Archimedean frames, revisited

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For Mel Henriksen, on his 80th birthday.

Abstract. This paper extends the notion of an archimedean frame to frames which are not necessarily algebraic. The new notion is called *joinfitness* and is *Choice-free*. Assuming the Axiom of Choice and for compact normal algebraic frames, the new and the old coincide.

There is a subfunctor from the category of compact normal frames with skeletal maps with joinfit values, which is almost a coreflection. Conditions making it so are briefly discussed.

The concept of an *infinitesimal* element arises naturally, and the join of suitably chosen infinitesimals defines the joinfit nucleus.

The paper concludes with mostly Choice-free applications of these ideas to commutative rings and their radical ideals.

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This research was generated in the course of work on epicompletions in archimedean frames with skeletal maps. The first part of that appears in [MZ07b], and is ongoing in [MZ07c] and [MZ07d]. The archimedean frames seemed sufficiently interesting on their own, to warrant special attention.

The concept of an archimedean lattice was first considered in [M73], in order to mimic the features of an archimedean lattice-ordered group on a frame-theoretic plane. Archimedean frames have since surfaced in a number of recent papers, usually relying upon the spatial nature of the frames involved. In this article we propose a pointfree approach.

To begin, we provide the necessary frame-theoretic background.

1. Frame-theoretic resources

The section begins with a catalogue of basic frame-theoretic vocabulary.

Definition 1.1. This information is provided as background, which the knowledgeable reader should be able to skip almost in its entirety. For additional information, we refer to [J82].

Throughout, A is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in A$, denote the set of elements of A less than or equal to (resp. greater than or equal to) x by $\downarrow x$ (resp. $\uparrow x$).

- A is *algebraic*: each $x \in A$ is a join of compact elements. Throughout, $\mathfrak{k}(A)$ stands for the set of compact elements in the lattice A.
- A has the *finite intersection property* (abbr. *FIP*): for any pair $a, b \in \mathfrak{k}(A)$ it follows that $a \wedge b \in \mathfrak{k}(A)$. Observe that $\mathfrak{k}(A)$ is always closed under taking finite suprema.
- A (algebraic) is *coherent*: 1 is compact and A has the FIP.
- A is a *frame*: the following distributive law holds for all $S \subseteq A$:

$$a \wedge \left(\bigvee S \right) = \bigvee \left\{ a \wedge s \, : \, s \in S \right\}.$$

  It is well known that an algebraic lattice is a frame as long as it is distributive.
- The operation $a \to b$ (in a frame A):

$$a \to b = \bigvee \left\{ x \in A \, : \, a \wedge x \le b \right\}.$$

  This is referred to as the *Heyting operation*.
- Put $x^{\perp} \equiv x \to 0$.

  $p \in A$ a *polar*: it is of the form $p = y^{\perp}$, for some $y \in A$. It is well known that the set $\mathcal{P}A$ of all polars forms a complete boolean algebra, in which infima agree with those in A.
- $a \preceq b$: (in a frame) $b \vee a^{\perp} = 1$; we say that a is *well below* b.
  - $x \in A$ is *regular*: $x = \vee \{ a \in A \, : \, a \preceq x \}$. A is *regular*: each element of A is regular.
  - $\mathrm{Reg}(A)$ denotes the subset of all regular elements of A.
- A frame A is *normal*: whenever $x \vee y = 1$, there exist $u \wedge v = 0$, such that $u \le x$ and $v \le y$, and $1 = x \vee v = u \vee y$.
- Let j be a closure operator j on a frame A.
  - j is *dense*: $j(0) = 0$.
  - $jA \equiv \{ x \in A \, : \, j(x) = x \}$. Note that j is dense if and only if $0 \in jA$.
  - j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$. It is well known that j is a nucleus $\iff$ in jA: $x \in jA$ implies that $a \to x \in jA$, for each $a \in A$.
  - When j is a nucleus, we also say that jA is *nuclear*.

We record a brief comment concerning frame homomorphisms and their adjoints.

Definition & Remarks 1.2. We start in the category $\mathfrak{Frm}$ of all frames and all frame homomorphisms. If $h : A \longrightarrow B$ is a $\mathfrak{Frm}$-morphism, then $h_* : B \longrightarrow A$ denotes its adjoint; that is, the map defined by

$$x \leq h_*(y) \iff h(x) \leq y, \quad \text{for all} \quad x \in A,\ y \in B.$$

The following are well known.

1. h_* preserves all infima.
2. $x \leq h_*(h(x))$, for each $x \in A$, and $h(h_*(y)) \leq y$, for each $y \in B$.
3. $h \cdot h_* \cdot h = h$ and $h_* \cdot h \cdot h_* = h_*$.
4. h is one-to-one if and only if $h_* \cdot h = 1_A$, and h is surjective if and only if $h \cdot h_* = 1_B$.
5. $j \equiv h_* \cdot h$ is a nucleus; jA is isomorphic to the image $h(A)$, and $h|_{jA}$ witnesses this, with inverse $h_*|_{h(A)}$.
6. h is *dense* if $h(x) = 0$ implies $x = 0$. Then the following are equivalent: (i) h is dense; (ii) j is dense; (iii) $h_*(0) = 0$.

We say that h is $*$-*dense* if $h_*(y) = 0$ implies that $y = 0$. Note that this follows if h is surjective, because h_* is one-to-one.

Next, we recall the notion of a skeletal map. The reader familiar with the corresponding terminology in topology, in the sense of [HS68] and [DPR81] will find the frame-theoretic counterpart natural enough.

Definition & Remarks 1.3. The frame homomorphism $h : A \longrightarrow B$ is *skeletal* if $x^{\perp\perp} = 1$ in A implies that $h(x)^{\perp\perp} = 1$. It is easy to verify that h is skeletal if and only if

$$x_1^\perp = x_2^\perp \implies h(x_1)^\perp = h(x_2)^\perp.$$

For convenience we shall say that x is *dense* in a frame, if $x^{\perp\perp} = 1$.

It is also easy to see that h is skeletal precisely when there is a (unique) frame homomorphism $\mathcal{P}(h) : \mathcal{P}A \longrightarrow \mathcal{P}B$ making the diagram below commute:

(1.3.1)
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "p_A"] & B \arrow[d, "p_B"] \\ \mathcal{P}A \arrow[r, "\mathcal{P}(h)"] & \mathcal{P}B \end{tikzcd}$$

In figure (1.3.1), p_A denotes the nucleus defined by $p_A(x) = x^{\perp\perp}$. (We do not decorate the $\perp$s to indicate which frame the complements are taken in.)

As noted in [BaP96], if one considers the subcategory $\mathfrak{FrmS}$ of frames with skeletal maps, then $\mathcal{P}$ turns into a functor — which is, evidently, a reflection ([HS79]).

We conclude this section with two basic observations about skeletal maps, plus a comment regarding the notational convention to be used in designating categories of frames in this paper.

The proof of the next lemma is straightforward, and is left to the reader.

Lemma 1.4. *If $g : A \longrightarrow B$ is any skeletal frame map, then*

$$g_*(0) = \bigvee \left\{ a \in A \,:\, g(a) = 0 \right\}$$

is a polar.

The second observation concerns factorizations of skeletal maps. First, in any category $\mathfrak{C}$ of frames, let us say that $h : A \longrightarrow B$ *has image factorization* if the image hA is a $\mathfrak{C}$-object, and the maps $\bar{h} : A \longrightarrow hA$ and $m : hA \longrightarrow B$ given by $\bar{h}(x) = h(x)$ and $m(y) = y$ are both in $\mathfrak{C}$. $\mathfrak{C}$ *has image factorization* if every morphism in $\mathfrak{C}$ has image factorization.

Lemma 1.5. *Every skeletal map has image factorization.*

Proof: Let $h : A \longrightarrow B$ be skeletal. We use the notation of the preamble to this lemma.

It is clear that hA is a frame, and that $\bar{h}$ is skeletal. To see that the inclusion m is skeletal as well, suppose that $y \in hA$ is dense; let $p = h_*(0)$. Note that $y = h(a)$, for some $a \geq p$; so if $a \wedge b = 0$, then $y \wedge h(b) = 0$, and we have $h(b) = 0$, whence $b \leq p \leq a$, and $b = 0$. This shows that a is dense in A, and so y is dense in B. □

Remark 1.6. We use German script to designate categories; $\mathfrak{Frm}$ stands for the parent category (of frames and frame homomorphisms). Further, we shall observe the following conventions:

- A $\mathfrak{K}$ (generally up front) indicates that the frames in the category question are compact.
- $\mathfrak{Reg}$ and $\mathfrak{N}$ that the frames are regular and, respectively, normal.
- $\mathfrak{Al}$ and $\mathfrak{Ch}$ in the name signals that the frames in the category are algebraic with the FIP and coherent, respectively, and in both instances the assumption is also that the maps of the category are coherent.
- An $\mathfrak{S}$ at the end of the name tells the reader that the morphisms are skeletal.

Thus, for example, $\mathfrak{KNAlS}$ denotes the category of all compact, normal, algebraic frames with FIP, together with skeletal, coherent maps. Lemma 1.5 states that $\mathfrak{FrmS}$ has image factorization.

2. Joinfitness

The objective of this section is to introduce the Choice-free alternative to archimedean frames. Let us first recall the original definition, assuming that A is an algebraic frame. Throughout, for any frame A, $\mathrm{Max}(A)$ denotes the set of all maximal elements of A.

For a discussion of archimedean frames the reader is referred to [MZ03, §6].

Definition 2.1. A is *archimedean* if, for each $c \in \mathfrak{k}(A)$, $\bigwedge \mathrm{Max}(\downarrow c) = 0$.

We begin with a pointfree characterization of archimedean frames, for which we do invoke the Axiom of Choice. For convenience, we record the following definition; the reason for this choice of terms is the resemblance the conditions bear to the well-established notions of "fitness" and "subfitness" in the literature.

Definition 2.2. Let A be a frame. If for each $0 < a \leq b \in A$, there exists a $c \in A$, with $c < b$, such that $b = a \vee c$, we say that A is *joinfit*. If A is algebraic and this condition holds with a, b and c compact, then we say that A is *finitely* joinfit.

It is straightforward that a joinfit algebraic frame is finitely joinfit. The distributive law also permits the following simplification: for A to be joinfit, it suffices that the above definition be satisfied for $b = 1$. Then it is easy to check that if A is compact, algebraic, and finitely joinfit, it is also joinfit.

If a given category $\mathfrak{C}$ is made up of objects which are joinfit, then $\mathfrak{Ar}$ will appear in the acronym for $\mathfrak{C}$. For example, $\mathfrak{NArS}$ stands for the category of normal, joinfit frames, with skeletal maps.

The following proposition and, indeed, all the results in this paper which use the Axiom of Choice, will be labelled with the abbreviation (AC).

Proposition 2.3 (AC)**.** *Suppose that A is an algebraic frame. Then A is archimedean if and only if it is finitely joinfit.*

PROOF: If A is archimedean, and $b \in \mathfrak{k}(A)$, then since $\bigwedge \mathrm{Max}(\downarrow b) = 0$, it follows that if $0 < a \leq b$, then there is an $m \in \mathrm{Max}(\downarrow b)$, such that $a \not\leq m$. Evidently, $a \vee m = b$. If a is compact, then a routine compactness argument establishes that a compact $c < b$ exists such that $b = a \vee c$.

Conversely, suppose A is finitely joinfit, and assume that $b \in A$ is compact. For each $a \in \mathfrak{k}(A)$, satisfying $0 < a \leq b$, let $c \in A$ such that $c < b$ and $a \vee c = b$. In particular, $a \not\leq c$, and appealing to Zorn's Lemma, as well as the compactness of b, there is an $m \in \mathrm{Max}(\downarrow b)$, with $c \leq m$, such that $a \not\leq m$. Since A is algebraic, this shows that A is archimedean. $\square$

We now proceed to characterize joinfitness in terms of the regular coreflection. We find the relationship between joinfit frames and the regular frames, through this coreflection, very striking. We preface this discussion with a review of the regular coreflection.

Definition & Remarks 2.4. A denotes an arbitrary frame.

It is well known that the subframe generated by any collection of regular subframes of A is again regular. Thus A has a largest regular subframe ρA. Since the image under a frame homomorphism of a regular frame is also regular, it follows that any frame map $g : F \longrightarrow A$, with F regular, must factor through ρA. It is then easily seen that ρ thus defines a monocoreflection of $\mathfrak{Frm}$ in $\mathfrak{Reg}$, the subcategory of all regular frames.

$\rho_A : \rho A \longrightarrow A$ will denote the canonical inclusion.

Evidently, each $x \in \rho A$ is regular, so that $\rho A \subseteq \mathrm{Reg}(A)$, but the two need not be the same. If the relation $\preceq$ *interpolates*, in the sense that

$$a \preceq b \implies \exists\, c \in A,\ a \preceq c \preceq b,$$

then it is well known that $\mathrm{Reg}(A) = \rho A$. In particular, this occurs if A is a normal frame.

We have now the promised characterization of joinfitness.

Proposition 2.5. *If ρA is $*$-dense in A then A is joinfit. If A is a normal frame the converse holds.*

Proof: Suppose that ρA is $*$-dense in A, and $0 < a \in A$. Then there is an $x > 0$ in A well below a; thus, there is a $y \in A$, disjoint from x, such that $a \vee y = 1$, and it is easy to show that $y < 1$. This witnesses the joinfitness of A.

Assume now that A is joinfit, normal, and that $0 < a \in A$. Pick $b < 1$ such that $a \vee b = 1$. Use the normality of A to find disjoint c and d such that $1 = a \vee c = d \vee b$, with $c \leq b$ and $d \leq a$. Then observe that $0 < d \preceq a$, whence

$$0 < (\rho_A)_*(a) = \bigvee \left\{ x \in A \,:\, x \preceq a \right\} \leq a,$$

proving that ρA is $*$-dense in A. □

Example 6.2 shows that the converse in Proposition 2.5 is hopeless without normality.

We now have the following characterization of archimedean frames.

Proposition 2.6 (AC)**.** *Suppose A is a compact normal algebraic frame. Then the following are equivalent.*

(a) *A is archimedean.*
(b) *ρA is $*$-dense in A.*
(c) *A is joinfit.*
(d) *A is finitely joinfit.*

Property (b) of Proposition 2.6 has an important consequence, using Lemma 2.2 and Theorem 2.3 of [HM07].

Corollary 2.7. *Suppose that A is a frame with ρA $*$-dense in A. Then ρ_A is skeletal and induces an isomorphism $\mathcal{P}(\rho_A)$.*

We record a curious — and, in [MZ07c], curiously important — property of the adjoint $(\rho_A)_*$ of the inclusion ρ_A. We emphasize that, in general, the adjoint of a frame homomorphism is not join-preserving.

Proposition 2.8. *Suppose that A is a normal frame. Then $(\rho_A)_*$ preserves finite joins, and, if A is compact, it also preserves suprema of updirected sets, and, consequently, is a frame homomorphism.*

PROOF: Let us abbreviate $r \equiv (\rho_A)_*$, for purposes of this proof. Since it is clear that $r(0) = 0$, it suffices to show that r preserves binary suprema, and to accomplish that it is enough to show that if $x \preceq a \vee b$ then there exist $s, t \in A$ such that $x \preceq s \vee t$, with $s \preceq a$ and $t \preceq b$. Now, if $x \preceq a \vee b$, there is a $y \in A$, disjoint from x, such that $y \vee a \vee b = 1$. Using normality, one produces disjoint u and t such that $u \leq y \vee a$, $t \leq b$, and

$$1 = u \vee b = y \vee a \vee t.$$

But then $t \preceq b$ and $x \preceq a \vee t$. Repeating the above moves, there is an s such that $s \preceq a$ and $x \preceq s \vee t$, as desired.

Next, suppose that A is also compact, and let S be an updirected set and $x \preceq \bigvee S$. Then $(\bigvee S) \vee x^\perp = 1$, and owing to the compactness, we have that $y \vee x^\perp = 1$, for some $y \in S$, whence $x \preceq y$. This shows that $r(\bigvee S) = \bigvee r(S)$, thus completing the proof. □

There is a converse to the preceding proposition, and we record the following remark in advance of it, to underscore the importance of a frame embedding $h : A \longrightarrow B$ having an adjoint which is a frame homomorphism. The reader may refer to [M07] for more on the subject.

Remark 2.9. A frame embedding $h : A \longrightarrow B$ whose adjoint has the feature that

$$x \vee y = 1 \implies h_*(x) \vee h_*(y) = 1,$$

is said to be a *capping of B by A*. It is shown in [M07, Lemma 1.3] that if h is such a capping, then A is normal if and only if B is normal.

We are now able to state the following.

Proposition 2.10. *Suppose that A is a frame. Then it is compact normal and joinfit if and only if there is an embedding $m : R \longrightarrow A$ with the following properties:*

- *R is compact and regular;*
- *m is a $*$-dense embedding;*
- *m_* is a frame homomorphism.*

Proof: That the conditions are necessary is accounted for by Proposition 2.8, with $R = \rho A$ and $m = \rho_A$.

As to the sufficiency, suppose there is such an embedding $m : R \longrightarrow A$. By Remark 2.9, A is normal. It is routine to show that A is compact, but note that it uses the full force of the hypothesis that m_* is a frame homomorphism. Finally, the joinfitness of A is proved as in the first part of the proof of Proposition 2.5. □

Proposition 2.10 provides all the motivation for the theorem that follows, which is the main theorem of the section. For the remainder of this section, let A stand for a normal frame. We also leave off the subscript A on the adjoint ρ_*.

Theorem 2.11. *Put*

$$\varphi A \equiv \left\{ x \in A \, : \, \rho_*(x)^{\perp} = x^{\perp} \right\}.$$

Then we have the following.

(a) *φA is a joinfit subframe of A, in which ρA is skeletally embedded.*
(b) *Suppose that B is a normal frame and $h : A \longrightarrow B$ is a skeletal frame map. Then the restriction h' of h maps φA into φB. h' is the unique frame map making the square below commute.*

(2.11.1)
$$\begin{tikzcd} \varphi A \arrow[r] \arrow[d, "h'"] & A \arrow[d, "h"] \\ \varphi B \arrow[r] & B \end{tikzcd}$$

(The unlabeled horizontal maps are inclusions.)

(c) *If L is a skeletally embedded, normal joinfit subframe of A, then $L \subseteq \varphi A$.*
(d) *φ defines a functor from the category $\mathfrak{NS}$ of normal frames with skeletal frame maps into the category $\mathfrak{Ar}$ of joinfit frames and all frame maps.*
(e) *If A is also compact, then φA is normal (and compact).*
(f) *Assume A is algebraic. Then*

$$\varphi A = \{ x \in A \, : \, \forall\, 0 < b \in \mathfrak{k}(A),\ b \leq x,\ \exists\, 0 < c \in \mathfrak{k}(A),\ c \leq b,\ c \preceq x \}.$$

Proof: To begin, let us observe that

$$\varphi A = \left\{ x \in A \, : \, \rho_*(x) \leq x \leq \rho_*(x)^{\perp\perp} \right\}.$$

Then note that we may as well assume that $0 < x \in \varphi A$ in the sequel. As ρA is regular, there is a $y \in \rho A$ such that $0 < y \preceq \rho_*(x)$. Then there is an $a \in \rho A$ with $y \wedge a = 0$ and $a < 1$, such that

$$1 = \rho_*(x) \vee a \leq x \vee a,$$

and it follows that if we can prove that φA is a subframe, then it will be joinfit.

As to that question, suppose that $S \subseteq \varphi A$; we need only show that $\bigvee S \leq \rho_*(\bigvee S)^{\perp\perp}$. To that end, observe that

$$\bigvee S \leq \bigvee \left\{ \rho_*(y)^{\perp\perp} \,:\, y \in S \right\} \leq \Big(\bigvee \left\{ \rho_*(y) \,:\, y \in S \right\} \Big)^{\perp\perp} \leq \rho_* \Big(\bigvee S \Big)^{\perp\perp},$$

whence $\bigvee S \in \varphi A$.

Further, suppose $u, v \in \varphi A$, and $\rho_*(u \wedge v) \wedge w = 0$. Then $\rho_*(u) \wedge \rho_*(v) \wedge w = 0$, so that $u \wedge \rho_*(v) \wedge w = 0$, and $u \wedge v \wedge w = 0$, proving that $u \wedge v \in \varphi A$, and thus that φA is indeed a subframe.

Finally, regarding (a) in the theorem, if $a \in \rho A$ is dense in ρA, and $a \wedge z = 0$, with $z \in \varphi A$, then $a \wedge \rho_*(z) = 0$, and so $\rho_*(z) = 0$, whence $z = 0$, which proves that ρA is skeletally embedded in φA.

For (b), the reader should observe that it suffices that $c \leq x \leq c^{\perp\perp}$, with $c \in \rho A$, to be able to deduce that $x \in \varphi A$. That, together with the assumption that h is skeletal and the fact that h carries ρA into ρB, implies the assertion about h'. Clearly, the diagram (2.11.1) commutes and h' is unique in this respect.

Regarding (c), if L is a skeletally embedded normal joinfit subframe of A, then by Proposition 2.5, $L = \varphi L$, and we may apply (b) to the inclusion of L in A.

The uniqueness of h' in (b) makes (d) an easy exercise, which is left to the reader.

Assume now that A is compact. To show that φA is normal, use Proposition 2.8: suppose $a \vee b = 1$ in φA; then $\rho_*(a) \vee \rho_*(b) = 1$, in ρA, which is compact and therefore normal. Thus, the witnesses to the normality, $u, v \in \rho A$, for $\rho_*(a)$ and $\rho_*(b)$ also witness for a and b. This proves (e).

Finally, we sketch (f). If $x \neq 0$ lies in φA, then $x \leq \rho_*(x)^{\perp\perp}$. This says that if $0 < b \leq x$ is compact, $b \wedge \rho_*(x) > 0$, so that a compact $c > 0$ exists such that $c \leq b \wedge \rho_*(x)$, and by [MZ07a, Lemma 1.4], necessarily, $c \preceq x$. The converse is almost a reversal of these steps, which is left to the reader. □

Remark 2.12. The preceding theorem is as interesting as it is disappointing. It appears to describe a coreflection, except for the following related issues. Here it is assumed that A is compact and normal.

- To begin, we do not seem to get φA to be normal without the assumption of compactness on A.
- The inclusion of φA in A need not be skeletal. Indeed, it is easy to see that φA is skeletally embedded in A precisely when ρA is skeletally embedded

in A. The latter occurs, by Corollary 2.7, if A is joinfit, in which case $\varphi A = A$.

However, ρA may be skeletally embedded without A being joinfit. Let A be the three-element frame in which $0 < a < 1$. This is clearly compact and normal, and $\rho A = \varphi A$ and skeletally embedded in A.

On the other hand, as Example 6.10 demonstrates, it is not difficult to construct an example in which A is coherent and ρA coincides with φA, yet is not skeletally embedded. The frames in which the regular coreflection lies skeletally will be studied elsewhere.

- Last, in Theorem 2.11(b), the restriction h' need not be skeletal. It is skeletal if ρA is skeletally embedded in A; we leave the verification to the reader.

In view of the preceding remarks, here is the best result we are prepared to formulate. The reader might also review the remarks in 1.6, on the naming of categories in this paper. We let $\mathfrak{KNS}^2$ designate the category of all compact normal frames in which the regular coreflection is skeletally embedded, together with all skeletal frame maps.

Corollary 2.13. *On the category $\mathfrak{KNS}^2$, the functor φ defines a monocoreflection in the full subcategory $\mathfrak{KNArS}$ of joinfit frames.*

Proof: If A is joinfit as well as compact and normal, and $h : A \longrightarrow B$ is a skeletal frame map into the $\mathfrak{KNS}^2$-object B, then by Theorem 2.11, h factorizes as $\rho'_B \cdot \varphi(h)$, with skeletal factors. $\square$

Regardless of whether the normal frame A lies in $\mathfrak{KNS}^2$ or not, we shall refer to φA as the *joinfit coreflection* of A, and to the functor φ as the joinfit coreflection.

3. Categories of joinfit frames: factorizations

We examine the behavior of joinfitness under formation of frame quotients in several categories of joinfit frames with skeletal maps. Some of these features might be expected by the reader who is familiar with [M73] and archimedean lattice-ordered groups; other properties may surprise. What bears underscoring is the pointfree and Choice-free approach.

The reader should note that the following proposition is false without the hypothesis of normality (Example 6.2). It is also worth pointing out what is not assumed here: the frames in question are not necessarily compact, so that Theorem 2.11 does not apply.

Proposition 3.1. *Suppose $g : A \longrightarrow B$ is a surjective skeletal frame map, and that A is normal. If A is joinfit then so is B.*

Proof: We use Proposition 2.5. For brevity, let us put $k = g_*(0)$. Now suppose $x > 0$ in B, and pick $a \in A$ such that $g(a) = x$. Observe that $a \not\leq k$, so that $c = a \wedge k^{\perp} > 0$, since $k \in \mathcal{P}A$. Since ρA is $*$-dense in A, there exists $0 < d \preceq c$. Finally, note that because $d \leq k^{\perp}$, $g(d) > 0$, and it is well known that

$$g(d) \preceq g(c) \leq g(a) = x,$$

which proves that ρB is $*$-dense in B. □

Remark 3.2. (a) If in Proposition 3.1 the map g is a so-called *closed quotient* of A — i.e., the induced frame map $x \vee g_*(0) \mapsto g(x)$ is an isomorphism — then B is joinfit without the additional assumption of normality on A.

As in the preceding proof, denote $k = g_*(0)$, and suppose $k < a$. Then there is a $b < 1$ such that $(a \wedge k^{\perp}) \vee b = 1$. Were $b \vee k = 1$, then we would have $b \geq k^{\perp} \geq a \wedge k^{\perp}$, which cannot be, since $b < 1$. Thus, $b \vee k < 1$ and this element witnesses the joinfitness for a in $\uparrow k \cong B$.

(b) We are unable to decide, but doubt, that with the hypotheses of Proposition 3.1 one can conclude that B is also normal. Nor, in a related question, are we able to settle whether the category $\mathfrak{NArS}$ has image factorization. Neither $\mathfrak{KArS}$ nor $\mathfrak{KArChS}$ have image factorization; see 6.2.

We turn now to a property enjoyed by frames of convex ℓ-subgroups of a lattice-ordered group, namely, disjointification. It is closely related to normality, and in compact algebraic frames implies normality.

Definition & Remarks 3.3. Let A be an algebraic frame. We say that A has the *disjointification property* (or, simply, that A is a frame *with disjointification*) if for each pair of compact elements $a, b \in A$ there exist disjoint $c, d \in \mathfrak{k}(A)$ such that $c \leq a$ and $d \leq b$, and $a \vee b = a \vee d = c \vee b$.

It is well known that A has disjointification if and only if $\downarrow a$ is a normal frame, for each $a \in \mathfrak{k}(A)$. Hence the reason that the disjointification property is alternatively referred to as *relative normality* (such as in [ST93]), and as *coherent normality* (in [Ba97]).

We denote the category of algebraic frames having the disjointification, with coherent maps, by $\mathfrak{Dj}$. It is a routine exercise to show that if $h : A \longrightarrow B$ is a surjective coherent frame homomorphism between algebraic frames, then if A has disjointification, so does B. Thus, $\mathfrak{Dj}$ has image factorization.

Our aim in this section is to show that $\mathfrak{DjArS}$, the category of algebraic joinfit frames with the FIP and disjointification, and skeletal, coherent maps, has image factorization.

Proposition 3.4. *Suppose that $h : A \longrightarrow B$ is a surjective coherent frame homomorphism between algebraic frames, which is also skeletal. If A is joinfit, with disjointification, then B has the same properties.*

Proof: The only thing that is required is a proof of the joinfitness of B. And that proof is similar in spirit to that of Proposition 3.1.

Let $k = h_*(0)$; this is a polar, according to Lemma 1.4. Suppose that $0 < y < z$ in $\mathfrak{k}(B)$; there exist compact elements of A, $0 < a < b$, such that $y = h(a)$ and $z = h(b)$. Now $a \not\leq k$, so that $a \wedge k^{\perp} > 0$, and there exists a $d \in \mathfrak{k}(A)$ such that $0 < d \leq a \wedge k^{\perp}$. Since A is joinfit, one may find $c < b$, with c compact, such that $d \vee c = b$. Owing to the disjointification in A, there exist disjoint u and v in A, with $u \leq d$ and $v \leq c$, such that $b = d \vee v = u \vee c$. Then

$$z = h(u) \vee h(c) = h(d) \vee h(v) = y \vee h(v).$$

We claim that $h(v) < z$. Else, if $h(v) = z$, then since $h(u)$ is disjoint to $h(v)$, it follows that $h(u) = 0$, which means that $u \leq k$. Coupled with the fact that $u \leq k^{\perp}$, this implies that $u = 0$, whence $b = c$, a contradiction. Thus, $h(v) < z$, proving that B is joinfit. $\square$

We have the desired result as an immediate consequence of this proposition and Lemma 1.5.

Corollary 3.5. $\mathfrak{DjArS}$ *has image factorization.*

We conclude this section with a comment for the reader steeped in category theory.

Remark 3.6. According to the dual of Theorem 37.1 of [HS79], in a category $\mathfrak{C}$ of frames having image factorization as well as coproducts, the full subcategory $\mathfrak{B}$ is monocoreflective relative to embeddings if and only if $\mathfrak{B}$ is closed under forming coproducts and quotients of $\mathfrak{C}$.

We will take up coproducts of frames with skeletal maps in [MZ07c]. It is shown there that joinfitness is closed under the formation of coproducts. Yet, our present joinfit coreflection is not recovered using this categorical approach, as there are problems with coproducts of normal frames, even assuming compactness.

4. The joinfit nucleus

According to Proposition 2.3, joinfitness is, roughly speaking, archimedeaneity without the privilege of Choice. Next, we introduce the "joinfit" nucleus; borrowing language from universal algebra, we say that $x \in A$ is *residually joinfit* if $\uparrow x$ is joinfit.

The reader is urged to compare residually joinfit elements with the upper-archimedean elements introduced in [MZ03], and, in particular, to juxtapose [MZ03, Lemma 6.2] and the following proposition.

Proposition 4.1. *Suppose A is any frame. Then*

(a) *the meet of residually joinfit elements of A is residually joinfit;*
(b) *if $a \in A$ and b is residually joinfit, then $a \to b$ is residually joinfit.*

Proof: (a) Let S be a set of residually joinfit elements of A, and $x \equiv \bigwedge S$. Suppose $x < a$; we require a c, with $x \leq c < 1$, such that $a \vee c = 1$. Now, there is a $y \in S$ such that $a \not\leq y$, and therefore, since $\uparrow y$ is joinfit, we have c, with $x \leq y \leq c < 1$, such that $1 = (a \vee y) \vee c = a \vee c$.

(b) Set $d = a \to b$. By the remark in 3.2(a), it suffices to show that the map $x \mapsto x \vee d$ from $\uparrow b \longrightarrow \uparrow d$ is a skeletal frame homomorphism. That it is a frame map is clear. Regarding the skeletal feature, suppose that $y \wedge z \leq b$ implies that $z \leq b$; it must be shown that whenever $(y \vee d) \wedge z \leq d$, it follows that $z \leq d$.

Now, if $(y \vee d) \wedge z \leq d$, we have

$$a \wedge y \wedge z \leq a \wedge [(y \vee d) \wedge z] \leq a \wedge d \leq b.$$

Therefore, $a \wedge z \leq b$, and it follows that $z \leq d$, as promised. □

The following is an immediate consequence of the foregoing.

Corollary 4.2. *If A is a joinfit frame, then every polar is residually joinfit.*

We are now able to define the "joinfit closure".

Definition & Remarks 4.3. Suppose A is a frame. For each $x \in A$ let $\mathfrak{j}(x)$ denote the least residually joinfit element of A exceeding x. By (a) in Proposition 4.1, $\mathfrak{j}$ is well defined and a closure operator, and (b) of that proposition insures that $\mathfrak{j}$ is a nucleus.

We wish to describe $\mathfrak{j}$ from below. We record a definition first, which ought to appeal to the reader's intuition about archimedeaneity. Suppose A is a frame and $s \in A$; we say that s is (an) *infinitesimal* if $s \vee x = 1$ implies that $x = 1$. It is immediate that infinitesimals are < 1. Further, the reader will easily be able to verify that

1. if $s' \leq s$, and s is infinitesimal, then s' is infinitesimal;
2. if $s, t \in A$ are infinitesimals, then so is $s \vee t$.

Note the obvious: that 0 is infinitesimal in any frame A, and the only infinitesimal if and only if A is joinfit. The first two properties of infinitesimals signify that the set $\inf(A)$ of all infinitesimals is a proper ideal of A.

Theorem 4.4. *Suppose that A is a frame. For each $a \in A$,*

$$\mathfrak{j}(a) \geq \bigvee \inf(\uparrow a).$$

If A is compact equality holds.

Proof: Denote $z = \mathfrak{j}(a)$ and $z' = \bigvee \inf(\uparrow a)$. Suppose $a \leq x$, with x infinitesimal in $\uparrow a$, yet $x \not\leq z$; necessarily, $z < 1$. Then it is easy to verify that $x \vee z < 1$ and that it is infinitesimal in $\uparrow z$, which contradicts that z is residually joinfit. This establishes that $z \geq z'$.

Now suppose that A is compact. Then z' is the largest infinitesimal in $\uparrow a$; a routine covering argument will verify this. So if $z > z'$ then, by the definition of z, there is a $u > z' \geq a$ such that $u \in \inf(\uparrow z')$. Then if $b \geq a$ and $b \vee u = 1$ then $b \vee z' \vee u = 1$, which implies that $b \vee z' = 1$, and, finally, that $b = 1$, proving that u is infinitesimal in $\uparrow a$, a contradiction. Therefore, $z = z'$, and the proof is complete. □

We remind the reader of the so-called *saturation* nucleus s on a compact frame A: $s(a)$ is the supremum of all $x \in A$ such that

$$x \vee y = 1 \implies a \vee y = 1.$$

The compactness of A then insures that $s(a) \vee y = 1$ implies that $a \vee y = 1$.

We then have the following corollary.

Corollary 4.5. *For any compact frame,* $\mathfrak{j} = s$.

Proof: If x is an infinitesimal above a, then (trivially), $x \vee y = 1$ implies that $a \vee y = 1$, which shows that $\mathfrak{j}(a) \leq s(a)$. If the inequality were strict, then, since $\mathfrak{j}(a)$ is residually joinfit, there is a $b \in A$, with $a \leq \mathfrak{j}(a) \leq b < 1$, such that $s(a) \vee b = 1$, which is nonsense. □

To the foregoing corollary we append the following remark, which uses comments of Banaschewski from [Ba02, §2], and, hopefully, will uncomplicate those of [MZ07a, 6.5].

Remark 4.6. Assume that A is a compact normal frame. Recall that the adjoint $\rho_* : A \longrightarrow \rho A$ is a frame homomorphism; it therefore makes sense to consider $t \equiv (\rho_*)_*$. Note that, for each $x \in \rho A$,

1. $$t(x) = \bigvee \Big\{ y \in A : \rho_*(y) = x \Big\}.$$
2. If $x \leq y \leq t(x)$, and $y \vee q = 1$ (in A), then
$$1 = x \vee \rho_*(q) \leq x \vee q,$$
whence $t(x) \leq s(x)$. On the other hand, by Corollary 4.5, $s(x) = \mathfrak{j}(x)$. Now, if $y \geq x$ is an infinitesimal in $\uparrow x$, then $t(y)$ is easily seen to be infinitesimal in $\{ a \in \rho A : a \geq x \}$, which implies that $t(y) = x$, since ρA is regular — by Proposition 5.3, if need be. This proves that $s(x) \leq t(x)$, and, thus, that $t = s|_{\rho A}$.
3. Banaschewski observes in [Ba02, 2.5] that t, viewed now as a map between ρA and the fixed set sA, is an isomorphism. The inverse is shown there to be $\rho_*|_{sA}$.

The reader, perhaps, has already recognized that, in the context of commutative rings with identity, $\mathfrak{j}(0)$ is the Jacobson radical. We comment briefly on that in §6.

Finally in this section, we outline what to do to describe the joinfit nucleus from within, when the frame A is algebraic but not necessarily compact, in order to "correct" the inequality in Theorem 4.4. Since we have relegated the applications of the material in this paper to ℓ-groups to another writing, there is no need to supply the details here.

Definition & Remarks 4.7. In this commentary A denotes an algebraic frame.

Let $a < b$ be compact elements. We say that a is *infinitesimal to* b, and write $a \ll b$, if $a \vee z = b$ implies that $z = b$. Note that, in view of the compactness of b, it suffices, when checking that $a \ll b$, to take z compact.

To simplify the notation, when $a \ll b$ in the quotient $\uparrow x$, we shall write $a \ll b\ (x)$.

Now here is the generalization of Theorem 4.4 we had in mind. The reader should recall that an algebraic frame which is joinfit is also finitely joinfit (2.2).

Theorem 4.8. *Suppose that A is an algebraic frame. Then, for each $a \in \mathfrak{k}(A)$,*

$$\mathfrak{j}(a) = \bigvee \Big\{ c \in \mathfrak{k}(A) \,:\, c \geq a,\ \exists\, b \in \mathfrak{k}(A),\ c \ll b\ (a) \Big\}.$$

Proof (Sketch): Once again, set $z = \mathfrak{j}(a)$ and

$$z' = \bigvee \Big\{ c \in \mathfrak{k}(A) \,:\, c \geq a,\ \exists\, b \in \mathfrak{k}(A),\ c \ll b\ (a) \Big\}.$$

To prove that $z \geq z'$, pick a compact $c \geq a$ and $b \in \mathfrak{k}(A)$ such that $c \ll b\ (a)$. If $c \nleq z$ and $a \leq d \in \mathfrak{k}(A)$, then the fact that $c \ll b\ (a)$ leads to the strict inequalities $z < c \vee z < b \vee z$, and eventually to $c \vee z \ll b \vee z\ (z)$, which contradicts that $\uparrow z$ is finitely joinfit. Thus, $z \geq z'$, as desired.

If $z > z'$, then there exist compact elements $a \leq c < b$ such that $z' < c \vee z'$ and $c \vee z' \ll b \vee z'\ (z')$. It is then straightforward that $c \ll b\ (a)$, which means that $c \leq z'$, and this is a contradiction. $\square$

5. "Hyper"- properties

Having examined the passage of joinfitness under selected surjective frame homomorphisms, we now proceed to the study of the opposite: frames for which every frame-homomorphic image is joinfit. The better known "fitness" properties now enter the picture; let us begin by reviewing them.

Definition & Remarks 5.1. Suppose that A is a frame. We say that A is

- *fit* if $a < b$ in A implies that there is a $z \in A$ such that $b \vee z = 1$ and $z \to a > a$;
- *subfit* if $a < b$ in A implies that there is a $z \in A$ such that $b \vee z = 1$ and $a \vee z < 1$.

Of the following, the first three are well known; see [PT01, p. 82]. The fourth is immediate from the definitions.

1. Every regular frame is fit.
2. A frame is fit if and only if every homomorphic image is subfit.
3. Every normal subfit frame is regular.
4. Every subfit frame is joinfit.

One should also observe that, without the assumption of normality, a subfit frame need not be regular: let X be an infinite set with the finite-complement topology; that is, the frame of open sets $\mathfrak{O}(X)$ is the collection of cofinite subsets. It is easy to see that $\mathfrak{O}(X)$ is fit, but not regular (and not normal).

In a concrete category $\mathfrak{C}$, let $\mathcal{P}$ be a property of objects in $\mathfrak{C}$. A $\mathfrak{C}$-object X is *hyper-*$\mathcal{P}$ if for each surjective $\mathfrak{C}$-morphism $g : X \longrightarrow Y$, Y has property $\mathcal{P}$.

Thus, regarding 5.1.2, the fit frames are the hypersubfit frames. In the sequel, the term "k-hyper-$\mathcal{P}$" refers to the property hyper-$\mathcal{P}$ restricted to closed surjective frame maps. The lemma which follows immediately gives the proposition that comes after.

Lemma 5.2. *If the frame A is k-hyperjoinfit, then it is subfit.*

PROOF: Suppose $a < b$ in A. In $\uparrow a$, which is joinfit by hypothesis, we have $a \leq u < 1$ such that $b \vee u = 1$. Then note that u witnesses the subfitness for the pair $a < b$. □

Proposition 5.3. *Suppose A is a frame; then the following are equivalent.*

(a) *A is hyperjoinfit.*
(b) *A is hypersubfit.*
(c) *A is fit.*

6. Applications to commutative rings

We consider commutative rings with identity. When A is such a ring, we denote by $\mathrm{Rad}(A)$ the frame of all radical ideals of A. Most of the rings exhibited here are, in fact, *semiprime*; that is, they have no nonzero nilpotent elements.

Definition & Remarks 6.1. Throughout these remarks A denotes a semiprime commutative ring with identity. To review, the ideal $\mathfrak{r}$ of A is *radical* if $x^2 \in \mathfrak{r}$ implies that $x \in \mathfrak{r}$. It is well known that $\mathrm{Rad}(A)$ is a coherent frame. Throughout this section $\langle S\rangle^A_{\mathrm{Rad}}$ denotes the radical ideal of A generated by $S \subseteq A$. We will omit the superscript when the ring of discourse is clear.

By a theorem of Hochster ([Ho69]), every coherent frame arises as $\mathrm{Rad}(A)$, for a suitable commutative ring A. It is Banaschewski, however, who in [Ba96] gives a Choice-free proof of this result.

(a) Now suppose $A \le B$ is a ring extension. Suppose that $\mathfrak{r} \in \mathrm{Rad}(A)$. Let $\varepsilon(\mathfrak{r}) = \langle B\mathfrak{r}\rangle^B_{\mathrm{Rad}}$; it is easily checked that $\varepsilon(\mathfrak{r})$ is the least radical ideal of B containing $\mathfrak{r}$. ε is a dense coherent frame map. Moreover,

- the adjoint $\varepsilon_* = \tau$, where $\tau(\mathfrak{s}) = \mathfrak{s} \cap A$, for each $\mathfrak{s} \in \mathrm{Rad}(B)$;
- let $j_{\mathrm{rad}} = \tau \cdot \varepsilon$; then $\mathfrak{r} \in j_{\mathrm{rad}}\,\mathrm{Rad}(A)$ if and only if

$$a^k = \sum_{i=1}^{m} b_i r_i, \quad \text{with} \quad b_i \in B,\ r_i \in \mathfrak{r}, \implies a \in \mathfrak{r};$$

ε and τ are referred to as *extension* and *trace*, respectively.

(b) An ideal $\mathfrak{r}$ of A is *dense* if $x\mathfrak{r} = \{0\}$ implies that $x = 0$. If the ring B extends A, then it is routine to verify that ε is skeletal if and only if the extension of each dense ideal of A is dense. Further, as is noted in [HM07], ε is $*$-dense precisely when B is a ring of quotients in the sense of Utumi ([U56]).

The following example is mentioned a number of times in §3.

Example 6.2. Let $A = \mathrm{Rad}(\mathbb{Z})$. As is well known, A consists of $\mathbb{Z}$ itself, $\{0\}$, and the ideals generated by the square-free integers. Every element of A is compact. Furthermore, it is easy to verify that A is joinfit. However, A is not normal, and, in fact, ρA is the frame of two elements, and, therefore, far from being $*$-dense in A. Thus, Proposition 2.5 fails without normality.

Moreover, consider the localization $\mathbb{Z}_{(2)}$ of all rational numbers with odd denominator. Let $B = \mathrm{Rad}(\mathbb{Z}_{(2)})$ and consider the extension $\varepsilon : A \longrightarrow B$. Note that B is the three-element frame which is not joinfit. Yet ε is a skeletal surjection, and this shows that Proposition 3.1 fails without the normality on the domain of the map. It also establishes that $\mathfrak{KArS}$ does not have image factorization: for in this case τ is a frame homomorphism, and because the rings in question are integral domains, τ is skeletal; thus $\tau \cdot \varepsilon$ witnesses the failure of image factorization in $\mathfrak{KArS}$, as well as in $\mathfrak{KArChS}$.

Remark 6.3. Let R be a commutative ring with 1, and $r \in R$. It is easy to check that $\langle\{r\}\rangle_{\mathrm{Rad}} \in \mathfrak{J}(R) \equiv \mathfrak{j}(\langle 0\rangle_{\mathrm{Rad}})$ if and only if $1 - rs$ is a multiplicative unit, for each $s \in R$. This is the classical condition which spells out when r belongs to the Jacobson radical $\mathfrak{J}(R)$. With Choice one also gets that $\mathfrak{J}(R)$ is the intersection of all the maximal ideals of R.

The regular elements of $\mathrm{Rad}(R)$ are the pure ideals of the ring R. Our references for this material are De Marco's [DM83] and [MZ07a].

Definition & Remarks 6.4. Let R be a commutative ring with identity. An ideal $\mathfrak{r}$ of R is *pure* if for each $a \in \mathfrak{r}$ there is a $c \in \mathfrak{r}$ such that $ca = a$. It is easy to see that if R is semiprime then every pure ideal is a radical ideal.

With Theorem 2.11(f) in mind, let us agree to call an ideal $\mathfrak{r}$ of R a *Gelfand ideal* if it contains a pure ideal densely, or, equivalently (per Lemma 6.5 below), if $\mathfrak{r} \in \varphi\,\mathrm{Rad}(R)$.

For the discussion ahead, it is also convenient to recall when $\mathrm{Rad}(R)$ is a normal frame. With the assumption of Choice, this happens precisely when every prime ideal is contained in a unique maximal ideal. These are referred to as the *pm-rings*. Without appealing to Choice, the normality of $\mathrm{Rad}(R)$ is rendered as follows: *If* $1 = x + y$ *in* R, *there exist* $r, s \in R$ *such that* $(1 - rx)(1 - sy) = 0$. Such rings go under the name *Gelfand ring*.

We have the following lemma.

Lemma 6.5. *Suppose that R is a semiprime commutative ring with identity and that $\mathfrak{r} \in \mathrm{Rad}(R)$. Then $\mathfrak{r}$ is regular if and only if it is pure.*

Proof: Suppose first that $\mathfrak{r}$ is pure and that $a \in \mathfrak{r}$. Then, for a suitable $c \in \mathfrak{r}$, $1 - c \in a^{\perp}$, so that $R = \mathfrak{r} + a^{\perp}$. We conclude that $a \in \langle c\rangle_{\mathrm{Rad}} \preceq \mathfrak{r}$, which implies that $\mathfrak{r}$ is regular.

Conversely, suppose that $\mathfrak{r}$ is regular. Pick $a \in \mathfrak{r}$, and let $c_1, \dots, c_k \in \mathfrak{r}$ such that $a \in \langle c_1, \dots, c_k\rangle_{\mathrm{Rad}} \preceq \mathfrak{r}$. Then $1 = x + y$, with $x \in \mathfrak{r}$ and $yc_i = 0$, for each $i = 1, \dots, k$. Since some power $a^n = r_1c_1 + \cdots + r_kc_k$, for suitable $r_i \in R$, we have $ya^n = 0$, and since R is semiprime, $ya = 0$, and $a = ax$, as desired. □

The preceding lemma and the remarks following Corollary 4.5 yield the following.

Corollary 6.6 (AC)**.** *Suppose R is a semiprime, Gelfand commutative ring with identity. The map ρ_*, restricted to the frame of ideals which are intersections of maximal ideals, defines an isomorphism onto the frame of pure ideals of R.*

Proof: Use the lemma, the remarks in 4.6, and the fact that the saturated ideals are precisely the intersections of maximal ideals ([Ba02, 2.1.3]). □

Lemma 6.5 and Proposition 2.5, together, also have the following consequence.

Corollary 6.7. *Suppose that R is a semiprime commutative ring with identity. If each nonzero radical ideal contains a nonzero pure ideal, then $\mathfrak{J}(R) = \{0\}$. The converse holds if R is also a Gelfand ring.*

Further, Theorem 2.11(f) has the following interpretation.

Corollary 6.8. *Suppose that R is a semiprime commutative Gelfand ring with* 1. *Then the radical ideal $\mathfrak{r}$ of R is Gelfand if and only if for each $0 \neq a \in \mathfrak{r}$ there exists a $b \in \mathfrak{r}$ such that $ab \neq 0$ and $eb = b$, for a suitable $e \in \mathfrak{r}$.*

The 'hyper' issues of the preceding section apply in commutative rings to give yet another characterization of commutative von Neumann regular rings.

Recall that a ring R is *von Neumann regular* if for each $a \in R$ there is an $x \in R$ such that $axa = a$. A commutative von Neumann regular ring is semiprime, and every ideal is radical. The following result should not be confused with the well known characterization of commutative von Neumann regular rings as being those in which every ideal is an intersection of maximal ideals. That one may be restated (Choice-free) by saying that the commutative ring with identity R is von Neumann regular if and only if every homomorphic image of R is a ring with trivial Jacobson radical.

Corollary 6.9. *Let R be a commutative semiprime Gelfand ring with 1. Then every semiprime homomorphic image has trivial Jacobson radical if and only if R is von Neumann regular.*

Proof: Put together Proposition 5.3 and the remark in 5.1.3. $\square$

We conclude with an example, promised in 2.12.

Example 6.10. Let $\mathbb{R}[[X]]$ denote the ring of formal power series in one variable with real coefficients. We remind the reader that this is a discrete valuation domain; the import of this information is that $\mathrm{Rad}(\mathbb{R}[[X]])$ is the three-element frame.

Now define a ring R as follows: let B be the ring of all real sequences, with pointwise operations, and R be the subring of $B \times \mathbb{R}[[X]]$ consisting of all (f, g) such that $f(n)$ is eventually equal to the constant g_0, where $g = \sum_{n=0}^{\infty} g_n X^n$. The important things to note about $A \equiv \mathrm{Rad}(R)$ are these:

1. A is coherent and normal, and, indeed, has disjointification.
2. The only nonzero infinitesimal is the ideal $\mathfrak{p}$ generated by $(0, X)$, which is, in fact, a polar such that $\rho_*(\mathfrak{p}) = 0$. $\mathfrak{p}^\perp$ is the ideal of all $(f, 0)$, with f eventually zero. It is a supremum of complemented compact elements, and therefore regular.
3. A proper ideal $\mathfrak{r}$ is pure if and only if either
   - $\mathfrak{r} \subseteq \mathfrak{p}^\perp$, and a supremum of compact complemented elements, or
   - $\mathfrak{r} \not\subseteq \mathfrak{p}^\perp$, and then it is complemented, and, in fact, the ideal generated by the idempotent element $(e, 1)$, where $e(n) = 1$ except for finitely many n.
4. Thus, $\mathfrak{p}^\perp$ is a maximal pure ideal, and dense in ρA, but not in A, so that ρA is not skeletally embedded in A.
5. ρA is isomorphic to the frame of open sets of the one-point compactification of discrete $\mathbb{N}$.
6. $\varphi A = \rho A$.

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