Some remarks on simple Bol loops

Gábor P. Nagy

Abstract. In this paper we make some remarks on simple Bol loops which were motivated by questions at the LOOPS'07 conference. We also list some open problems on simple loops.

Keywords: Bol loop, Bruck loop, simple loop

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1. Introduction

A loop Q satisfying the *right Bol identity* $((xy)z)y = x((yz)z)$ is a *right Bol loop*, or simply Bol loop. In the last three decades, the existence of finite simple Bol loops was considered as one of the main open problems of the theory of loops, see [21]. Recently the author constructed two classes of simple Bol loops. The first construction [18] uses exact factorizations of groups in order to produce simple Bol G-loops. The resulting simple loops can be infinite or finite, and in the latter case, both even and odd orders can occur.

The second construction [19] gives an infinite class of finite simple Bol loop of exponent 2. The smallest member of this class has order 96 and was found by using the information given by M. Aschbacher [1] on the structure of minimal non-solvable Bol loops of exponent 2.

At the LOOPS'07 conference in Prague, the following questions on Bol loops were posed to the author.

Question 1 (A. Greil, München)**.** *Let Q_2 be the 2-dimensional simple Bruck loop defined on the hyperbolic plane and let Q_3 be the 3-dimension simple Bol G-loop obtained by the exact factorization of $PSL_2(\mathbb{R})$. Is Q_2 isomorphic to a subloop of Q_3?*

Question 2 (V. Shcherbacov, Chişinău)**.** *Let Q be a finite Bol loop which admits a fixed-point-free automorphism of prime order. Is Q then solvable?*

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Question 3 (H. Kiechle, Hamburg). *Are there finite non-associative Bol loops with transitive automorphism groups?*

It turns out that all these questions have negative answers. Question 3 was recently answered by M. Aschbacher [2]. We will present another proof of the non-existence of finite Bol loops with transitive automorphism group which uses the classification of 1-factorizations of complete graphs with doubly transitive automorphism groups. We finish this paper with a few open problems which are related to simple Bol loops.

2. Representation of Bol loops by loop folders

Let Q be a loop, $\mathrm{RSec}(Q)$ the set of right translations

$$R_x : y \mapsto yx$$

of Q, $\mathrm{RMlt}(Q)$ the *right multiplication group* generated by $\mathrm{RSec}(Q)$ and $\mathrm{RInn}(Q)$ the stabilizer of 1 in $\mathrm{RMlt}(Q)$. $\mathrm{RSec}(Q)$ and $\mathrm{RInn}(Q)$ are called the *right section* and *right inner mapping group* of Q, respectively. It is well known that Q is a Bol loop if and only if for every $R_a, R_b \in \mathrm{RSec}(Q)$,

$$R_a R_b R_a \in \mathrm{RSec}(Q)$$

holds.

Definition 2.1. The triple (G, H, K) is a *loop folder* if G is a group, $H \leq G$, $1 \in K$ and K is a system of right coset representatives for each conjugate H^g of H in G. The triple (G, H, K) is a *Bol loop folder* if it is a loop folder such that $aba \in K$ for all $a, b \in K$.

For a (Bol) loop Q, the triple $(\mathrm{RMlt}(Q), \mathrm{RSec}(Q), \mathrm{RInn}(Q))$ is a (Bol) loop folder. Conversely, let (G, H, K) be a loop folder. For $a, b \in K$, let c be the unique element in $Hab \cap K$; the binary operation $a \circ b = c$ turns K into a loop. Moreover, $(K, \circ)$ is Bol if and only if the loop folder is Bol.

Remark. The terminology "loop folder" is due to M. Aschbacher [1]. The correspondence between loops and loop folders was first used by R. Baer [4], hence the name *Baer correspondence*.

The loop folder of a loop is not uniquely determined. In general, we do not require that

$$\mathrm{core}_G(H) = \bigcap_{g \in G} H^g = 1$$

and $G = \langle K \rangle$; if this is the case then we speak of a *faithful loop folder*. The faithful loop folder of a loop Q is isomorphic to $(\mathrm{RMlt}(Q), \mathrm{RSec}(Q), \mathrm{RInn}(Q))$, hence unique. To see this, let us assume that (G,H,K) is a loop folder and $G=\langle K\rangle$. We denote the associated loop $(K,\circ)$ by Q. The right multiplication group $\mathrm{RMlt}(Q)$ is a homomorphic image of G with kernel $\mathrm{core}_G(H)$. In particular, if $\mathrm{core}_G(H)=1$ then G and $\mathrm{RMlt}(Q)$ are isomorphic. (Cf. [20, Proposition 1.13] or [1, Remark 1.1 and Example 1.2].)

Automorphisms of loops can also be described within the frame of the Baer correspondence. Let (G,H,K) be a loop folder and α an automorphism of G normalizing H and K. Then the restriction of α to K is a loop automorphism:

$$a\circ b=c \Leftrightarrow Hab=Hc \Leftrightarrow Ha^\alpha b^\alpha = Hc^\alpha \Leftrightarrow a^\alpha \circ b^\alpha = c^\alpha.$$

Our final lemma is useful for Bol loop folders.

Lemma 2.2. *Let G be a group, $H\le G$ a subgroup and $1\in K\subseteq G$ is a system of right coset representatives of H in G. Assume that $a^{-1}, aba\in K$ for all $a,b\in K$. Then K is a system of right coset representatives of each conjugate H^g of H in G. In particular, the triple (G,H,K) is a Bol loop folder.*

Proof: We define the operation $x*y$ on K by $H(x*y)=Hxy$, then

$$x*((y*z)*y)=x*(yzy)=((x*y)*z)*y$$

and $x=(x*y)*y^{-1}=(x*y^{-1})*y$ hold for all $x,y,z\in K$. By [8, Lemma 2], $(K,*)$ is a right Bol loop. $\square$

3. 2-dimensional subloops and exact factorizations

We first define the 2-dimensional hyperbolic loop by loop folders. Put

$$\begin{aligned}
G_2 &= PSL_2(\mathbb{R}),\\
H_2 &= \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid t\in\mathbb{R} \right\},\\
K_2 &= \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a>0, ac-b^2=1 \right\}.
\end{aligned}$$

Observe that K_2 is given by positive definite symmetric matrices. The polar decomposition of real matrices implies that K_2 is a system of right coset representatives of H_2 in G_2 with $aba\in K_2$ for $a,b\in K_2$. By Lemma 2.2, the triple (G_2,H_2,K_2) is a Bol loop folder; we will denote the associated Bol loop by Q_2. Clearly, the loop folder (G_2,H_2,K_2) is faithful.

Let M be a proper subgroup of $PSL_2(\mathbb{R})$ containing H_2. Since the 2-dimensional subgroups of $PSL_2(\mathbb{R})$ are the Borel subgroups and H_2 is not contained in any Borel subgroup, $\dim(M)=1$ and H_2 is the connected component of M. Moreover, straightforward calculation shows that $N_{G_2}(H_2)=H_2$. Therefore $M=H_2$ is a maximal subgroup of $G_2=PSL_2(\mathbb{R})$.

Lemma 3.1. *Let (G_0, H_0, K_0) be a Bol loop folder such that $G_0 = \langle K_0 \rangle$ and let us assume that the associated Bol loop is isomorphic to the 2-dimensional hyperbolic loop Q_2. Then H_0 is a maximal subgroup of G_0.*

Proof: As the loop folder (G_2, H_2, K_2) of Q_2 is faithful, we have a surjective homomorphism $G_0 \to G_2$ mapping H_0 to H_2. The kernel $\mathrm{core}_{G_0}(H_0)$ of this homomorphism is contained in H_0. Since H_2 is maximal in G_2, the lemma follows. □

The following concept is important on its own in the theory of groups.

Definition 3.2. Let A be a group and B, C subgroups of A. We say that BC is an *exact factorization* of A if $A = BC$ and $B \cap C = 1$. The exact factorization $A = BC$ is *faithful* if $\mathrm{core}_A(B) = \mathrm{core}_A(C) = 1$.

Let $A = BC$ be an exact factorization and define the following:

$$\begin{cases} G = A \times A, \\ H = \{(b, c) \mid b \in B, c \in C\}, \\ K = \{(a, a^{-1}) \mid a \in A\}. \end{cases} \tag{1}$$

By [18, Proposition 3.2], the triple (G, H, K) is a Bol loop folder and the associated loop Q is a G-loop. Moreover, the underlying set of Q can be naturally identified with the underlying set of the group A. In particular, if A, B, C are Lie groups then Q is a differentiable loop and $\dim(Q) = \dim(A)$.

Lemma 3.3. *Let $A = BC$ be an exact factorization and define the loop folder (G, H, K) and the loop Q as in (1). Put $B^* = B \times A$, $C^* = A \times C$ and $K_B = B^* \cap K$, $K_C = C^* \cap K$. Then,*

$$(B^*, H, K_B) \text{ and } (C^*, H, K_C)$$

are subloop folders of (G, H, K) and the associated subloops are isomorphic to B and C, respectively.

Proof: Let us first observe that

$$K_B = \{(b, b^{-1}) \mid b \in B\}$$

and that (A, C, B) is a Bol loop folder corresponding to the group B. Define the projections $\pi_i : G = A \times A \to A$. Then π_2 maps B^* to A, H to C and K_B to B, that is, π_2 induces a homomorphism between the loop folders

$$(B^*, H, K_B) \text{ and } (A, C, B).$$

Moreover, $\ker \pi_2 \cap B^* = B \times 1 \le H$, hence (B^*, H, K_B) and (A, C, B) determine the same loop. Finally, (A, C, B) and $(\langle B \rangle, B \cap C, B) = (B, 1, B)$ determine the same loop again, which is isomorphic to the group B. A similar argument proves that the subloop corresponding to the loop folder (C^*, H, K_C) is isomorphic to C. $\square$

Let us now put $A = PSL_2(\mathbb{R})$,

$$B = \left\{ \pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid t \in \mathbb{R} \right\},$$
$$C = \left\{ \pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\}$$

of A. Let (G_3, H_3, K_3) be the Bol loop folder defined by (1) and let us denote by Q_3 the associated simple differentiable Bol G-loop; $\dim(Q_3) = 3$. As the loop folder (G_3, H_3, K_3) is faithful we can identify G_3, H_3 and K_3 with $\mathrm{RMlt}(Q_3), \mathrm{RInn}(Q_3)$ and $\mathrm{RSec}(Q_3)$, respectively.

Theorem 3.4. *The* 3*-dimensional simple Bol G-loop Q_3 has no subloop isomorphic to the* 2*-dimensional hyperbolic loop Q_2.*

Proof: Let us assume that the subloop $\tilde{Q} \le Q_3$ is isomorphic to Q_2 and define the set
$$\tilde{K} = \{R_x \mid x \in \tilde{Q}\}$$
of right translations corresponding to the elements of $\tilde{Q}$. Put $\tilde{G} = \langle \tilde{K} \rangle$ and $\tilde{H} = \tilde{G} \cap H_3$. Then $(\tilde{G}, \tilde{H}, \tilde{K})$ is a loop folder for $\tilde{Q}$ satisfying $\tilde{G} = \langle \tilde{K} \rangle$. By Lemma 3.1, $\tilde{H}$ is a maximal subgroup in $\tilde{G}$.

Define the subgroups

$$M = \{(a, c) \mid a \in A, c \in C\} \quad \text{and} \quad \tilde{M} = \tilde{G} \cap M$$

of G_3. Since $M \cong A \times C$, $\dim(M) = 5$ and $\dim(\tilde{M}) \ge \dim(\tilde{G}) - 1$. However, $H_3 \le M$ implies $\tilde{H} \le \tilde{M}$. As $\dim(\tilde{H}) = \dim(\tilde{G}) - 2$, we obtain that $\tilde{M}$ contains $\tilde{H}$ properly. This is only possible if $\tilde{M} = \tilde{G}$, that is, $\tilde{G} \le M$.

By Lemma 3.3, $(M, H_3, M \cap K_3)$ is a subloop folder and the corresponding subloop is isomorphic to the group C. As $\tilde{Q}$ is not isomorphic to a subgroup of C, we obtained a contradiction. $\square$

4. Finite simple Bol loops with fixed-point-free automorphisms

In [19], the author constructed an infinite class of simple Bol loops of exponent 2. The smallest member of the this class has order 96; its loop folder (G, H, K) is given by the following procedure, for details see [19].

Let S_n be the symmetric group on n elements. The number of even and odd involutions of S_5 is 15 and 10, respectively. These involutions are permuted by a fixed point free Sylow 5-subgroup of S_5. The normalizer of a 5-Sylow of S_5 has order 20. The group S_5 has a unique central extension such that transpositions lift to elements of order 2 and the even involutions lift to elements of order 4; this central extension is usually denoted by $2.S_5$.

Let A be a group, $B \lhd A$ and $A/B \cong C$. Then we say that A is an *extension* of B by C. If A has a subgroup S such that $A = BS$ and $B \cap S = 1$ the extension is said to be *split*. In this case, A is the semidirect product of B and C.

There is a unique group G with the properties (P1) and (P2):

- (P1) G contains a minimal normal subgroup J_0 of order 2^4 such that $G/J_0 \cong 2.S_5$;
- (P2) G is a non-split extension of J_0 by $2.S_5$ and a split extension of G' by the group of order 2.

Let us denote by $J = O_2(G)$ the largest normal 2-subgroup of G. Then $J_0 \leq J$ and J is an elementary abelian group of order 2^5. (Otherwise the Frattini subgroup of J would be properly contained in J_0, contradicting the minimality of J_0.) Furthermore, the minimality of J_0 implies $J_0 \leq G'$. Since $(2.S_5)'$ has index 2 in $2.S_5$, we obtain that $G/G' \cong C_2$.

Lemma 4.1. (1) *Let P be a Sylow 5-subgroup of G and put $H = N_G(P)$. Then H is isomorphic to the semidirect product $C_8 \rtimes C_5$. In particular, $|H| = 40$.*

(2) *Let $g \to g^+$ denote the surjective homomorphism $G \to G/J \cong S_5$. There is an involution $c \in G$ such that c^+ is a transposition. The conjugacy class c^G of c in G has size 80.*

(3) *Let us put $K = J_0 \cup c^G$. Then (G, H, K) is a Bol loop folder corresponding to a simple Bol loop of exponent 2 of order 96.*

Proof: See [19, Theorem 3.6]. □

Theorem 4.2. *Let us define G, H, K as before and let Q be the simple Bol loop corresponding to the loop folder (G, H, K). Let a be an element of order 5 in H and let us denote by α the inner automorphism of G induced by a. Then $H^\alpha = H$ and $K^\alpha = K$, thus, α induces an automorphism $\tilde{\alpha}$ of Q. The order of $\tilde{\alpha}$ is 5 and it has no fixed point in $Q \setminus \{1\}$.*

Proof: As K is invariant under conjugation in G, α acts on K. Clearly, $H^\alpha = H$, thus, $\tilde{\alpha}$ is well defined. We have already seen that the action of $\tilde{\alpha}$ is equivalent with the action of α on K. It is therefore enough to show that a does not centralize any element of $K \setminus \{1\}$. However, if $y \in K \cap C_G(a)$ then $y \in H = N_G(\langle a \rangle)$ which implies $y = 1 = K \cap H$. □

5. A new proof on finite Bol loops with transitive automorphism group

In [2, Theorem 1], M. Aschbacher proved that a finite Bol loop with transitive automorphism group is an elementary abelian p-group for some prime p. As pointed out in [17], this result also follows from Theorems 1 and 3 of P.J. Cameron and G. Korchmáros [6]. In this section, we present this second proof.

Let Q denote a finite Bol loop such that $\mathrm{Aut}(Q)$ acts transitively on $Q^\sharp = Q \setminus \{1\}$. It is well known that Bol loops are power-associative. Therefore, the orders of elements are well defined and in the case of a finite Bol loop, the orders divide the order of the loop. In particular, each element of $Q^\sharp$ has order p for some prime p. Let us first consider the case when p is odd.

The next lemma is rather *folklore*. It is more general and can be useful in other context, too.

Lemma 5.1. *Let Q be a finite right Bol loop in which every non-trivial element has order p for some odd prime p. Then* $\mathrm{RMlt}(Q)$ *is a p-group and Q is solvable.*

Proof: Let G be the subgroup of $\mathrm{RMlt}(Q) \times \mathrm{RMlt}(Q)$ which is defined by the set

$$\{(R_x, R_x^{-1}) \mid x \in Q\}.$$

Due to the right inverse property of Q, the map $\sigma : (x,y) \mapsto (y,x)$ leaves G invariant, hence $\sigma \in \mathrm{Aut}(G)$. We consider σ as an element of the semidirect product $G \ltimes \langle\sigma\rangle$ and claim that the conjugacy class σ^G consists of the elements $\sigma^{(R_x, R_x^{-1})}$. Indeed, using

$$\sigma^{(R_x, R_x^{-1})} = \sigma(R_{x^2}, R_{x^2}^{-1}), \tag{2}$$

one can show by a somewhat tedious calculation that

$$\sigma^{(R_x, R_x^{-1})(R_y, R_y^{-1})} = \sigma^{(R_z, R_z^{-1})}$$

holds with $z = ((yx^2)y)^{\frac{1}{2}}$. Notice that $x \mapsto x^{\frac{1}{2}}$ is well defined since every element has odd order.

As the orders of the elements x, R_x and $(R_{x^2}, R_{x^2}^{-1})$ are the same odd prime p, we see that the product of two conjugates of σ has always odd order p. By [7, Satz 4.1], the order of G is a power of p. This implies $\mathrm{RMlt}(Q)$ to be a p-group since it is a homomorphic image of G. We still have to show that Q is solvable. As $\mathrm{RMlt}(Q)$ is nilpotent, the right inner mapping group of Q is contained in a normal subgroup N of $\mathrm{RMlt}(Q)$. It is straightforward to show that the map

$$Q \to \mathrm{RMlt}(Q)/N, \qquad x \mapsto R_x N$$

is a surjective homomorphism from Q to the p-group $\mathrm{RMlt}(Q)/N$. Since the latter is solvable, Q is solvable as well. □

Remark. The above lemma holds also for right Bol loops in which the orders of the elements are powers of p; the proof can be used without any change. It was open for a long time if the solvability of Q can be strengthened to nilpotence. The fact that this is not possible was very recently shown by M. Kinyon [14] who constructed a right Bol loop of order 27 and exponent 3 with trivial center.

We can now come to the case of Bol loops of odd order with transitive automorphism groups.

Lemma 5.2. *Let Q be a finite right Bol loop with a transitive automorphism group. Assume that every element of Q has order p with odd prime p. Then Q is an elementary abelian p-group.*

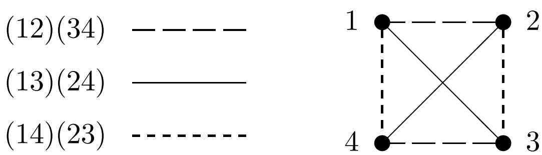
Proof: Let Q' denote the commutator-associator subloop of Q. By Lemma 5.1, Q' is properly contained in Q and $Q \setminus Q' \neq \emptyset$. As Q' is a characteristic normal subloop of Q, we obtain that all non-trivial elements are in $Q \setminus Q'$, thus $Q' = 1$. Hence, Q is an abelian group in which all elements have the same order. This proves the lemma. □

Let us now turn to the much harder case when Q consists of elements of order 2. For this, we need the concept of 1-factorization of graphs (also called a minimal edge coloring).

Definition 5.3. Let $\Gamma = (V, E)$ be a finite simple graph. A subset $F \subseteq E$ of edges of Γ is a 1-*factor* if for any $v \in V$ there is a unique edge $e \in F$ containing v. A 1-*factorization* of Γ is a partition of the set E of edges into 1-factors $F_1, \dots, F_k$.

It is easy to see that the 1-factors of Γ correspond to the involutorial permutations of the set V of vertices. Let n be a positive even integer, put $V = \{1, \dots, n\}$ and denote by K_n the complete graph on V. Let $\mathscr{F} = \{F_1, \dots, F_{n-1}\}$ be a 1-factorization K_n. Let $U_{\mathcal{F}}$ be the set consisting of id_V and the involutorial permutations $u_1, \dots, u_{n-1}$ corresponding to the 1-factors $F_1, \dots, F_{n-1}$, respectively. Then U is a *sharply transitive set* on V, that is, for each $x, y \in V$, there is a unique element $u \in U$ with $x^u = y$.

Conversely, let $U = \{\mathrm{id}_V, u_1, \dots, u_{n-1}\}$ be a sharply transitive set on $V = \{1, \dots, n\}$ such that $u_i^2 = \mathrm{id}_V$ for each $i = 1, \dots, n-1$. Then, the corresponding 1-factors determine a 1-factorization of K_n. The following picture illustrates the correspondence between the 1-factorization of the complete graph K_4 and the involutions of the elementary abelian group of order 4.



The next lemma explains the relationship between Bol loops of exponent 2 and 1-factorizations of the complete graphs.

Lemma 5.4. *Let Q be a finite Bol loop of exponent 2 such that* $\mathrm{Aut}(Q)$ *acts transitively on $Q^\sharp = Q \setminus \{1\}$. Then the set of right multiplication maps of Q determines a 1-factorization $\mathscr{F}$ of the complete graph K_n with $n = |Q|$. Moreover, the automorphism group of $\mathscr{F}$ acts doubly transitively on the vertices of K_n.*

Proof: Clearly, the set of right translations of Q consists of involutions and forms a sharply transitive set on Q. Let us identify the elements of Q and the vertices of the complete graph K_n. The 1-factors can be indexed with the elements of $Q^\sharp$: For $x \in Q^\sharp$, the 1-factors F_x consists of the edges $\{y, yx\}$, $y \in Q$.

Let us take an arbitrary element $a \in Q$, then

$$\begin{aligned} F_x^{R_a} &= \{\{y, yx\} \mid y \in Q\}^{R_a} \\ &= \{\{ya, (yx)a\} \mid y \in Q\} \\ &= \{\{ya, (((ya)a)x)a = (ya)((ax)a)\} \mid y \in Q\} \\ &= F_{(ax)a}. \end{aligned}$$

Thus, every right multiplication map R_a is an automorphism of $\mathscr{F}$. Similarly, one shows that for any $\alpha \in \mathrm{Aut}(Q)$, $F_x^\alpha = F_{x^\alpha}$ holds. This implies

$$\mathrm{RMlt}(Q), \mathrm{Aut}(Q) \leq \mathrm{Aut}(\mathscr{F}).$$

Let G be the subgroup of $\mathrm{Sym}(Q)$ generated by $\mathrm{RMlt}(Q)$ and $\mathrm{Aut}(Q)$; clearly G is transitive on Q and $G \leq \mathrm{Aut}(\mathscr{F})$. Moreover, the stabilizer G_1 of 1 contains $\mathrm{Aut}(Q)$, hence G is doubly transitive on Q. □

We are now able to proof our theorem.

Theorem 5.5. *Let Q be a finite right Bol loop and assume that* $\mathrm{Aut}(Q)$ *acts transitively on $Q^\sharp = Q \setminus \{1\}$. Then Q is an elementary abelian p-group for some prime p.*

Proof: We have seen that all non-trivial elements of Q have prime order p. Lemma 5.2 proves the theorem in the case $p > 2$. Assume $p = 2$ and suppose that Q is not elementary abelian. Then by Lemma 5.4, we construct a unique 1-factorization $\mathscr{F}$ of the complete graph K_n with $n = |Q|$ such that $\mathrm{Aut}(\mathscr{F})$ is doubly transitive on the vertices. [6, Theorem 3] implies that $n = 6, 12$ or 28 and in all cases $\mathscr{F}$ is unique. As the construction of the 1-factorization is explicitly given in [6, Proposition 3], it is straightforward to verify that these 1-factorizations do not correspond to Bol loops. This contradiction proves our theorem. □

6. Open problems on simple loops

In this section, we present five problems which are related to simple Bol loops. The first two problems were proposed by the author at the LOOPS'07 conference in Prague.

Problem 1. *Are there simple Bol loops which are neither G-loops nor isotopes of Bruck loops?*

Problem 1 is not reduced to finite Bol loops. By the author's best knowledge all known simple Bol loops are either G-loops or isotopes of Bruck loops. Here by *Bruck loop* we mean a right Bol loop with the automorphic inverse property. We mention that all groups and all known simple Moufang loops are G-loops.

A sufficient but not necessary condition of the simplicity of a loop is that its right multiplication group acts primitively. Trivial examples of simple loops with imprimitive right multiplication groups are non-abelian simple groups. For non-associative simple Moufang loops, the left, right and full multiplication groups coincide, hence they act primitively on the loop. All known non-Moufang finite simple Bol loops have imprimitive right multiplication groups. This motivates the following question of Grishkov and Zavarnitsine [9].

Problem 2. *Is there a finite simple non-Moufang Bol loop where the right multiplication group acts primitively on the loop?*

The first step towards settling Problem 2 could be the following result of E.K. Loginov [16]: If the right multiplication group of a Bol loop Q is a finite simple group of Lie type then Q is a simple Moufang group. Moreover, as Bol loops of prime power order are solvable, $\mathrm{RMlt}(Q)$ cannot be a primitive group of affine type.

The next problem is *folklore* for different loop classes. Lagrange's Theorem was shown for finite Moufang loops by A. Grishkov and Zavarnitsine [9]. It is known that in an equationally defined class of loops, the problem can be reduced to simple loops.

Problem 3. *Does Lagrange's Theorem hold for finite Bol loops?*

Let $\mathbb{O}$ be a non-degenerate octonion algebra over the field k. It is well known that modulo the center, the elements of norm 1 form a simple Moufang loop; we call them *classical*. Liebeck's theorem [15] says that all finite simple non-associative Moufang loops are classical with finite ground field. This result was recently extended to locally finite simple Moufang loops by J.I. Hall [10]. Moreover, it also holds for differentiable Moufang loops, see [11, IX.6.31. Theorem]. However, no non-classical simple Moufang loops are known.

Problem 4. *Do there exist non-classical infinite simple Moufang loops?*

Our last problem was proposed by M. Kinyon at the Mile High Conference 2005 in Denver. It is not directly related to Bol loops. The loop Q is said to be an *A-loop* if all inner maps of Q are automorphisms of Q. Clearly, all groups are A-loops.

Problem 5. *Do there exist finite simple non-associative A-loops?*

For dealing with this problem one has to consider the full multiplication group $G = \mathrm{Mlt}(Q)$ of the loop Q and represent Q by *connected transversals*. As Q is simple, G acts primitively on Q. The typical situation for finite simple loops is that G contains $\mathrm{Alt}(Q)$. However, it is easy to show that for a loop of order at least 5, the automorphism group cannot act 3-transitively. Hence, the transitivity degree of G is at most 3, and so G cannot contain $\mathrm{Alt}(Q)$.

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Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

Current address:
Mathematisches Institut, Universität Würzburg, Am Hubland, D-97070 Würzburg, Germany

E-mail: nagyg@math.u-szeged.hu

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