Manifolds admitting stable forms

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Abstract. In this note we give a direct method to classify all stable forms on $\mathbb{R}^n$ as well as to determine their automorphism groups. We show that in dimensions 6, 7, 8 stable forms coincide with non-degenerate forms. We present necessary conditions and sufficient conditions for a manifold to admit a stable form. We also discuss rich properties of the geometry of such manifolds.

Keywords: stable forms, automorphism groups

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1. Introduction

Special geometries defined by a class of differential forms on manifolds are in the center of the interest of geometers again. These interests are motivated by the fact that such a setting of special geometries unifies many known geometries as symplectic geometry and geometries with special holonomy [12], as well as other geometries arising in the M-theory [8], [20]. A series of papers by Hitchin [10], [11] and his school [21], etc., opened a new way to these special geometries. Among them they studied geometries associated with certain stable 3-forms in dimensions 6, 7 and 8 (see the definition of a stable form in Section 2 after Proposition 2.2.)

To classify the stable forms on $\mathbb{R}^n$ one could use the classification by Sato and Kimura [13] of the stable forms on $\mathbb{C}^n$ (they are partial cases of prehomogeneous spaces) and to find the corresponding real forms of the complex stable forms. We note that the Sato and Kimura classification does not include the list of the automorphism groups of the complex stable forms. We also have noticed a proof by Witt in [21] attempting to define the automorphism group of the real stable form of PSU(3)-type, but unfortunately this proof is incomplete (see Remark 4.8 below).

In Sections 2, 3 we study some properties of stable forms. In Section 4 we classify stable forms on $\mathbb{R}^n$ and we determine their automorphism groups. Our classification is based on the Djokovic work [6]. In Sections 5, 6, 7 we present certain necessary conditions as well as some sufficient conditions for a manifold to admit a stable form. We also discuss the rich structure of manifolds admitting

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stable forms in Sections 5, 6, 8. In particular we show that for $n = 7$ or 8 the tangent bundle of any manifold M^n which admits a stable 3-form has a canonical structure of a real simple Malcev algebra bundle.

2. Multi-symplectic forms and stable forms

We recall that a k-form γ on a vector space V^n over a field F is called *multi-symplectic*, if the following map

$$I_\gamma : V \to \Lambda^{k-1}(V^n)^* : v \mapsto v \rfloor \gamma$$

is injective.

Clearly, a 2-form is multi-symplectic if and only if it is symplectic.

A multi-symplectic form is generic in the following sense. For any k-form γ we can define its *rank*, denoted by $\rho(\gamma)$, as the minimal dimension of the subspace $W \subset V^*$ such that $\gamma \in \Lambda^k W$.

2.1 Lemma. *A k-form γ on V^n is multi-symplectic if and only if its rank is n.*

Proof: It is easy to see that if the rank of γ is less than n, then the linear map I_γ has a non-trivial kernel. On the other hand, if I_γ has the non-trivial kernel, then γ can be represented as a k-form in the dual space of the kernel. In fact we have that the dimension of kernel of I_γ is equal to $n - \rho(\gamma)$. □

From now on we shall assume that $F = \mathbb{C}$ or $\mathbb{R}$. In these cases the space $\Lambda^k(V^n)^*$ has the natural topology induced from F.

2.2 Proposition. *The set of multi-symplectic k-forms is open and dense in the space of all k-forms.*

Proof: The equation for $\gamma \in \Lambda^k(V^n)^*$ defining that I_γ has non-trivial kernel is an algebraic equation, thus the set of non-multi-symplectic k-forms is a closed subset in $\Lambda^k(V^n)^*$. It is also easy to check that for any k there exists a multi-symplectic k-form on V^n. Hence the statement follows. □

Clearly the multi-symplecticity is invariant under the action of the group $\mathrm{GL}(F^n)$. We shall say that a k-form γ is *stable* if the orbit $\mathrm{GL}(F^n)(\gamma)$ is open in the space $\Lambda^k(V^n)^*$. By Proposition 2.2 the set of multi-symplectic k-forms has non-trivial intersection with the orbit of any stable form. Hence it immediately follows

2.3 Corollary. *A stable form is multi-symplectic.*

The converse statement is true for $k = 2$ or $k = n - 2$. If $k = 3$ and $n = 7$, $F = \mathbb{R}$, it is known that there are 8 types of $\mathrm{GL}(\mathbb{R}^7)$-orbits of multi-symplectic 3-forms but only two of them are stable.

We say that two forms are equivalent (or of the same type), if they are in the same orbit of $\mathrm{GL}(V^n)$-action. Clearly, a real form is stable if and only if its complexification is stable. We also know that each complex orbit has a finite number of real forms [1, Proposition 2.3]. Thus the classification of real stable forms is equivalent to the classification of complex stable forms plus the classification of the real forms of the complex stable forms. The classifications of complex stable forms is a part of the Sato-Kimura classification of prehomogeneous spaces [13].

3. Symmetric bilinear forms associated to a 3-form on $\mathbb{R}^8$

In this section we associate to a 3-form ω^3 on $\mathbb{R}^8$ several symmetric bilinear forms which are invariants of ω^3. We prove that the only non-degenerate 3-forms (see definition below, after formula (3.4)) are the stable forms. With each stable form we shall associate a Lie algebra structure on $\mathbb{R}^8$.

We denote by I the natural isomorphism $I : \mathbb{R}^8 \otimes \Lambda^8(\mathbb{R}^8)^* \to \Lambda^7(\mathbb{R}^8)^*$:

$$I(v \otimes \theta) = v \rfloor \theta, \tag{3.1}$$

where $\theta \in \Lambda^8(\mathbb{R}^8)$ is a volume form.

Let ω be a 3-form on $\mathbb{R}^8$. We associate with ω a symmetric bilinear map $S : \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8 \otimes \Lambda^8(\mathbb{R}^8)^*$ as follows

$$S^\omega(v, w) = I^{-1}((v\rfloor\omega) \wedge (w\rfloor\omega) \wedge \omega). \tag{3.2}$$

Equivalently

$$S^\omega(v, w) = -\sum_{i=1}^{8} e_i \otimes ((v\rfloor\omega) \wedge (w\rfloor\omega) \wedge \omega \wedge e_i^*) \tag{3.2.a}$$

for any basis (e_i) in $\mathbb{R}^8$ and its dual basis (e_i^*).

For each $v \in \mathbb{R}^8$ we define a linear map $L_v^\omega : \mathbb{R}^8 \to \mathbb{R}^8 \otimes \Lambda^8(\mathbb{R}^8)^*$ by letting the first variable in S^ω to be v

$$L_v^\omega(w) = S^\omega(v, w). \tag{3.3}$$

Now we shall define a symmetric linear form $B^\omega(v, w) : \mathbb{R}^8 \times \mathbb{R}^8 \to (\Lambda^8(\mathbb{R}^8)^*)^2$ as follows

$$B^\omega(v, w) = \mathrm{Tr}(L_v^\omega \circ L_w^\omega) \in (\Lambda^8(\mathbb{R}^8)^*)^2. \tag{3.4}$$

We say that ω is *non-degenerate*, if the reduced trace form $\langle B^\omega, \rho^2\rangle$ is non-degenerate, for some choice of $\rho \in \Lambda^8(\mathbb{R}^8) \setminus \{0\}$.

Let G_ω be the automorphism group of ω. Let us consider the component $G_\omega^+ := G_\omega \cap \mathrm{Gl}^+(\mathbb{R}^8)$.

3.5 Proposition. *The bilinear forms S^ω and B^ω are* $\mathrm{Gl}(\mathbb{R}^8)$*-equivariant in the following sense. For any $g \in \mathrm{Gl}(\mathbb{R}^8)$ we have*

$$S^{g^*(\omega)}(X,Y) = g^*(S^\omega(g^{-1}X, g^{-1}Y)), \tag{3.5.1}$$

$$B^{g^*(\omega)}(X,Y) = g^*(B^\omega(g^{-1}X, g^{-1}Y)). \tag{3.5.2}$$

If ω is non-degenerate, then the group G^+_ω is a subgroup of $\mathrm{SL}(\mathbb{R}^8)$*. The group G_ω preserves the reduced trace form $\langle B^\omega, \rho^2\rangle$ for any choice of $\rho \in \Lambda^8(\mathbb{R}^8)$.*

Proof: The computation of (3.5.1) and (3.5.2) is straightforward, so we omit them. The symmetric form $B^\omega(v,w)$ can be considered as a linear map $B^\omega : (\mathbb{R}^8) \to (\mathbb{R}^8)^* \otimes (\Lambda^8(\mathbb{R}^8)^*)^2$. Let us consider the associated linear map

$$\det(B^\omega) : \Lambda^8(\mathbb{R}^8) \to \Lambda^8((\mathbb{R}^8)^* \otimes (\Lambda^8(\mathbb{R}^8)^*)^2) = \Lambda^8((\mathbb{R}^8)^*)^{17}. \tag{3.5.3}$$

If B^ω is non-degenerate, then the map $\det(B^\omega)$ is not trivial. From (3.5.2) we deduce that the map $\det B^\omega$ is G^+_ω-invariant map. So for any $g \in G^+_\omega$ we get from (3.5.3)

$$\det g = (\det g^{-1})^{17}.$$

Since $\det g > 0$ we conclude that $\det g = 1$. Now using (3.5.2) we get the last statement immediately. $\square$

3.6 Proposition. (i) *The trace form B^ω is compatible with the multiplication S^ω in the following sense*

$$B^\omega(S^\omega(a.b), c) = B^\omega(a, S^\omega(b,c)).$$

(ii) *The trace form B^ω is non-degenerate, if and only if ω is stable.*

Proof: The first statement follows immediately from the definition. The second statement could be derived from the result of Sato and Kimura [13]. Here we give a straightforward proof of this fact. We observe that if ω_1 and ω_2 are the real forms of the same complex 3-form, then their trace forms are also the real forms of the trace form for the complex 3-form (all these bilinear forms S^ω and B^ω can be defined for any vector space V over an arbitrary field.) Thus to check how many real 3-forms are non-degenerate we need to check only 22 representatives of 3-forms in the Djokovic classification [6]. Furthermore we know that a non-degenerate 3-form must be multi-symplectic. Thus it suffices to compute the trace form of 13 multi-symplectic 3-forms in tables XI-XXIII in the Djokovic classification. We wrote a program for computing the trace form B^ω to run it under Maple. We denote by $e_1^* \wedge \cdots \wedge e_8^*$ by θ, where e_i^* are the coordinate 1-forms on $\mathbb{R}^8$. We shall use θ to make a (reduced) multiplication $V \times V \to V$

$$(vw \rfloor \theta) = (v \rfloor \omega) \wedge (w \rfloor \omega) \wedge \omega \tag{3.7}$$

Clearly we have

$$S^{\omega}(v,w) = vw \otimes \theta. \tag{3.8}$$

We define structure constants A^k_{ij} by

$$e_i e_j = \sum_k A^k_{ij} e_k. \tag{3.9}$$

Then

$$S^{\omega}(e_i,e_j) = \sum_k A^k_{ij} e_k \otimes \theta. \tag{3.9.a}$$

Now let us compute

$$\begin{aligned} B^{\omega}(e_l,e_m) = & \sum_n (S(e_l,S(e_m,e_n)),e_n^*) \\ \overset{3.2.a}{=} & \sum_{k,n} \langle e_k \otimes (e_l\rfloor\omega)\wedge(e_m e_n\rfloor\omega)\wedge\omega\wedge e_k^*\otimes\theta, e_n^*\rangle \\ = & \sum_{n,p} (e_l\rfloor\omega)\wedge A^p_{mn}(e_p\rfloor\omega)\wedge\omega\wedge e_n^*\otimes\theta \\ = & \sum_{n,p} A^n_{lp}\cdot A^p_{m,n}\otimes(\theta)^2. \end{aligned} \tag{3.10}$$

The result is that the only stable forms numerated by XXIIIa, XXIIIb, XXIIIc by Djokovic have non-degenerate trace forms.

Below we shall compute explicitly the reduced multiplication forms as well as the reduced trace forms $\langle B^{\phi_i},(\theta^*)^2\rangle$ for stable forms ϕ_i on $\mathbb{R}^8$ from the Djokovic classification.

(Form XXIIIa): $\phi_1 = e^{124} + e^{134} + e^{256} + e^{378} + e^{157} + e^{468}$.
(Form XXIIIb): $\phi_2 = e^{135} + e^{245} + e^{146} - e^{236} + e^{127} + e^{348} + e^{678}$.
(Form XXIIIc): $\phi_3 = e^{135} - e^{146} + e^{236} + e^{245} + e^{347} + e^{568} + e^{127} + e^{128}$.

The reduced multiplication table for the form XXIIIa is:

$$\begin{bmatrix} 0 & -e_1 & e_1 & 3e_2-3e_3 & -3e_8 & 0 & -3e_6 & 0 \\ -e_1 & -2e_2 & -2e_2+2e_3 & -e_4 & -e_5 & -e_6 & 2e_7 & 2e_8 \\ e_1 & -2e_2+2e_3 & 2e_3 & e_4 & -2e_5 & -2e_6 & e_7 & e_8 \\ 3e_2-3e_3 & -e_4 & e_4 & 0 & 0 & 3e_7 & 0 & 3e_5 \\ -3e_8 & -e_5 & -2e_5 & 0 & 0 & 3e_3 & -3e_4 & 0 \\ 0 & -e_6 & -2e_6 & 3e_7 & 3e_3 & 0 & 0 & 3e_1 \\ -3e_6 & 2e_7 & e_7 & 0 & -3e_4 & 0 & 0 & -3e_2 \\ 0 & 2e_8 & e_8 & 3e_5 & 0 & 3e_1 & -3e_2 & 0 \end{bmatrix}.$$

The reduced trace form for the form XXIIIa is:

$$\begin{bmatrix} 0 & 0 & 0 & -30 & 0 & 0 & 0 & 0 \\ 0 & 20 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 20 & 0 & 0 & 0 & 0 & 0 \\ -30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 & -30 & 0 \end{bmatrix}.$$

The reduced multiplication table for the form XXIIIb is:

$$\begin{bmatrix} 6\,e_8 & 0 & -3\,e_6 & 3\,e_5 & -e_1 & 3\,e_2 & -3\,e_4 & 0 \\ 0 & 6\,e_8 & -3\,e_5 & -3\,e_6 & -e_2 & -3\,e_1 & 3\,e_3 & 0 \\ -3\,e_6 & -3\,e_5 & 6\,e_7 & 0 & -e_3 & -3\,e_4 & 0 & 3\,e_2 \\ 3\,e_5 & -3\,e_6 & 0 & 6\,e_7 & -e_4 & 3\,e_3 & 0 & -3\,e_1 \\ -e_1 & -e_2 & -e_3 & -e_4 & -2\,e_5 & 2\,e_6 & 2\,e_7 & 2\,e_8 \\ 3\,e_2 & -3\,e_1 & -3\,e_4 & 3\,e_3 & 2\,e_6 & -6\,e_5 & 0 & 0 \\ -3\,e_4 & 3\,e_3 & 0 & 0 & 2\,e_7 & 0 & 0 & 3\,e_5 \\ 0 & 0 & 3\,e_2 & -3\,e_1 & 2\,e_8 & 0 & 3\,e_5 & 0 \end{bmatrix}.$$

The reduced trace form for the form XXIIIb is:

$$\begin{bmatrix} 0 & 0 & 0 & -60 & 0 & 0 & 0 & 0 \\ 0 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 30 & 0 \end{bmatrix}.$$

The reduced multiplication table for the form XXIIIc is:

$$\begin{bmatrix} 6\,e_7-6\,e_8 & 0 & 3\,e_6 & 3\,e_5 & 3\,e_4 & 3\,e_3 & e_1 & -e_1 \\ 0 & 6\,e_7-6\,e_8 & -3\,e_5 & 3\,e_6 & -3\,e_3 & 3\,e_4 & e_2 & -e_2 \\ 3\,e_6 & -3\,e_5 & 6\,e_8 & 0 & -3\,e_2 & 3\,e_1 & e_3 & 2\,e_3 \\ 3\,e_5 & 3\,e_6 & 0 & 6\,e_8 & 3\,e_1 & 3\,e_2 & e_4 & 2\,e_4 \\ 3\,e_4 & -3\,e_3 & -3\,e_2 & 3\,e_1 & -6\,e_7 & 0 & -2\,e_5 & -e_5 \\ 3\,e_3 & 3\,e_4 & 3\,e_1 & 3\,e_2 & 0 & -6\,e_7 & -2\,e_6 & -e_6 \\ e_1 & e_2 & e_3 & e_4 & -2\,e_5 & -2\,e_6 & 2\,e_7 & 2\,e_7-2\,e_8 \\ -e_1 & -e_2 & 2\,e_3 & 2\,e_4 & -e_5 & -e_6 & 2\,e_7-2\,e_8 & -2\,e_8 \end{bmatrix}.$$

The reduced trace form for the form XXIIIc is:

$$\begin{bmatrix} 60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 60 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 60 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 20 \end{bmatrix}.$$

□

3.11 Proposition. *Each stable form ϕ defines a Lie algebra structure $[\cdot,\cdot]_\phi$ on $\mathbb{R}^8$ by the following formula*

$$\big\langle [X,Y]_\phi, Z\big\rangle_\phi = \phi(X,Y,Z), \tag{3.11.1}$$

where $\langle\cdot,\cdot\rangle_\phi$ denotes a reduced trace form of ϕ. Moreover the Lie algebra $[\cdot,\cdot]_{\phi_i}$ is the non-compact real form of $\mathrm{sl}(3,\mathbb{C})$ *for $i = 1, 2$ and the Lie algebra $[\cdot,\cdot]_{\phi_3}$ is the compact real form of* $\mathrm{sl}(3,\mathbb{C})$.

Proof: First we note that the anti-symmetric bracket $[\cdot,\cdot]_\phi$ satisfies the following invariant property. For each $g \in \mathrm{Gl}(\mathbb{R}^8)$ we have

$$[X,Y]_{g^*\phi} = g([g^{-1}(X), g^{-1}(Y)])_\phi. \tag{3.12}$$

Hence if the Jacobi identity holds at a form ϕ, it also holds at any point in the orbit $\mathrm{GL}(\mathbb{R}^8)(\phi)$, moreover these Lie brackets are equivalent. Secondly we notice that the bracket $[\cdot,\cdot]_\phi$ can be extended linearly over $\mathbb{C}$ and this complexification is the anti-symmetric bracket defined by the complexification of the form ϕ according to the same formula (3.11.1). Thus to verify the Jacobi identity for 3 stable forms ϕ_i, $i = \overline{1,3}$, it suffices to verify for one of them.

Next, we shall show that the forms ϕ_i are equivalent to the Cartan forms on the real form of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$ and the trace form of one of the Cartan forms is a multiple of the Killing form. Hence we shall get that the skew-symmetric multiplication defined in (3.11.1) coincides up to a non-zero constant with the Lie bracket on the Lie algebra.

Taking into account Proposition 3.6.ii we observe that to show the equivalence of the complex Cartan form on $\mathrm{sl}(3,\mathbb{C})$ to the stable forms $\phi_i \otimes \mathbb{C}$ it suffices to show that one of the real Cartan forms is stable.

Now we compute the reduced trace formula for the Cartan form on the algebra su(3)

$$\rho_3(X,Y,Z) = \langle [X,Y], Z\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the Killing form on su(3). We use the following explicit expression taken from [21] for a multiple of the form ρ_3:

$$(-1/\sqrt{3})^3\rho_3 = e^{123} + (1/2)(e^{147} - e^{156} + e^{246} + e^{257} + e^{345} - e^{367}) \\ + (\sqrt{3}/2)(e^{845} + e^{867})$$

where (e_i) are an orthonormal basis in su(3) and e^{ijk} denotes the form $e^i \wedge e^j \wedge e^k$. A direct computation (also using Maple) gives us the following multiplication table for $(4/3)\cdot(-1/\sqrt{3})^3\rho_3$

$$\begin{bmatrix}
2e_8 & 0 & 0 & \sqrt{3}e_6 & \sqrt{3}e_7 & \sqrt{3}e_4 & \sqrt{3}e_5 & 2e_1\\
0 & 2e_8 & 0 & -\sqrt{3}e_7 & \sqrt{3}e_6 & \sqrt{3}e_5 & -\sqrt{3}e_4 & 2e_2\\
0 & 0 & 2e_8 & \sqrt{3}e_4 & \sqrt{3}e_5 & -\sqrt{3}e_6 & -\sqrt{3}e_7 & 2e_3\\
\sqrt{3}e_6 & -\sqrt{3}e_7 & \sqrt{3}e_4 & \sqrt{3}e_3 - e_8 & 0 & \sqrt{3}e_1 & -\sqrt{3}e_2 & -e_4\\
\sqrt{3}e_7 & \sqrt{3}e_6 & \sqrt{3}e_5 & 0 & \sqrt{3}e_3 - e_8 & \sqrt{3}e_2 & \sqrt{3}e_1 & -e_5\\
\sqrt{3}e_4 & \sqrt{3}e_5 & -\sqrt{3}e_6 & \sqrt{3}e_1 & \sqrt{3}e_2 & -\sqrt{3}e_3 - e_8 & 0 & -e_6\\
\sqrt{3}e_5 & -\sqrt{3}e_4 & -\sqrt{3}e_7 & -\sqrt{3}e_2 & \sqrt{3}e_1 & 0 & -\sqrt{3}e_3 - e_8 & -e_7\\
2e_1 & 2e_2 & 2e_3 & -e_4 & -e_5 & -e_6 & -e_7 & -2e_8
\end{bmatrix}$$

and we compute easily from here (also by using Maple) that the reduced trace formula for $(-1/\sqrt{3})^3\rho_3$ is equal to $(45/4)$ (diag). So the trace formula is a multiple of the Killing form.

Once we know that the reduced trace form is a multiple of the Killing form, we get the equivalence of the complex Cartan form and the form $\phi_i \otimes \mathbb{C}$. Since the only reduced trace form of ϕ_3 is of signature 0, we conclude that the ϕ_3 is equivalent to ρ_3. Now it follows immediately that the Lie bracket for ϕ_3 defined in (3.11.1) coincides with Lie bracket on su(3), since the reduced trace form is a multiple of the Killing form. Thus the Lie bracket for ϕ_3 satisfies the Jacobi identity. Hence the Jacobi identity for all other ϕ_1, ϕ_2 also holds. This proves the first statement of Proposition 3.11.

It remains to determine that ϕ_1 is equivalent to the Cartan form on $\mathrm{sl}(3,\mathbb{R})$ and ϕ_2 is equivalent to the Cartan form on $\mathrm{su}(1,2)$. We know that the reduced trace form of the Cartan form on $\mathrm{sl}(3,\mathbb{R})$ is a bilinear symmetric non-degenerate form which is invariant under the automorphism group $\mathrm{Aut}(\mathrm{sl}(3,\mathbb{R}))$ of the Lie algebra $\mathrm{sl}(3,\mathbb{R})$, since the Cartan form is invariant under the action of $\mathrm{Aut}(\mathrm{sl}(3,\mathbb{R}))$. Hence the reduced trace form of the Cartan form on $\mathrm{sl}(3,\mathbb{R})$ is a multiple of the Killing form, in particular it has signature (3,5). Now we know that the signature of the reduced trace form of ϕ_1 is (3,5) and the signature of the reduced trace form of ϕ_2 is of signature (4,4). This proves the second statement of Proposition 3.11. $\square$

4. Classification of real stable forms

We observe that the stability of a k-form is preserved under the Poincaré isomorphism $\Lambda^k(V^n)^* \to \Lambda^{n-k}(V^n)$. We shall use notation $e^{12\cdots k}$ for the form $e^1 \wedge e^2 \wedge \cdots \wedge e^k$. We also use notation G_γ for the isotropy group of γ under the action of $\mathrm{Gl}(\mathbb{R}^n)$ and by $\mathfrak{g}_\gamma$ the Lie algebra of G_γ.

4.1 Theorem. *Suppose that $3 \le k \le n-k$.*

(i) *Then a stable k-form γ on $\mathbb{R}^n$ exists, if and only if $k=3$ and $6 \le n \le 8$. Furthermore*

(ii) *if $n=6$, then γ is equivalent to one of the following forms:*
$\gamma_1 = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$ *with* $G_{\gamma_1} = \mathrm{SL}(\mathbb{R}^3) \times \mathrm{SL}(\mathbb{R}^3) \times \mathbb{Z}_2$;
$\gamma_2 = \mathrm{Re}\,(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)$ *with* $G_{\gamma_2} = \mathrm{SL}(\mathbb{C}^3)$,

(iii) *if $n=7$, then γ is equivalent to one of the following forms:*
$\omega_1 = e^{123} - e^{145} + e^{167} + e^{246} + e^{257} + e^{347} - e^{356}$ *with* $G_{\omega_1} = G_2$;
$\omega_2 = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}$ *with* $G_{\omega_2} = \tilde{G}_2$,

(iv) *if $n=8$, then γ is equivalent to one of the following forms:*
$\phi_1 = e^{124} + e^{134} + e^{256} + e^{378} + e^{157} + e^{468}$ *with* $G_{\phi_1} = \mathrm{SL}(3,\mathbb{R}) \times \mathbb{Z}_2$;
$\phi_2 = e^{135} + e^{245} + e^{146} - e^{236} + e^{127} + e^{348} + e^{678}$ *with* $G_{\phi_2} = \mathrm{PSU}(1,2) \times \mathbb{Z}_2$;
$\phi_3 = e^{135} - e^{146} + e^{236} + e^{245} + e^{347} + e^{568} + e^{127} + e^{128}$ *with* $G_{\phi_3} = \mathrm{PSU}(3) \times \mathbb{Z}_2$.

Proof: We first show that if $4 \le k \le n-k$ then there is no stable form. It suffices to show that in this case we have

$$\dim \Lambda^k(\mathbb{R}^n) \ge n^2 + 1 = \dim(\mathrm{Gl}(V^n)) + 1. \tag{4.2}$$

Clearly we have under the assumption that $4 \le k \le n-k$

$$\dim \Lambda^k(\mathbb{R}^n) \ge \dim \Lambda^4(\mathbb{R}^n).$$

Therefore (2.2) is a consequence of the following equality

$$f(n) := n^3 - 6n^2 - 13n - 6 \ge 1, \quad \text{for} \quad n \ge 8. \tag{4.3}$$

Since $f'(n) > 0$ for all $n \ge 8$ it suffices to check (4.3) for $n = 8$ which is an easy exercise. To complete the proof of Theorem 4.1(i) we need to show that stable 3-forms exist for $n = 6, 7, 8$ and not for $n \ge 9$. But this is an well-known fact for $n = 6, 7$ and it follows from the classification of 3-forms on $\mathbb{R}^8$ by Djokovic [6]. To show that there is no stable 3-form in $\mathbb{R}^n$ if $n \ge 9$, we can repeat the argument above to show that in this case $\dim \Lambda^3(\mathbb{R}^n) > \dim \mathrm{Gl}(\mathbb{R}^n)$.

(ii) This classification is already well-known, see [10] for a wonderful treatment.

(iii) This classification follows from the list of Bureš and Vanžura of multi-symplectic 3-forms in dimension 7 [2] together with their automorphism groups. The groups G_{ω_i} have been first determined by Bryant [3].

(iv) We shall complete this classification from the last table in [6]. In that table Djokovic supplied us only the Lie algebras g_{ϕ_i}, for $i = 1, 2, 3$. We shall recover G_{ϕ_i} from g_{ϕ_i} by using the following Lemmas 4.4 and 4.5.

4.4 Lemma. *Group* $\mathrm{Gl}^+(\mathbb{R}^8)$ *acts transitively on the orbit* $\mathrm{Gl}(\mathbb{R}^8)(\phi_i)$, *for* ϕ_i *being one of the forms in Theorem* 4.1(iv).

Proof: It suffices to show that the intersection $G_{\phi_i} \cap \mathrm{Gl}^-(\mathbb{R}^8)$ is not empty, where $\mathrm{Gl}^-(\mathbb{R}^8)$ denotes the orientation reversing component of $\mathrm{Gl}(\mathbb{R}^n)$.

– For ϕ_1 this intersection contains the following element $\sigma_{23} \cdot \sigma_{57} \cdot \sigma_{68} \cdot I_1 \cdot I_4$. Here σ_{ij} denotes the orientation reversing linear transformation which permutes the basic vectors v_i and v_j and leaves all other basic vectors fixed, and I_j denotes the orientation reversing linear transformation which acts as $-\mathrm{Id}$ on the line $v_j \otimes \mathbb{R}$ and leaves all other basic vectors fixed.

– For ϕ_2 this intersection contains the following element $\sigma_{12} \cdot \sigma_{34} \cdot I_6 \cdot I_7 \cdot I_8$.

– For ϕ_3 this intersection contains $\sigma_{34} \cdot \sigma_{56} \cdot I_1 \cdot I_7 \cdot I_8$. □

4.5 Lemma. *The group* $\mathrm{Gl}^+_\phi := \mathrm{Gl}^+(\mathbb{R}^8) \cap G_\phi$ *is connected for* ϕ_i *being one of the forms from Theorem* 4.1(iv).

Proof: We use the observation obtained in Section 3 that all three forms ϕ_i are the Cartan forms

$$\rho(X, Y, Z) = \langle X, [Y, Z] \rangle$$

on the Lie algebra sl$(3,\mathbb{R})$, su$(1,2)$ and su(3), where $\langle\cdot,\cdot\rangle$ denotes the Killing form. Hence, it follows that

$$\mathrm{Aut}(\mathfrak{g}_{\phi_i}) \subset G_{\phi_i}. \tag{4.6}$$

In Proposition 3.11 we have also defined a way to recover the structure of the corresponding Lie algebra from ϕ_i. Since all the reduced bilinear forms are invariant with respect to G_{ϕ_i} we get

$$G_{\phi_i} \subset \mathrm{Aut}(\mathfrak{g}_{\phi_i}). \tag{4.7}$$

Finally the structure of $\mathrm{Aut}(\mathfrak{g}_{\phi_i})$ is well-known, see e.g. [16] and the references therein. Thus we get Lemma 4.5 from (4.6) and (4.7). □

Actually the proof of Lemma 4.5 implies Lemma 4.4. Nevertheless the proof of Lemma 4.4 gives us explicitly an element in $\mathrm{Gl}^-(\mathbb{R}^8)\cap G_{\phi_i}$. This completes the proof of Theorem 4.1. □

4.8 Remark. In his thesis [21] Witt gave a proof that the component $G^+_{\phi_3}$ is PSU(3). His proof is incomplete, since he used implicitly without a proof the fact that the component $G^+_{\phi_3}$ preserves the Killing metric on su(3). (His method is to associate the Cartan form to a bilinear form with values in $\mathbb{R}^8$ by using a fixed basis of $\mathbb{R}^8$. A detailed analysis shows that such a use is equivalent to giving a linear map from $(\mathbb{R}^8)^*$ to $\mathbb{R}^8$ and in the given case of Witt, that map is an isomorphism defined by the Killing metric.)

We say that a differentiable form γ on a manifold M^n is *stable*, if at each $x\in M$ the form $\gamma(x)$ is stable.

4.9 Proposition. *If a connected manifold M^n admits a differentiable stable form γ^3, then for all $x\in M^n$ the form $\gamma(x)$ has the same type. In particular M^n admits a $G_{\gamma(x)}$ structure. Conversely, if M^n admits a G_γ structure, then it admits a differentiable form of γ type.*

Proof: For each $x\in M^n$ denote by $U(x)$ the set of all points $y\in M^n$ such that $\gamma^3(y)$ has the same type as $\gamma^3(x)$. Clearly $U(x)$ is an open subset in M^n. Suppose that $U(x)\neq M^n$. Then the closure $\bar U(x)$ contains an point y which is not in $U(x)$. Clearly $\gamma(y)$ also has the same type as $\gamma(x)$ since $U(y)$ has a non-empty intersection with $U(x)$. Thus $y\in U(x)$ which is a contradiction. The last statement follows from the fact that the transition functions on $G(x)$-manifold preserve the form $\gamma(x)$. □

5. Stable 3-forms on 6-manifolds

5.1. *Obstruction for the existence of a G_{γ_1}-structure.*

If a non-orientable manifold M^6 admits a G_{γ_1}-structure, then its orientable double covering shall admit G_{γ_1}-structure. Now we shall consider only orientable manifolds M^6 and so only the identity component of G_{γ_1}. Clearly M^6 admits an $\mathrm{SL}(3)\times\mathrm{SL}(3)$-structure, if and only if it admits a distribution of oriented 3-planes on M^6.

We denote by $\rho_2 : H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Z}_2)$ the modulo 2 reduction.

5.1.1 Proposition. *Suppose that a closed manifold M^6 admits an $\mathrm{SL}(3)\times\mathrm{SL}(3)$-structure. Then its Euler class vanishes. Assume that $H^4(M^6,\mathbb{Z})$ has no 2-torsion, the Euler class $e(M^6)$ vanishes and M^6 satisfies moreover the following condition* (P). *There are classes $c_1, c_2 \in H^2(M,\mathbb{Z})$ such that*

$$p_1(M^6) = c_1^2 + c_2^2,\ \rho_2(c_1+c_2) = w_2(M^6). \tag{P}$$

Then M^6 admits an $\mathrm{SL}(3)\times\mathrm{SL}(3)$*-structure.*

Proof: The first statement is well-known, since the Euler class of an oriented 3-dimensional vector bundle is a 2-torsion, and $H^6(M,\mathbb{Z})$ has no 2-torsion. Let us assume that an orientable manifold M^6 with vanishing Euler class has no 2-torsion in $H^4(M,\mathbb{Z})$ and, moreover, that M^6 satisfies condition (P). Let V be a non-vanishing vector field on M^6. Since M^6 satisfies condition (P), there is an almost complex structure J on M^6 such that $c_1(J) = c_1 + c_2$, where c_1 and c_2 satisfies condition (P). Let W^4 be a J-invariant sub-bundle of TM^6 which is complement to V and JV. Clearly $p_1(W^4) = p_1(M^6)$. Let L_1 and L_2 be the complex line bundles over M^6 with the first Chern classes c_1 and c_2 satisfying condition (P). Then $p_1(W^4) = p_1(M^6) = p_1(L_1 \oplus L_2)$ and $w_2(W^4) = w_2(M^6) = w_2(L_1 \oplus L_2)$. Hence according to [19, Lemma 1], W^4 and $L_1 \oplus L_2$ are stably isomorphic. Next we compute that

$$\begin{aligned} e(W^4) = c_2(W^4) = c_2(TM^6,J) = \frac{1}{2}(c_1^2(TM^6,J) - p_1(TM^6)) \\ = c_1 \cdot c_2 = e(L_1 \oplus L_2). \end{aligned}$$

Hence, taking into account [19, Lemma 2], W^4 and $L_1 \oplus L_2$ are isomorphic as real vector bundles. Thus TM^6 is the sum of two 3-dimensional vector bundles.

□

5.1.2 Remark. (i) In 5.3 we discuss regular maximally non-integrable G_{γ_1}-structures. If a G_{γ_1}-structure is degenerate, but still regular, then it is easy to see that M^6 satisfies the condition (P).

(ii) If M^6 admits 3 linearly independent vector fields, then it admits also an $\mathrm{SL}(3)\times\mathrm{SL}(3)$-structure. In [18] Thomas gave a necessary and sufficient condition for an orientable 6-manifold to admit 3 linearly independent vector fields, namely M^6 has vanishing Euler class and vanishing Stiefel-Whitney class w_4.

5.2. *Obstruction for the existence of a G_{γ_2}-structure.*

5.2.1 Proposition. *A manifold M^6 admits an* $\mathrm{SL}(3,\mathbb{C})$*-structure, if and only if it is orientable and spinnable.*

PROOF: Clearly a 6-manifold M^6 admits an $\mathrm{SL}(3,\mathbb{C})$-structure, if and only if M^6 admits an almost complex structure of vanishing first Chern class. In particular M^6 must be orientable and spinnable. On the other hand, if M^6 is orientable and spinnable, then M^6 admits an $\mathrm{SL}(3,\mathbb{C})$-structure, since it admits an almost complex structure, whose first Chern class is an integral lift of w_2. Thus the necessary and sufficient condition for M^6 to admit an $\mathrm{SL}(3,\mathbb{C})$-structure is the vanishing of the Stiefel-Whitney classes $w_1(M^6)$ and $w_2(M^6)$. □

5.3. *Maximally non-integrable 3-forms of γ_1-type.*

Every 3-form γ_1 on M^6 defines a pair of two oriented transversal 3-distributions D_1 and D_2 together with volume forms on each D_i as follows. Recall that at every point $x \in M$ we can write $\gamma_1 = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$. The union $D_1 \cup D_2$ is defined uniquely as the set of all vectors $v \in T_xM$ such that $\operatorname{rank}(v\rfloor\gamma_1) = 2$, or equivalently, $(v\rfloor\gamma_1)^2 = 0$. The orientation (the volume form) of D_1 and D_2 is defined by the restriction of γ_1 to each distribution D_i. Conversely, a pair of two transversal oriented 3-distributions D_1 and D_2 on M^6 together with their volume form defines a 3-form of γ_1-type as follows. Let their volume forms be α_1 and α_2 respectively. Now we define $\gamma_1 = p_1^*(\alpha_1) + p_2^*(\alpha_2)$, where $p_1 : TM \to D_1$ and $p_2 : TM \to D_2$ are the projections defined by D_i.

We call the structure (M^6, γ_1) *regular*, if the dimensions of the distributions $[D_i, D_i]$ defined by γ_1 are constant over M^6. We shall call a regular G_{γ_1}-structure *maximal non-integrable*, if at least one of the distributions D_i is maximal non-integrable in the sense that $D_i + [D_i, D_i] = TM$.

At this place we note that the labeling D_1 and D_2 is well-defined only locally. Globally we may be not able to distinguish, which of the two planes is the D_1. This ambiguity can be removed, if M^6 is simply connected, since in this case the two line bundles $\det D_1$ and $\det D_2$ can be distinguished.

We can describe the maximal non-integrability of D_i in terms of γ_1 as follows. Write $\omega_1 = p_1^*(\alpha_1)$, $\omega_2 = p_2^*(\alpha_2)$. Locally we can write $\omega_1 = p_1^*(e^1 \wedge e^2 \wedge e^3)$, $\omega_2 = p_2^*(e^4 \wedge e^5 \wedge e^6)$.

5.3.1 Proposition. *There is a volume form $D^3\omega_2 \in \Lambda^3(\Lambda^2(D_1))^*$ defined in local coordinates as follows:*

$$D^3(\omega_2) = i_1^*(\mathrm{d}\,p_2^*(e^4) \wedge \mathrm{d}\,p_2^*(e^5) \wedge \mathrm{d}\,p_2^*(e^6)),$$

where $i_1 : D_i \to TM$ is the embedding, and $\mathrm{d}\,p_2^*(e^i)$ are considered as elements of $(\Lambda^2 TM)^*$. This expression does not depend on the choice of local 1-forms e^i

considered as 1*-forms on* D_2*. This volume form is not zero, if and only if* D_1 *is maximal non-integrable.*

Proof: We first show that, if f^4, f^5, f^6 is another co-frame in D_2, so that $(f^4, f^5, f^6) = g(e^4, e,^5, e^6)$ for $g \in \mathrm{Gl}(D_2)$ then

$$(5.3.2) \qquad i_1^*(\mathrm{d}\, p_2^*(f^4) \wedge \mathrm{d}\, p_2^*(f^5) \wedge \mathrm{d}\, p_2^*(f^6)) \\ = (\det g) \cdot i_1^*(\mathrm{d}\, p_2^*(e^4) \wedge \mathrm{d}\, p_2^*(e^5) \wedge \mathrm{d}\, p_2^*(e^6)).$$

Proposition 5.3.1 is a local statement, so it suffices to prove it on a small disk $B^6 \subset M^6$. We denote by A the open dense subset in the gauge transformation group $\Gamma(B^6 \times \mathrm{Gl}(D_2))$ which is defined by the condition that (f^4, e^5, e^6) and (f^4, f^5, e^6) are also a co-frames on D_2. Then we have $g = g_3 \circ g_2 \circ g_1$, where g_1 sends (e^4, e^5, e^6) to (f^4, e^5, e^6), g_2 sends (f^4, e^5, e^6) to (f^4, f^5, e^6) and $g_3 = g \circ g_1^{-1} \circ g_2^{-1}$. Now it is straightforward to check (5.3.2) for each g_1, g_2, g_3. Hence (5.3.2) holds on the open dense set A. Since the LHS and RHS of (5.3.2) are continuous mappings, the equality (5.3.2) holds on the whole $\mathrm{Gl}(D_2)$. This proves the first statement. The second statement now follows by direct calculations in local coordinates. □

Our study of maximally non-integrable G_{γ_1}-structures is motivated by its relation with the parabolic geometry. This structure is a generalization of the famous Cartan 2-distribution in a 5-manifold and it has a canonical conformal structure [4]. The Lie algebra of the automorphism group $\mathrm{Aut}(M^6, \gamma_1)$ as well as local invariants of (M^6, γ_1) can be calculated using the theory of filtered manifolds (see e.g. [22]).

6. Stable 3-forms on 7-manifolds

6.1. *Topological conditions for the existence of a stable* 3*-form on a* 7*-manifold.*

The sufficient and necessary condition for the existence of a G_2-structure on a 7-manifold M^7 has been established by Gray [9]. A manifold admits a G_2-structure, if and only if it is both orientable and spinnable, i.e. the first two Stiefel-Whitney classes vanish.

It has been observed in [15] that a closed 7-manifold M^7 admits a $\tilde{G}_2$-structure, if and only if it is orientable and spinnable. The closedness condition originates from the Dupont work [7] using the K-theory, which implies the reduction of the SO(7)-structure on M^7 to an SO(3) × SO(4)-structure.

The geometry of G_2-manifolds has been intensively studied, but the geometry of $\tilde{G}_2$-manifolds is barely explored. In [14] we have constructed the first example of a stably non-homogeneous closed 7-manifold which admits a closed 3-form of $\tilde{G}_2$-type.

6.2. *Malcev algebra structure on 7-manifolds admitting stable 3-forms.*

Any stable 3-form ϕ in dimension 7 defines a reduced symmetric bilinear form by the formula [3]

$$\langle V, W\rangle_\phi = \langle (V\rfloor\phi)\wedge(W\rfloor\phi)\wedge\phi, \rho\rangle$$

where ρ is some non-zero element in $\Lambda^8(\mathbb{R}^7)$. Let us define a multiplication $x\circ_\phi y$ on $\mathbb{R}^7$ by the following formula:

$$\left\langle x\circ_\phi y, z\right\rangle_\phi = \phi(x, y, z).$$

With Peter Nagy we have discovered that the skew-symmetric multiplication $x\circ_\phi y$ defines the structure of the simple Malcev algebra A^* on $\mathbb{R}^7$ whose corresponding Moufang loop is S^7 for $\phi = \omega_1$ in Theorem 4.1 (resp. the pseudo sphere $S_{(4,4)}(1)$ of the unit vector in the vector space $\mathbb{R}^8$ with the metric with the signature $(4,4)$ for $\phi = \omega_2$). Malcev algebras are generalization of Lie algebras, see [17] for more information, in particular the structure of the simple Malcev algebras A^* on $\mathbb{R}^7$.

Thus the tangent bundle TM^7 has the canonical structure of the simple Malcev algebra bundle.

7. Stable 3-forms on 8-manifolds

As before we assume that M^8 is orientable, since we can go to the orientable double covering, if necessary.

The maximal compact subgroup of $G^+_{\phi_1}$ is SO(3) which is included in SO(8) via the adjoint representation. The maximal compact subgroup of $\mathrm{PSU}(1,2) = \mathrm{SU}(1,2)/\mathbb{Z}_3$ is $S(U(1)\times U(2))/\mathbb{Z}_3$. The subgroups SO(3) and $S(U(1)\times U(2))/\mathbb{Z}_3$ are subgroups of $\mathrm{PSU}(3) = \mathrm{SU}(3)/\mathbb{Z}_3$. Thus any orientable 8-manifold M^8 admitting a 3-form of ϕ_1-type or of ϕ_2-type admits also a 3-form of ϕ_3-type. In particular M^8 must be orientable and spinnable. Now for any spinnable manifold M^8 we define the characteristic class $q_1(M)$ as follows.

Denote by q_1 the spin characteristic class in $H^4(\mathrm{B\,Spin}(\infty), \mathbb{Z})$ corresponding to $-c_2 \in H^4(\mathrm{BSU}(\infty), \mathbb{Z})$. For any spin-bundle ξ over M we denote by $q_1(\xi)$ the pull-back of q_1. We set $q_1(M) := q_1(TM)$.

As before $\rho_2 : H^2(M^8, \mathbb{Z}) \to H^2(M^8, \mathbb{Z}_2)$ denotes the modulo 2 reduction. The following proposition is essentially a reformulation of Corollary 6.4 in [5].

7.1 Proposition. *A closed orientable 8-manifold M^8 admits a stable 3-form, if and only if it satisfies the following conditions*

(a) $$w_2(M^8) = 0 = e(M^8),$$

(b) $$w_6(M^8) \in \rho(H^6(M^8, \mathbb{Z})),$$

(c) $$p_2(M^8) = -q_1(M^8)^2 \quad \textit{and} \quad \frac{(q_1(M^8))^2}{9}[M^8] = 0 \mod 6.$$

In fact Corollary 6.4 in [5] is formulated as a necessary and sufficient condition for a manifold to admit a PSU(3)-structure. But we have seen that the necessary condition for a manifold M^8 to admit a PSU(3)-structure is also a necessary condition for a manifold to admit a $\mathrm{SL}(3,\mathbb{R})$-structure or a $\mathrm{PSU}(1,2)$-structure.

8. Further remarks

8.1. It is easy to see that our construction of natural bilinear forms works also for 3-forms on space $\mathbb{R}^{3n+2}$. In the same way (this is already noticed first by Bryant for $\mathbb{R}^7$, in [2] this form has been computed for all except one multi-symplectic 3-form) we can associate to any 3-form ω on $\mathbb{R}^{3n+1}$ a bilinear form with values in $\Lambda^{3n+1}(\mathbb{R}^{3n+1})^*$, and it descends to a bilinear form if the 3-form is non-degenerate; we can also associate to any 3-form ω on $\mathbb{R}^{3n}$ a linear map from $\mathbb{R}^{3n}$ to $\mathbb{R}^{3n} \otimes \Lambda^{3n}(\mathbb{R}^{3n})^*$, and this linear map descends to a linear map $\mathbb{R}^{3n} \to \mathbb{R}^{3n}$, if the 3-form ω is non-degenerate (this is noticed by Hitchin for $\mathbb{R}^6$). We have not yet tested, if non-degenerate 3-forms exist in higher dimensions. In low dimensions 6, 7, 8 they coincide with stable forms.

8.2. Let ω^3 be a stable 3-form on M, $\dim M \geq 7$. Then there is the canonical inclusion G_ω to $O(k,l)$. So if a manifold M admits a stable form $\omega^3 \neq \gamma_i$, $i = 1,2$, it also admits a canonical (pseudo)-Riemannian metric. The curvature of this (pseudo)-Riemannian metric is a differential invariant of manifold (M,ω^3). Using these metrics and existing stable forms we can construct new differential forms which appear in other special geometries. Now we shall call a manifold (M,ω^3) stable, if ω^3 is stable. Stable 8-manifolds (M^8,ω^3) seem to us especially interesting, since the bundle TM^8 has the canonical commutative multiplication as well as the structure of Lie algebra bundle defined in Proposition 3.11. We conjecture that the algebra $\mathbb{R}^8$ with the commutative multiplication defined by ϕ_i is a simple algebra. We have a partial proof for that conjecture in the case of ϕ_2. The stable form ϕ_i also defines the volume form on M^8 and therefore according to Djokovic it defines the graded E_8-structure on the bundle $\oplus_{i=1}^3(\Lambda^i(TM) \oplus \Lambda^i(T^*M)) \oplus \mathrm{End}(TM)$.

8.3. Suppose that M is a compact manifold and ω^3 is a stable 3-form on M. As we have seen from 8.2 if $\dim M \geq 7$, then the automorphism group $\mathrm{Aut}(M,\omega^3)$ is a finite dimensional Lie group. If γ_1 is maximal non-integrable, then the automorphism group (M^6,γ_1) is also a finite dimensional Lie group. If γ_1 is degenerate, then the automorphism group $\mathrm{Aut}(M^6,\gamma_1)$ can be infinite dimensional. An example is $M^6 = S^1(\theta^1) \times S^1(\theta^2) \times \Sigma_1 \times \Sigma_2$ and $\omega^3 = d\theta^1 \wedge \sigma^1 + d\theta^2 \wedge \sigma^2$, where σ^i is the volume element on the surface Σ_i. Finally the automorphism group $\mathrm{Aut}(M^6,\gamma_2)$ is also finite dimensional, since $\mathrm{SL}(3,\mathbb{C})$ is elliptic.

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