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Vector-valued modular forms associated to linear ordinary differential equations

Min Ho Lee

Abstract. We consider a class of linear ordinary differential equations determined by a modular form of weight one, and construct vector-valued modular forms of weight two by using solutions of such differential equations.

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1. Introduction

Modular forms are complex-valued functions defined on the Poincaré upper half plane $\mathcal{H}$ satisfying a certain transformation formula with respect to an action of a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, and they play a major role in modern number theory. Modular forms have also been studied in connection with problems in many other areas of pure and applied mathematics such as cryptography, coding theory, gauge theory, string theory, and conformal field theory.

Vector-valued modular forms for Γ are functions on $\mathcal{H}$ with values in a finite-dimensional complex vector space satisfying a transformation formula with respect to a representation of the group Γ, and they are related to many topics in number theory. For example, they occur naturally in connection with Jacobi forms (cf. [2]) or the cohomological interpretation of modular forms of Eichler [1] and Shimura [5]. Vector-valued modular forms of weight two can be expressed in terms of derivatives of a modular form by using a method developed by Kuga and Shimura [3], and certain types of such modular forms correspond to usual modular forms of higher weight.

In this paper we consider vector-valued modular forms associated to a certain class of linear ordinary differential equations. Such differential equations are determined by a modular form φ of weight one, and their connections with modular forms as well as with elliptic surfaces were studied by P. Stiller (see e.g. [6]). For example, modular forms of weight higher than two can be expressed in terms of solutions of those differential equations and φ. We construct vector-valued modular forms of weight two by combining this result with the above-mentioned method of Kuga and Shimura.

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2. Vector-valued modular forms

In this section we describe relations between vector-valued meromorphic modular forms for a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ and usual scalar-valued ones. In particular, we review the method of Kuga and Shimura [3] of constructing vector-valued modular forms of weight two by using derivatives of a usual scalar-valued modular form.

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the Poincaré upper half plane on which $\mathrm{SL}(2, \mathbb{R})$ acts by linear fractional transformations, so that we may write

$$\gamma z = \frac{az + b}{cz + d}$$

for $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a Fuchsian group of the first kind, that is, a discrete subgroup such that the quotient space $\Gamma \backslash \mathcal{H}^*$ is compact, where $\mathcal{H}^*$ denotes the union of $\mathcal{H}$ and the set of cusps of Γ (see e.g. [4]). Let $\rho : \Gamma \to \mathrm{GL}(\ell, \mathbb{C})$ be a representation of Γ in the complex vector space $\mathbb{C}^\ell$ for some positive integer ℓ.

Definition 2.1. Let k be an integer, and consider meromorphic functions $f : \mathcal{H} \to \mathbb{C}$ and $\Psi : \mathcal{H} \to \mathbb{C}^\ell$. Then f is a *meromorphic modular form of weight* k *for* Γ and Ψ is a *vector-valued meromorphic modular form of weight* k *for* Γ *with respect to* ρ if they are meromorphic at the cusps and satisfy

$$f(\gamma z) = (cz + d)^k f(z), \quad \Psi(\gamma z) = (cz + d)^k \rho(\gamma) \Psi(z)$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We shall denote by $M_k(\Gamma)$ and $\mathbf{M}_k(\Gamma, \rho)$ the space of modular forms of weight k for Γ and the space of vector-valued modular forms of weight k for Γ with respect to ρ, respectively.

If m is a positive integer, we denote by $\rho_m : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{GL}(m+1, \mathbb{C})$ the m-th symmetric tensor power of the standard representation of $\mathrm{SL}(2, \mathbb{R})$ in $\mathbb{C}^2$. Thus, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$, then we have

$$\begin{aligned} &\rho_m(\gamma)(u^m, u^{m-1}v, \ldots, uv^{m-1}, v^m)^T \\ &\qquad = ((au + bv)^m, (au + bv)^{m-1}(cu + dv), \ldots \\ &\qquad\qquad \ldots, (au + bv)(cu + dv)^{m-1}, (cu + dv)^m)^T \end{aligned}$$

for all $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathbb{C}^2$, where $(\cdot)^T$ denotes the transpose of the row vector $(\cdot)$. By restricting ρ_m to Γ we obtain a representation of Γ in $\mathbb{C}^{m+1}$, which we also denote by ρ_m.

Definition 2.2. Given a positive integer m, we define the matrix-valued function $\widehat{\rho}_m : \mathcal{H} \to \mathrm{GL}(m+1, \mathbb{C})$ on $\mathcal{H}$ associated to ρ_m by

$$\widehat{\rho}_m(z) = \rho_m\left(\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)\right)$$

for all $z \in \mathcal{H}$.

Let α and β be even integers with $\alpha > 0$ and

$$-(\alpha - 2) \leq \beta \leq \alpha + 2. \tag{2.1}$$

We set

$$\delta = \frac{\alpha + 2 - \beta}{2},$$

and for each nonnegative integer $k \leq \delta$ denote by $\eta_{k,\alpha,\beta}$ the rational number defined by

$$\eta_{k,\alpha,\beta} = \begin{cases} 0 & \text{if } k < 1 - \beta; \\ \frac{(k+\alpha-\delta)!}{k!(\beta+k-1)!} & \text{if } k \geq 1 - \beta. \end{cases} \tag{2.2}$$

Given a meromorphic function $f : \mathcal{H} \to \mathbb{C}$, we use its derivatives of various orders as well as the numbers $\eta_{k,\alpha,\beta}$ to define the finite sequence $\{\phi_{\ell,\alpha,\beta}\}_{\ell=0}^{\alpha}$ of functions on $\mathcal{H}$ by

$$\phi_{\ell,\alpha,\beta}(z) = \begin{cases} 0 & \text{if } \ell < \alpha - \delta; \\ \eta_{\ell-\alpha+\delta,\alpha,\beta} f^{(\ell-\alpha+\delta)}(z) & \text{if } \ell \geq \alpha - \delta \end{cases}$$

for $z \in \mathcal{H}$ and $0 \leq \ell \leq \alpha$.

Definition 2.3. We define the vector-valued function $\Phi_f : \mathcal{H} \to \mathbb{C}^{\alpha+1}$ associated to a meromorphic function $f : \mathcal{H} \to \mathbb{C}$ by

$$\Phi_f(z) = \widehat{\rho}_\alpha(z)(\phi_{0,\alpha,\beta}(z), \phi_{1,\alpha,\beta}(z), \ldots, \phi_{\alpha,\alpha,\beta}(z))^T \tag{2.3}$$

for all $z \in \mathcal{H}$.

Theorem 2.4. *If $f \in M_\beta(\Gamma)$, then the associated $\mathbb{C}^{\alpha+1}$-valued function Φ_f given by (2.3) is a vector-valued meromorphic modular form belonging to $\mathbf{M}_2(\Gamma, \rho_\alpha)$.*

Proof: This follows from [3, Theorem 3]. □

Remark 2.5. Let $\Psi : \mathcal{H} \to \mathbb{C}^{\alpha+1}$ be a vector-valued meromorphic function which can be written in the form

$$\Psi(z) = f(z)(z^\alpha, z^{\alpha-1}, \ldots, z, 1)^T$$

for all $z \in \mathcal{H}$, where f is a meromorphic function on $\mathcal{H}$. Then it can be easily shown that Ψ is a vector-valued modular form belonging to $\mathbf{M}_2(\Gamma, \rho_\alpha)$ if and only if f is a modular form belonging to $M_{\alpha+2}(\Gamma)$.

3. Differential equations and modular forms

In this section we review connections between meromorphic modular forms of one variable and a certain class of linear ordinary differential equations following closely the work of Stiller in [6]. We use the method of Kuga and Shimura [3] to construct vector-valued modular forms of weight two determined by solutions of such differential equations.

Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a Fuchsian group of the first kind as in Section 2, and fix a meromorphic modular form $\varphi \in M_1(\Gamma)$ of weight one for Γ. Then the associated compact Riemann surface $X = \Gamma\backslash\mathcal{H}^*$ may be considered as an algebraic curve over $\mathbb{C}$. We denote by $K(X)$ the function field of the algebraic curve X, and choose a nonconstant element x of $K(X)$. If the functions $\varphi(z)$ and $z\varphi(z)$ on $\mathcal{H}$ are regarded as functions on X, they satisfy a second order homogeneous linear ordinary differential equation $\mathcal{D}_{\varphi,X} f = 0$ on X with

$$\mathcal{D}_{\varphi,X} = \frac{d^2}{dx^2} + P_X(x)\frac{d}{dx} + Q_X(x) \tag{3.1}$$

that has regular singular points, where $P_X(x)$ and $Q_X(x)$ are elements of $K(X)$. Given an element $f \in K(X)$, we see easily that

$$\frac{df}{dx} = \frac{df}{dz}\frac{dz}{dx}, \qquad \frac{d^2 f}{dx^2} = \left[\frac{d^2 f}{dz^2} - \frac{df}{dz} \cdot \frac{d}{dz}\log\frac{dx}{dz}\right]\left(\frac{dz}{dx}\right)^2,$$

where z is the standard coordinate in $\mathbb{C}$. Using this, we can pull the differential operator (3.1) back via the natural projection $\mathcal{H}^* \to X = \Gamma\backslash\mathcal{H}^*$. Then the homogeneous equation $\mathcal{D}_{\varphi,X} f = 0$ on X is equivalent to the equation $\mathcal{D}_\varphi f = 0$ on $\mathcal{H}$ with

$$\mathcal{D}_\varphi = \frac{d^2}{dz^2} + P(z)\frac{d}{dz} + Q(z), \tag{3.2}$$

where $P(z)$ and $Q(z)$ are meromorphic functions on $\mathcal{H}$ given by

$$P(z) = P_X(x(z))\frac{dx}{dz} - \frac{d}{dz}\log\frac{dx}{dz}, \qquad Q(z) = Q_X(x(z))\left(\frac{dx}{dz}\right)^2$$

(see [6, p. 63]). Thus the functions $z\varphi(z)$ and $\varphi(z)$ for $z \in \mathcal{H}$ are linearly independent solutions of the associated homogeneous equation $\mathcal{D}_\varphi f = 0$, and the regular singular points of $\mathcal{D}_\varphi$ coincide with the cusps of Γ (see [6] for details). If m is a positive integer, we denote by $S^m\mathcal{D}_\varphi$ the linear ordinary differential operator of order $m+1$ such that the solutions of the corresponding homogeneous equation $S^m\mathcal{D}_\varphi f = 0$ are of the form

$$f(z) = \sum_{i=0}^{m} c_i (z\varphi(z))^{m-i}(\varphi(z))^i = \sum_{i=0}^{m} c_i z^{m-i}\varphi(z)^m \tag{3.3}$$

for some constants $c_i \in \mathbb{C}$.

We now consider a more general linear ordinary differential operator of order n of the form

$$\mathcal{D} = \frac{d^n}{dx^n} + P_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + P_1\frac{d}{dx} + P_0,$$

where $P_i \in K(X)$ for $0 \leq i \leq n-1$. Let $S \subset X$ be the set of singular points of $P_0, \ldots, P_{n-1}$, and let $X_0 = X - S$. We choose a base point $x_0 \in X_0$ and let $\omega_1, \ldots, \omega_n$ be a basis for the space of local solutions of $\mathcal{D}f = 0$ near x_0. Then the Wronskian

$$W_{\mathcal{D}} = \det M_{\mathcal{D}}, \tag{3.4}$$

is the determinant of the $n \times n$ matrix $M_{\mathcal{D}} = (d^{j-1}\omega_i/dx^{j-1})$ whose (i,j) entry is $d^{j-1}\omega_i/dx^{j-1}$ for $1 \leq i, j \leq n$. Given $x \in X$, let $\eta = \{\eta_1, \ldots, \eta_{n-1}\}$ be the set of $n-1$ local solutions of $\mathcal{D}f = 0$ near x, and let A_η be the $(n-1)\times(n-1)$ matrix whose (i,j) entry is $d^{j-1}\eta_i/dx^{j-1}$ for $1 \leq i, j \leq n-1$. Then a function $\psi \in K(X)$ is said to satisfy the *residue conditions with respect to* $\mathcal{D}$ if the differential $(A_\eta\psi/W)dx$ has zero residue at every $x \in X_0 = X - S$ for each set η of $n-1$ local solutions of $\mathcal{D}f = 0$ near x.

Definition 3.1. An element $\psi \in K(X)$ is said to satisfy the *parabolic residue conditions with respect to* $\mathcal{D}$ if it satisfies the residue conditions and if for each η the differential $(A_\eta\psi/W)dx$ has zero residue at every singular point $x \in S$ whenever A_η is single-valued.

Theorem 3.2. *Let n and ν be integers with $1 \leq \nu \leq n$. Let $\psi \in K(X)$ satisfy the parabolic residue conditions with respect to $S^{2\nu}\mathcal{D}_\varphi$, and let $\mathfrak{S}(\psi)$ be a solution of the differential equation $S^{2\nu}\mathcal{D}_\varphi f = \psi$. We define a vector-valued function $\Phi : \mathcal{H} \to \mathbb{C}^{2n+1}$ by*

$$\Phi(z) = \widehat{\rho}_{2n}(z)(\phi_0(z), \phi_1(z), \ldots, \phi_{2n}(z))^T$$

for all $z \in \mathcal{H}$, where

$$\phi_\ell = \begin{cases} 0 & \text{if } \ell < n+\nu \\ \frac{\ell!(\varphi^{-2\nu}\mathfrak{S}(\psi))^{(\nu+1+\ell-n)}}{(\ell-n-\nu)!(\ell+\nu-n+1)!} & \text{if } \ell \geq n+\nu. \end{cases} \tag{3.5}$$

Then Φ is a vector-valued meromorphic modular form belonging to $\mathbf{M}_2(\Gamma, \rho_{2n})$.

Proof: Given a solution $\mathfrak{S}(\psi)$ of the differential equation $S^{2\nu}\mathcal{D}_\varphi f = \psi$, if we set

$$\Xi_{\nu,\varphi}(\psi) = \frac{d^{2\nu+1}}{dz^{2\nu+1}}\left(\frac{\mathfrak{S}(\psi)}{\varphi^{2\nu}}\right), \tag{3.6}$$

we see easily that $\Xi_{\nu,\varphi}(\psi)$ is independent of the choice of the solution $\mathfrak{S}(\psi)$. Furthermore, it is known that $\Xi_{\nu,\varphi}(\psi)$ is a meromorphic modular form belonging to $M_{2\nu+2}(\Gamma)$ (see [6]). We now apply Theorem 2.4 for $\alpha = 2n$, $\beta = 2\nu + 2$, and $f = \Xi_{\nu,\varphi}(\psi)$. Thus we have

$$\delta = (\alpha + 2 - (2\nu + 2))/2 = (2n - 2\nu)/2 = n - \nu.$$

We set

$$\phi_\ell = \begin{cases} 0 & \text{if } \ell < n + \nu \\ \eta_{\ell-n-\nu} f^{(\ell-n-\nu)} & \text{if } \ell \geq n + \nu, \end{cases}$$

where

$$\eta_k = \frac{(k+n+\nu)!}{k!(2\nu+k+1)!}$$

for each $k \geq 0$. Here we have

$$\begin{aligned} f^{(\ell-n-\nu)} &= \left(\frac{\mathfrak{S}(\psi)}{\varphi^{2\nu}}\right)^{(2\nu+1+\ell-n-\nu)} = \left(\frac{\mathfrak{S}(\psi)}{\varphi^{2\nu}}\right)^{(\nu+1+\ell-n)}, \\ \eta_{\ell-n-\nu} &= \frac{(\ell-n-\nu+n+\nu)!}{(\ell-n-\nu)!(2\nu+\ell-n-\nu+1)!} \\ &= \frac{\ell!}{(\ell-n-\nu)!(\ell+\nu-n+1)!}, \\ \Xi_{\nu,\varphi}(\psi) &= \frac{d^{2\nu+1}}{dz^{2\nu+1}}\left(\frac{\mathfrak{S}(\psi)}{\varphi^{2\nu}}\right). \end{aligned} \tag{3.7}$$

Thus we obtain a sequence $\{\phi_\ell\}_{\ell=0}^{2n}$, where ϕ_ℓ is given by (3.5). $\square$

Example 3.3. We consider the case where $n = 3$ and $\nu = 1$. From (3.5) we obtain

$$\phi_\ell = \begin{cases} 0 & \text{if } \ell < 4 \\ \frac{\ell}{(\ell-4)!}(\varphi^{-2}\mathfrak{S}(\psi))^{(\ell-1)} & \text{if } \ell \geq 4. \end{cases}$$

On the other hand, we see that $\widehat{\rho}_6(z) = (a_{k,\ell}(z))$ is a 7×7 matrix with

$$a_{k,\ell}(z) = \binom{7-k}{\ell-k} z^{\ell-k}$$

for all $z \in \mathcal{H}$ and $1 \leq k, \ell \leq 7$, assuming that $\binom{u}{0} = 1$ and $\binom{u}{v} = 0$ for all $u \geq 0$ and $v < 0$. Thus if we set

$$\begin{aligned} \psi_k(z) = 2(7-k)(6-k)z^{5-k}\left(\frac{\mathfrak{S}(\psi)}{\varphi^2}\right)^{(3)} \\ + 5(7-k)z^{6-k}\left(\frac{\mathfrak{S}(\psi)}{\varphi^2}\right)^{(4)} + 3z^{7-k}\left(\frac{\mathfrak{S}(\psi)}{\varphi^2}\right)^{(5)} \end{aligned}$$

for $k = 5, 6, 7$, then the function $\Phi : \mathcal{H} \to \mathbb{C}^7$ given by

$$\Phi(z) = (0, 0, 0, 0, \psi_5(z), \psi_6(z), \psi_7(z))^T$$

for all $z \in \mathcal{H}$ is a vector-valued meromorphic modular form belonging to $\mathbf{M}_2(\Gamma, \rho_6)$.

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Department of Mathematics, University of Northern Iowa, Cedar Falls,
Iowa 50614, USA

E-mail: lee@math.uni.edu

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