. *A finite groupoid G is subdirectly irreducible in $\mathbf{D}_{1,1}$ if and only if G is isomorphic to either $\mathcal{G}_0$ or $\mathcal{G}_1$.*

PROOF: It is easy to see that $\mathcal{G}_0$ and $\mathcal{G}_1$ are subdirectly irreducible in $\mathbf{D}_{1,1}$ because 0 and 1 cannot be separated by *proper* homomorphisms, i.e., homomorphisms that are not isomorphisms.

We prove the "only if" part.

(i) Let $G \in \mathbf{D}_{1,1}$ and let $J = \{m \in M : |G_m| > 1\}$. Notice that, for every groupoid G that is subdirectly irreducible in $\mathbf{D}_{1,1}$, we have $|J| \leqslant 1$.

Indeed, let there exist $m_1, m_2 \in M$ such that $m_1 \neq m_2$ and $|G_{m_1}|, |G_{m_2}| > 1$. For $j = 1, 2$, consider the map ψ_j defined by the rule

$$\psi_j(x) = \begin{cases} x, & x \notin G_{m_j}, \\ m_j, & x \in G_{m_j}. \end{cases} \tag{3}$$

Since m_j is a maximal element and G_{m_j} is a non-singleton orbit, ψ_j is a proper homomorphism, $j = 1, 2$. It is easy to see that $\ker \psi_1 \cap \ker \psi_2$ is the equality relation Δ_G, i.e., the homomorphisms ψ_1 and ψ_2 separate points of G. Therefore, if $|J| > 1$ then G is not subdirectly irreducible.

(ii) If $J = \emptyset$ then $G \in \mathbf{D}_{0,1}$, i.e., G is a left zero band. Each subdirectly irreducible groupoid in $\mathbf{D}_{0,1}$ is isomorphic to $\mathcal{G}_0$. In the sequel, we only consider subdirectly irreducible groupoids in $\mathbf{D}_{1,1}$ that are not left zero bands and assume that $|J| = 1$, i.e.,

$$G = \bigcup_{1 \leqslant i \leqslant n} G_i, \quad \text{where } |G_1| > 1 \text{ and } G_i = \{g_i\} \text{ for } i > 1.$$

(iii) Let $x, y \in G$ and let $x \neq y$. We show that x and y are separated by homomorphisms to $\mathcal{G}_1$.

If either $x = g_i$ or $y = g_i$, $2 \leqslant i \leqslant n$, then it suffices to consider the homomorphism ψ_1 from (3).

Assume that $x, y \in G_1$ and $y \nleqslant x$. Define a map φ_{xy} as follows:

$$\varphi_{xy}(a) = \begin{cases} 0, & a \leqslant x, \\ 1, & \text{either } \ a \in G_1 \text{ with } a \nleqslant x \ \text{ or } \ a = g_k \ \text{ with } \ xg_k = x, \\ 2, & a = g_k \ \text{ with } \ xg_k \neq x. \end{cases}$$

It is clear that φ_{xy} is a map from G onto $\mathcal{G}_1$ and $\varphi_{xy}(x) = 0 \neq 1 = \varphi_{xy}(y)$. It remains to prove that φ_{xy} is a homomorphism.

We show that $\varphi_{xy}(ab) = \varphi_{xy}(a)\varphi_{xy}(b)$. Three cases are possible.

(a) Let $\varphi_{xy}(a) = 0$, i.e., let $a \leqslant x$.

If $b \in G_1$ then $ab = a$ and $\varphi_{xy}(a)\varphi_{xy}(b) = 0 \cdot z = 0 = \varphi_{xy}(a) = \varphi_{xy}(ab)$, where $z \in \{0, 1\}$.

If $\varphi_{xy}(b) = 1$ and $b \notin G_1$ then $b = g_i$ with $xg_i = x$. Since $a \leqslant x$, we have $ab = ag_i \leqslant xg_i = x$ by (2). Hence, $\varphi_{xy}(ab) = 0 = 0 \cdot 1 = \varphi_{xy}(a)\varphi_{xy}(b)$.

If $\varphi_{xy}(b) = 2$ then $b = g_i$ with $xg_i \neq x$. Assume that $ab = ag_i \leqslant x$. Since $a \leqslant x$, there exist $y_1, \dots, y_n \in G$ such that $ay_1 \dots y_n = x$. We obtain $xg_i = ay_1 \dots y_n g_i = ag_i y_1 \dots y_n \leqslant xy_1 \dots y_n = x$ by using (L), (2), and (1′). Hence, $xg_i \leqslant x$. By definition, $x \leqslant xg_i$, which implies $x = xg_i$, a contradiction. Thus, $ab \nleqslant x$ and $\varphi_{xy}(ab) = 1 = 0 \cdot 2 = \varphi_{xy}(a)\varphi_{xy}(b)$.

(b) Let $a \in G_1$ and let $a \nleqslant x$.

For every $b \in G$, we have $ab \in G_1$ and $ab \nleqslant x$. Since $1 \cdot z = 1$ in $\mathcal{G}_1$, we obtain $\varphi_{xy}(ab) = 1 = 1 \cdot z = \varphi_{xy}(a) \cdot \varphi_{xy}(b)$ for every $b \in G$.

(c) Let $a = g_i$.

For every $b \in G$, we have $ab = a$. Since $1 \cdot z = 1$ and $2 \cdot z = 2$ in $\mathcal{G}_1$, we obtain $\varphi_{xy}(ab) = t = t \cdot z = \varphi_{xy}(a) \cdot \varphi_{xy}(b)$ for every $b \in G$, where $t \in \{1, 2\}$.

Thus, if $|G| > 3$ then all points of G are separated by proper homomorphisms to $\mathcal{G}_1$; hence, G cannot be subdirectly irreducible in $\mathbf{D}_{1,1}$. □

Lemma 6. *If $G \in \mathbf{D}_{1,1} \setminus \mathbf{D}_{0,1}$ then $\mathcal{G}_1$ is embeddable into G.*

Proof: Since $G \notin \mathbf{D}_{0,1}$, there exist $a, b \in G$ such that $ab \neq a$. Define a map from $\mathcal{G}_1$ into G as follows:

$$0 \mapsto a, \quad 1 \mapsto ab, \quad 2 \mapsto ba.$$

It is easy to see that this is the required embedding. □

Theorem 7. *The lattice* $\mathsf{L_q}(\mathbf{D}_{1,1})$ *is a three-element chain.*

PROOF: Since $\mathbf{D}_{1,1}$ is locally finite and has finitely many finite subdirectly irreducible groupoids, there are no infinite subdirectly irreducible groupoids in $\mathbf{D}_{1,1}$. By the Birkhoff Subdirect Representation Theorem and Theorem 5, $\mathbf{D}_{1,1}$ is the quasivariety generated by $\mathcal{G}_1$. The lattice $\mathsf{L_q}(\mathbf{D}_{0,1})$ is a two-element chain. By Lemma 6, if a subquasivariety $\mathbf{K}$ of $\mathbf{D}_{1,1}$ contains a groupoid G that is not a left zero band then $\mathbf{K} = \mathbf{D}_{1,1}$. □

3. Concluding remarks

We have proven that the variety **Dm** is $\mathcal{Q}$-universal. It is easy to see that the method used in the proof of Theorem 4 does not allow us to prove that some subvariety of the form $\mathbf{D}_{i,n}$ is $\mathcal{Q}$-universal. Indeed, the family $(\mathcal{A}_i)_{i<\omega}$ does not belong to such a subvariety. We have also shown that the variety $\mathbf{D}_{1,1}$ is not $\mathcal{Q}$-universal. The following problem seems to be of an interest: Determine the borderline between simple and $\mathcal{Q}$-universal varieties of differential groupoids.

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