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Ternary quasigroups and the modular group

Jonathan D.H. Smith

Abstract. For a positive integer n, the usual definitions of n-quasigroups are rather complicated: either by combinatorial conditions that effectively amount to Latin n-cubes, or by $2n$ identities on $n+1$ different n-ary operations. In this paper, a more symmetrical approach to the specification of n-quasigroups is considered. In particular, ternary quasigroups arise from actions of the modular group.

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1. Quasigroups

For a positive integer n, a (*combinatorial*) *n-quasigroup* is a set Q equipped with an n-ary *multiplication* operation

$$\mu : Q^n \to Q;\ (x_n, \dots, x_1) \mapsto x_n \dots x_1\mu$$

such that, for an $(n+1)$-tuple

$$(x_n, \dots, x_1, x_0) \tag{1.1}$$

of elements of Q required to satisfy the condition

$$x_n \dots x_1\mu = x_0, \tag{1.2}$$

specification of any n coordinates of (1.1) determines the remaining one uniquely. Note that a combinatorial 1-quasigroup is just a set Q with a permutation (self-bijection) $\mu : Q \to Q$, or in other words a dynamical system with state space Q and invertible transition operator μ.

For each index $1 \leq i \leq n$, and for each choice $x_n, \dots, x_{i+1}, x_{i-1}, \dots, x_1$ of fixed elements of an n-quasigroup Q, a *translation*

$$T_i(x_n, \dots, x_{i+1}, x_{i-1}, \dots, x_1) : Q \to Q;\ x_i \mapsto x_n \dots x_1\mu \tag{1.3}$$

is defined. The combinatorial definition of an n-quasigroup means precisely that each translation is a permutation of the underlying set Q.

The combinatorial definition of n-quasigroups may be reformulated in algebraic terms of operations and identities. An (*equational*) *n-quasigroup* $(Q, \mu, \mu^1, \dots, \mu^n)$ is a set Q equipped with n-ary operations $\mu, \mu^1, \dots, \mu^n$ satisfying the identities

$$x_n \dots x_{i+1} (x_n \dots x_1\mu) x_{i-1} \dots x_1\mu^i = x_i \tag{1.4}$$

and

$$x_n \dots x_{i+1} (x_n \dots x_1\mu^i) x_{i-1} \dots x_1\mu = x_i \tag{1.5}$$

for each $1 \le i \le n$. The operations $\mu^1, \dots, \mu^n$ are described as *divisions*. Note that the identity (1.4) gives the injectivity of the translation (1.3), while (1.5) gives its surjectivity. Thus each equational n-quasigroup $(Q, \mu, \mu^1, \dots, \mu^n)$ yields a combinatorial n-quasigroup (Q, μ). Conversely, a combinatorial n-quasigroup (Q, μ) yields an equational n-quasigroup $(Q, \mu, \mu^1, \dots, \mu^n)$, defining

$$x_n \dots x_{i+1} x_0 x_{i-1} \dots x_1\mu^i = x_i$$

if and only if (1.2) holds.

2. Groups

For a positive integer n, consider the group M_n presented as

$$\langle \sigma, \tau \mid \sigma^n = \tau^2 = 1 \rangle.$$

In other words, M_n is the free product of two cyclic groups, one $\langle \sigma \rangle$ of order n, and one $\langle \tau \rangle$ of order 2.

Example 2.1. For $n = 1$, M_1 is just the cyclic group $\langle \tau \rangle$ of order 2.

Example 2.2. For $n = 2$, M_2 is the *infinite dihedral group* ([2, p. 133]). Recall that the *dihedral group* D_d of degree d and order $2d$ (the group of symmetries of the regular d-gon) may be presented as

$$\langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^d = 1 \rangle \tag{2.1}$$

([2, (1.53)]).

Example 2.3. For $n = 3$, M_3 is the *modular group* $\mathsf{SL}_2(\mathbb{Z})/\{\pm I_2\}$ ([8, p. 128]). For each element

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of $\mathsf{SL}_2(\mathbb{Z})$, a matrix of determinant 1 with integral entries, write the corresponding coset $\{\pm A\}$ in M_3 as

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}.$$

Setting

$$\sigma = \begin{Bmatrix} 0 & -1 \\ 1 & 1 \end{Bmatrix} \quad \text{and} \quad \tau = \begin{Bmatrix} 0 & -1 \\ 1 & 0 \end{Bmatrix},$$

one has $\sigma^3 = \tau^2 = 1$, and $\mathsf{SL}_2(\mathbb{Z})/\{\pm I_2\}$ is generated freely by σ and τ, subject to these order relations ([2, (7.25)], [8, p. 131]).

Lemma 2.4. *Consider the symmetric group $S_{n+1} = \{0, 1, \ldots, n\}!$.*

(a) *For $n \geq 1$, the group S_{n+1} is a quotient of M_n.*
(b) $S_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^3 = 1 \rangle$.
(c) $S_4 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = (\sigma\tau)^4 = 1 \rangle$.

Proof: (a): Apply the First Isomorphism Theorem to the surjective homomorphism

$$r : M_n \to S_{n+1};\ \sigma \mapsto (1\ 2\ \ldots\ n),\ \tau \mapsto (0\ 1). \tag{2.2}$$

(b): This is the case $d = 3$ of (2.1).
(c): See [2, (1.59)]. □

3. Spaces

For a positive integer n, an *n-ary space* (G, σ, τ) is a set G equipped with maps

$$\sigma : G \to G;\ g \mapsto \sigma g \tag{3.1}$$

and

$$\tau : G \to G;\ g \mapsto \tau g \tag{3.2}$$

satisfying $\sigma^n = \tau^2 = 1$. The map σ is known as the *shift*, while the map τ is known as the *inversion*. Note that n-ary spaces are left M_n-sets.

Example 3.1. For each positive integer n, each set G furnishes a trivial n-ary space, on which both σ and τ are the identity map id_G.

Example 3.2. For n=1, each group G provides a unary space, with $\tau g = g^{-1}$ for g in G.

Example 3.3. For n=2, the binary spaces are the *reflexion-inversion spaces* of [9], the shift being described as *reflexion* in this case.

(a) For a field F, take $G = F \smallsetminus \{0, 1\}$. Then G is a binary space, with $\sigma g = 1 - g$ and $\tau g = g^{-1}$ for points g of G ([9, Example 3.3]).
(b) The symmetric group S_3 is a binary space. Taking $\sigma = (1\ 2)$ and $\tau = (0\ 1)$, the maps (3.1) and (3.2) are interpreted as left multiplications within S_3 — compare Lemma 2.4(b).

Example 3.4. The symmetric group S_4 is a ternary space. Taking $\sigma = (1\ 2\ 3)$ and $\tau = (0\ 1)$, the maps (3.1) and (3.2) are interpreted as left multiplications within S_4 — compare Lemma 2.4(c).

Example 3.5. Let R be a unital ring, and let U be a group of units in R. For a positive integer n, consider $G = U^n$. Define

$$\sigma(u_n, \dots, u_2, u_1) = (u_{n-1}, \dots, u_1, u_n)$$

and

$$\tau(u_n, \dots, u_2, u_1) = \left(-u_n {u_1}^{-1}, \dots, -u_2 {u_1}^{-1}, {u_1}^{-1}\right)$$

for a point $(u_n, \dots, u_1)$ of G. Then G becomes an n-ary space.

4. Hyperquasigroups

For a positive integer n, an *n-hyperquasigroup* (or *n-ary hyperquasigroup*) is a pair (Q, G) consisting of a set Q and an n-ary space G, with an n-ary *action*

$$Q^n \times G \to Q;\ (x_n, \dots, x_1, g) \mapsto x_n \dots x_1 \underline{g} \tag{4.1}$$

of G on Q, such that the (n-)*hypercommutative law*

$$x_n \dots x_2 x_1 \underline{g} = x_{n-1} \dots x_1 x_n \underline{\sigma g} \tag{4.2}$$

and the (n-)*hypercancellation law*

$$x_n \dots x_2 (x_n \dots x_1 \underline{g})\, \underline{\tau g} = x_1 \tag{4.3}$$

are satisfied for all $x_1, \dots, x_n$ in Q and g in G.

Remark 4.1. A hyperquasigroup (Q, G) may be construed as a two-sorted or heterogeneous algebra ([4], [6]), with the n-ary space operations σ and τ on the sort G, and (4.1) as a third operation.

Example 4.2. For each positive integer n, and for each n-ary space G, the empty set forms an n-hyperquasigroup $(\emptyset, G)$. The actions (4.1) reduce to $\mathrm{id}_\emptyset$.

Example 4.3. For each positive integer n, consider the trivial n-ary space $\emptyset$ as in Example 3.1. Let Q be a set. Then $(Q, \emptyset)$ forms an n-hyperquasigroup, with (4.1) as the insertion $\emptyset \hookrightarrow Q$. The hypercommutativity (4.2) and hypercancellation (4.3) are vacuously satisfied.

Example 4.4. For $n = 1$, let G be a group, construed as a unary space according to Example 3.2. Consider a right G-set Q. For g in G and x in Q, define the unary action $x\underline{g} = xg$. The hypercommutativity is trivial, while the hypercancellation is just $(xg)g^{-1} = x$. Thus (Q, G) is a unary hyperquasigroup.

Example 4.5. For each positive integer n, consider the trivial n-ary space $\{1\}$.

(a) For $n = 1$, each set Q forms a unary hyperquasigroup $(Q, \{1\})$ as a $\{1\}$-set for the trivial group $\{1\}$, according to Example 4.4.

(b) For $n = 2$, a binary hyperquasigroup $(Q, \{1\})$ is just a totally symmetric quasigroup, with multiplication $x_1 x_2 \underline{1}$.

(c) For any positive n, let Q be an abelian group of exponent 2. Then $(Q, \{1\})$ forms an n-hyperquasigroup with

$$x_1 x_2 \dots x_n \underline{1} = x_1 x_2 \dots x_n$$

for $x_1, \dots, x_n$ in Q.

Example 4.6. For $n = 2$, binary hyperquasigroups reduce to hyperquasigroups in the sense of [9].

(a) For a field F, consider the binary space $G = F \smallsetminus \{0, 1\}$ of Example 3.3(a). For a vector space Q over F, define the binary action

$$Q^2 \times G \to Q;\ (x_2, x_1, g) \mapsto x_2(1 - g) + x_1 g.$$

Then (Q, G) forms a binary hyperquasigroup ([9, Proposition 5.1]).

(b) Let $(Q, \cdot, /, \backslash)$ be a (binary) quasigroup, and let $G = S_3$, construed as a binary space according to Example 3.3(b). Then (Q, G) is a binary hyperquasigroup under the operations

$$\begin{array}{lll} xy\,\underline{1} = x \cdot y, & xy\,\underline{\sigma\tau\sigma} = x/y, & xy\,\underline{\tau} = x\backslash y, \\ xy\,\underline{\sigma} = y \cdot x, & xy\,\underline{\tau\sigma} = y/x, & xy\,\underline{\sigma\tau} = y\backslash x \end{array}$$

([9, Proposition 5.2]).

Example 4.7. For a positive integer n and a unital ring R, consider the n-ary space G of Example 3.5. Let Q be a unital right R-module. Define the n-ary action

$$x_n \dots x_1\, \underline{(u_n, \dots, u_1)} = x_n u_n + \dots + x_1 u_1$$

for x_i in Q and $(u_n, \dots, u_1)$ in G. Then (Q, G) is an n-ary hyperquasigroup.

The meaning of hypercommutativity in an n-hyperquasigroup is immediate. The significance of hypercancellation is interpreted as follows (compare [5], [9] for the binary case).

Proposition 4.8. *Let (Q, G) be an n-hyperquasigroup. For each point g in G, define*

$$\widehat{g} : Q^n \to Q^n;\ (x_n, \dots, x_2, x_1) \mapsto (x_n, \dots, x_2, x_n \dots x_1 \underline{g}).$$

Then $\widehat{\tau g}$ is the two-sided inverse of $\widehat{g}$ in the semigroup of selfmaps on the set Q^n.

Proof: The equation $\widehat{g}\,\widehat{\tau g} = \mathrm{id}_{Q^n}$ is immediate from (4.3), while $\widehat{\tau g}\,\widehat{g} = \mathrm{id}_{Q^n}$ follows from (4.3) with g replaced by τg, recalling $\tau\tau g = g$. $\square$

Remark 4.9. For an n-ary operation

$$Q^n \to Q;\ (x_n, \ldots, x_1) \mapsto x_n \ldots x_1\omega$$

on a set Q, the invertibility of the map

$$\widehat{\omega} : Q^n \to Q^n;\ (x_n, \ldots, x_2, x_1) \mapsto (x_n, \ldots, x_2, x_n \ldots x_1\omega)$$

does not mean that (Q, ω) is a (combinatorial) n-quasigroup. For example, the binary projection

$$\pi_1 : Q^2 \to Q;\ (x_0, x_1) \mapsto x_1$$

has $\widehat{\pi_1} = \mathrm{id}_{Q^2}$.

5. From hyperquasigroups to quasigroups

By Proposition 4.5 and Remark 4.9, hypercancellativity alone is insufficient for a quasigroup. The following theorem shows that quasigroups are obtained from the combination of hypercommutativity and hypercancellativity. The binary case appeared as [9, Theorem 6.1]. The proof of the general case given here is conceptually simpler, although the details are more complex.

Theorem 5.1. *For a positive integer n, let (Q, G) be an n-hyperquasigroup. Then for each element g of the n-ary space G, there is an equational n-quasigroup*

$$\left(Q, \underline{g}, \underline{\tau g}, \ldots, \underline{\sigma^{i-1}\tau\sigma^{1-i}g}, \ldots, \underline{\sigma^{n-1}\tau\sigma^{1-n}g}\right)$$

with multiplication $\underline{g}$ and divisions $\underline{\sigma^{i-1}\tau\sigma^{1-i}g}$ for $1 \leq i \leq n$.

Proof: The identities (1.4) and (1.5) must be established for $1 \leq i \leq n$, with $\mu = \underline{g}$ and $\mu^i = \underline{\sigma^{i-1}\tau\sigma^{1-i}g}$. Consider the hypercancellativity

$$x_n \ldots x_2 \left(x_n \ldots x_1 \underline{g}\right) \underline{\tau g} = x_1 \tag{5.1}$$

as in (4.3). Applying hypercommutativity $i-1$ times to the inner operation of (5.1) yields

$$x_n \ldots x_2 \left(x_{n-(i-1)} \ldots x_2 x_1 x_n \ldots x_{n-(i-2)} \underline{\sigma^{i-1} g}\right) \underline{\tau g} = x_1.$$

Applying hypercommutativity $i-1$ times to the outer operation then gives

$$x_{n-(i-1)} \ldots x_2 \left(x_{n-(i-1)} \ldots x_2 x_1 x_n \ldots x_{n-(i-2)} \underline{\sigma^{i-1} g}\right) x_n \ldots$$
$$\ldots x_{n-(i-2)}\, \underline{\sigma^{i-1}\tau g} = x_1.$$

Replacing x_k by

$$\begin{cases} x_{k+(i-1)} & \text{for} \quad 1 \le k \le n-(i-1), \\ x_{k+(i-1)-n} & \text{for} \quad n-(i-2) \le k \le n \end{cases}$$

yields

$$x_n \dots x_{i+1} \left(x_n \dots x_1 \, \underline{\sigma^{i-1} g} \right) x_{i-1} \dots x_1 \, \underline{\sigma^{i-1} \tau g} = x_i. \tag{5.2}$$

Replace g in (5.2) by $\sigma^{1-i} g$ to obtain

$$x_n \dots x_{i+1} \left(x_n \dots x_1 \, \underline{g} \right) x_{i-1} \dots x_1 \, \underline{\sigma^{i-1} \tau \sigma^{1-i} g} = x_i,$$

which is (1.4). Finally, replace g in (5.2) by $\tau\sigma^{1-i} g$ to obtain

$$x_n \dots x_{i+1} \left(x_n \dots x_1 \, \underline{\sigma^{i-1} \tau \sigma^{1-i} g} \right) x_{i-1} \dots x_1 \, \underline{g} = x_i,$$

which is (1.5). □

Corollary 5.2. *For a positive integer n, let (Q, G) be an n-hyperquasigroup. Then each point g of the n-ary space G yields a combinatorial n-quasigroup $(Q, \underline{g})$.*

6. The structure theorem

Let n be a positive integer. In the symmetric group $S_{n+1} = \{0, 1, \dots, n\}!$, consider the involution

$$\alpha = (2\ n)(3\ n-1) \dots \begin{cases} \dots \left(\frac{n}{2}\ \frac{n+4}{2}\right), & n \text{ even;} \\ \dots \left(\frac{n+1}{2}\ \frac{n+3}{2}\right), & n \text{ odd.} \end{cases}$$

Define a surjective homomorphism

$$M_n \to S_{n+1};\ \pi \mapsto \overline{\pi} \tag{6.1}$$

by concatenating the surjective homomorphism r of (2.2) with conjugation by the permutation α in S_{n+1}. In particular,

$$\overline{\sigma} = (1\ 2\ \dots\ n)^{\alpha} = (1\ n\ \dots\ 2) \tag{6.2}$$

and

$$\overline{\tau} = (0\ 1)^{\alpha} = (0\ 1). \tag{6.3}$$

Lemma 6.1. *Let (Q, G) be an n-hyperquasigroup. Then*

$$x_n \dots x_2 x_1 \underline{g} = x_0 \quad \Leftrightarrow \quad x_{n\overline{\pi}} \dots x_{2\overline{\pi}} x_{1\overline{\pi}} \underline{\pi g} = x_{0\overline{\pi}} \tag{6.4}$$

for each element π of M_n, point g in G, and elements $x_0, \dots, x_n$ of Q.

Proof: The equivalence (6.4) holds trivially for $\pi = 1$. Suppose that it holds for a certain element π of M_n. Then

$$\begin{aligned} & x_{n\overline{\pi}} \dots x_{2\overline{\pi}} x_{1\overline{\pi}}\, \underline{\pi g} = x_{0\overline{\pi}} \\ \Leftrightarrow \quad & x_{(n-1)\overline{\pi}} \cdots x_{1\overline{\pi}} x_{n\overline{\pi}}\, \underline{\sigma\pi g} = x_{0\overline{\pi}} \\ \Leftrightarrow \quad & x_{n\overline{\sigma\pi}} \dots x_{2\overline{\sigma\pi}} x_{1\overline{\sigma\pi}}\, \underline{\sigma\pi g} = x_{0\overline{\sigma\pi}} \end{aligned}$$

by the hypercommutativity (4.2) and (6.2). Thus the equivalence (6.4) holds for $\sigma\pi$ in M_n. Again,

$$\begin{aligned} & x_{n\overline{\pi}} \dots x_{2\overline{\pi}} x_{1\overline{\pi}}\, \underline{\pi g} = x_{0\overline{\pi}} \\ \Leftrightarrow \quad & x_{n\overline{\pi}} \dots x_{2\overline{\pi}} x_{0\overline{\pi}}\, \underline{\tau\pi g} = x_{1\overline{\pi}} \\ \Leftrightarrow \quad & x_{n\overline{\tau\pi}} \dots x_{2\overline{\tau\pi}} x_{1\overline{\tau\pi}}\, \underline{\pi g} = x_{0\overline{\tau\pi}} \end{aligned}$$

by the hypercancellativity (4.3) and (6.3) Thus the equivalence (6.4) holds for $\tau\pi$ in M_n. By induction, it follows that the equivalence (6.4) holds for each element of M_n. □

Let (Q, G) be an n-hyperquasigroup. Set

$$\underline{G} = \{\underline{g} : Q^n \to Q \mid g \in G\}.$$

By Lemma 6.1, the action

$$M_n \to \underline{G}!; \ \pi \mapsto (\underline{g} \mapsto \underline{\pi g})$$

factorizes through the homomorphism (6.1) to S_{n+1}. Thus the set $\underline{G}$ of n-ary operations on Q is an S_{n+1}-set. For a point g in the space G, Corollary 5.2 yields n-quasigroups $(Q, \underline{\pi g})$ given by the S_{n+1}-orbit of $\underline{g}$. The various n-quasigroups in a given orbit are described as mutual *conjugates* or *parastrophes*. For binary quasigroups, these concepts are well known ([1, Example II.6.1], [7]). For unary quasigroups, as invertible dynamical systems, conjugation corresponds to time reversal.

The structure of $(Q, \underline{G})$ may now be summarized as follows (compare [9, Theorem 6.7] for the binary case).

Theorem 6.2. *Let n be a positive integer. Then each n-hyperquasigroup (Q, G) yields an algebra structure $(Q, \underline{G})$ consisting of the union of mutually disjoint sets of conjugate n-quasigroup operations.*

Remark 6.3. Let $(Q, \varphi, \varphi^1, \dots, \varphi^n)$ be an n-quasigroup. Consider M_n as an n-ary space (M_n, σ, τ) given by the free left M_n-set, so that the actions (3.1) and (3.2) are the left multiplications by σ and τ in the group M_n. Use the specification

$$x_n \dots x_2 x_1 \underline{1} = x_n \dots x_2 x_1 \varphi$$

together with (6.4) to define an n-ary action of M_n on Q. A comparison with Theorem 5.1 and its proof shows that

$$\varphi^i = \underline{\sigma^{i-1} \tau \sigma^{1-i}}$$

for $1 \leq i \leq n$. One then obtains (Q, M_n) as a hyperquasigroup. Within this hyperquasigroup, the n-quasigroup $(Q, \underline{1})$ yielded by Theorem 5.1 realizes the given n-quasigroup (Q, φ). By Theorem 6.2, the n-quasigroups $(Q, \underline{g})$ for g in M_n are the conjugates of (Q, φ).

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Department of Mathematics, Iowa State University, Ames, Iowa 50011-2064, U.S.A.

E-mail: jdhsmith@math.iastate.edu
URL: `http://orion.math.iastate.edu/jdhsmith/`

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