Möbius gyrovector spaces in quantum information and computation

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Abstract. Hyperbolic vectors, called gyrovectors, share analogies with vectors in Euclidean geometry. It is emphasized that the Bloch vector of Quantum Information and Computation (QIC) is, in fact, a gyrovector related to Möbius addition rather than a vector. The decomplexification of Möbius addition in the complex open unit disc of a complex plane into an equivalent real Möbius addition in the open unit ball $\mathbb{B}^2$ of a Euclidean 2-space $\mathbb{R}^2$ is presented. This decomplexification proves useful, enabling the resulting real Möbius addition to be generalized into the open unit ball $\mathbb{B}^n$ of a Euclidean n-space $\mathbb{R}^n$ for all $n \geq 2$. Similarly, the decomplexification of the complex 2×2 qubit density matrix of QIC, which is parametrized by the real, 3-dimensional Bloch gyrovector, into an equivalent (in a specified sense) real 4×4 matrix is presented. As in the case of Möbius addition, this decomplexification proves useful, enabling the resulting real matrix to be generalized into a corresponding matrix parametrized by a real, n-dimensional Bloch gyrovector, for all $n \geq 2$. The applicability of the n-dimensional Bloch gyrovector with $n = 3$ to QIC is well known. The problem as to whether the n-dimensional Bloch gyrovector with $n > 3$ is applicable to QIC as well remains to be explored.

Keywords: quantum information, Bloch vector, density matrix, hyperbolic geometry, gyrogroups, gyrovector spaces

Classification: 51M10, 51P05, 81P15

1. Introduction

Hyperbolic vectors, called gyrovectors, share analogies with vectors in Euclidean geometry, as shown in Section 3 for vectors, in Figure 1, and for gyrovectors, in Figure 2.

In the standard model of Euclidean geometry vectors are added by either (i) ordinary vector addition, or by the (ii) parallelogram addition law, where the two kinds of addition in (i) and (ii) are identically *equal* to one another.

In full analogy, in the Poincaré ball model of hyperbolic geometry gyrovectors are added by either (i) Möbius vector addition, as we show in Section 2, or by the (ii) hyperbolic parallelogram (gyroparallelogram) addition law, as shown in [24, Figures 4–5] and [23], [26]. By a remarkable disanalogy, here the two kinds of addition in (i) and (ii) are *different*.

Möbius addition stems from the well-known Möbius transformation of the complex open unit disc of a complex plane, as shown in Section 2. In order to extend it from the complex open unit disc $\mathbb{D}$ of a complex plane to the open unit ball $\mathbb{B}^n$ of a Euclidean n-space $\mathbb{R}^n$, for any $n \geq 2$, we *decomplexify* it in Section 2. The resulting real Möbius addition turns out to be susceptible of obvious generalization into higher dimensions. Furthermore, the generalized Möbius addition admits scalar multiplication, giving rise to Möbius gyrovector spaces, presented in Section 3. Möbius gyrovector spaces are studied in [20], [23], [26] where it is shown that they form the algebraic setting for the Poincaré ball model of hyperbolic geometry just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

The 3-dimensional Bloch gyrovector of QIC is presented in Section 4 as the parameter of the so called *qubit density matrix*, which is the complex, Hermitian 2×2 matrix (21). In order to extend it from 3 dimensions to $n > 3$ dimensions we decomplexify the complex qubit density matrix in Section 5, thus recovering a real matrix which is susceptible of obvious generalization into higher dimensions, as shown in Section 6. The resulting generalized real qubit density matrix is parametrized by the n-dimensional Bloch gyrovector, which is regulated by Möbius addition in the ball $\mathbb{B}^n$ of the Euclidean n-space $\mathbb{R}^n$. By discovering the n-dimensional Bloch gyrovector that parametrizes a generalized qubit density matrix, and which is regulated algebraically by a Möbius gyrovector space we have completed the task we face in this paper. It remains to explore whether the n-dimensional Bloch gyrovector with $n > 3$ is applicable to QIC by extending the use of the 3-dimensional Bloch gyrovector in the study of two-level quantum systems to the use of the n-dimensional Bloch gyrovector in the study of higher-level quantum systems.

2. Möbius addition in the disc and the ball

Möbius transformations of the disc $\mathbb{D}$,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \tag{1}$$

of the complex plane $\mathbb{C}$ offer in [24], [27] an elegant way to introduce the grouplike loops known as *gyrogroups*. Ahlfors' book [1], *Conformal Invariants: Topics in Geometric Function Theory*, begins with a presentation of the Möbius self-transformation of the complex open unit disc $\mathbb{D}$,

$$z \mapsto e^{i\theta}\frac{a+z}{1+\overline{a}z} = e^{i\theta}(a\oplus z), \tag{2}$$

$a, z \in \mathbb{D}$, $\theta \in \mathbb{R}$, where $\overline{a}$ is the complex conjugate of a [7, p. 211], [10, p. 185], [18, pp. 177–178]. Suggestively, the *polar decomposition* (2) of Möbius transformation of the disc gives rise to *Möbius addition*, $\oplus$,

$$a\oplus z = \frac{a+z}{1+\overline{a}z}\,. \tag{3}$$

Möbius addition, $\oplus$, in the disc $\mathbb{D}$ is neither commutative nor associative. It is, however, both a gyrocommutative and a gyroassociative binary operation that possesses the grouplike structure known as a gyrocommutative gyrogroup, which we motivate and define below.

Being noncommutative, Möbius addition gives rise to *gyrations* $\mathrm{gyr}[a,b]$, for all $a,b\in\mathbb{D}$, which are defined by the equation

$$\mathrm{gyr}[a,b] = \frac{a\oplus b}{b\oplus a} = \frac{1+a\overline{b}}{1+\overline{a}b}\,. \tag{4}$$

Gyrations, in turn, give rise to the *gyrocommutative law* of Möbius addition,

$$a\oplus b = \mathrm{gyr}[a,b](b\oplus a). \tag{5}$$

Being unimodular complex numbers, gyrations represent rotations of the disc $\mathbb{D}$ about its center. The inverse gyration is, again, a gyration,

$$(\mathrm{gyr}[a,b])^{-1} = \mathrm{gyr}[b,a] \tag{6}$$

for all $a,b\in\mathbb{D}$. Furthermore, gyrations respect Möbius addition in the disc in the sense that gyrations and Möbius addition are interchangeable,

$$\mathrm{gyr}[a,b](x\oplus y) = \mathrm{gyr}[a,b]x\oplus \mathrm{gyr}[a,b]y \tag{7}$$

for all $a,b,x,y\in\mathbb{D}$.

The gyrocommutative law (5) of Möbius addition is not terribly surprising since it follows immediately from the gyration definition (4). But, we are not finished. Coincidentally, the gyrations give rise to the left and right gyroassociative laws as well,

$$\begin{aligned} a\oplus(b\oplus c) &= (a\oplus b)\oplus \mathrm{gyr}[a,b]c \\ (a\oplus b)\oplus c &= a\oplus(b\oplus \mathrm{gyr}[b,a]c) \end{aligned} \tag{8}$$

for all $a,b,c\in\mathbb{D}$, as one can readily check.

Coincidences in mathematics are not accidental. The coincidences in our study of Möbius addition in the disc lead us to the tip of a giant iceberg, the notion of gyrogroups and gyrovector spaces. Gyrations endow the Möbius disc groupoid $(\mathbb{D},\oplus)$ with a rich grouplike structure. Furthermore, gyrations possess their own rich structure, which reveals itself in important gyration identities as, for instance, the left and right loop properties,

$$\begin{aligned} \mathrm{gyr}[a,b] &= \mathrm{gyr}[a\oplus b,b] \\ \mathrm{gyr}[a,b] &= \mathrm{gyr}[a,b\oplus a] \end{aligned} \tag{9}$$

for all $a, b \in \mathbb{D}$, as one can straightforwardly check. Many other gyration identities, along with their applications, are found in [20], [23], [26].

Thus, we are led by Möbius addition in the disc to the discovery of the generalized commutative group that we naturally call a *gyrocommutative gyrogroup*, in which commutativity and associativity find a natural extension into gyrocommutativity and gyroassociativity by means of special automorphisms called gyrations. Taking the key features of Möbius addition in the disc as axioms, the formal definition of gyrogroups follows.

Definition 1 (Gyrogroups)**.** A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

$(G1)$ $\qquad 0 \oplus a = a$

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a, satisfying

$(G2)$ $\qquad \ominus a \oplus a = 0.$

Moreover, for any $a, b, c \in G$ there exists a unique element $\mathrm{gyr}[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law

$(G3)$ $\qquad a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c.$

The map $\mathrm{gyr}[a, b] : G \to G$ given by $c \mapsto \mathrm{gyr}[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,

$(G4)$ $\qquad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus),$

and the automorphism $\mathrm{gyr}[a, b]$ of G is called the gyroautomorphism, or the gyration, of G generated by $a, b \in G$. The operator $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called the gyrator of G. Finally, the gyroautomorphism $\mathrm{gyr}[a, b]$ generated by any $a, b \in G$ possesses the left loop property

$(G5)$ $\qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b].$

The gyrogroup axioms $(G1)$–$(G5)$ in Definition 1 are classified into three classes.

(1) The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.
(2) The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.
(3) The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$. In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 2 (Gyrocommutative Gyrogroups)**.** A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law

$(G6)$ $\qquad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$

for all $a, b \in G$.

It is shown in [24] that gyrogroups are loops, so that they are of interest in loop theory. Gyrogroups abound in group theory, as demonstrated in [8], [9], [6]. Some first gyrogroup theorems, some of which are analogous to group theorems, are presented in [23, Chapter 2]. Thus, in particular, the gyrogroup left identity and left inverse are identical with their right counterparts, and the resulting identity and inverse are unique, as in group theory. Furthermore, the left gyroassociative law and the left loop property of gyrogroups are associated with corresponding right counterparts.

Owing to the presence of complex numbers in Möbius addition law (3) in the complex open unit disc $\mathbb{D}$, Möbius addition in its complex form cannot be extended into higher dimensions. In contrast, its real counterpart can be extended to higher dimensions as we will find below. We therefore wish to translate Möbius addition law (3) into its real counterpart. To obtain the translation we identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way,

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2, \tag{10}$$

$i = \sqrt{-1}$, so that the inner product and the norm in $\mathbb{R}^2$ are given by the equations

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \tfrac{1}{2}(\bar{u}v + u\bar{v}), \\ \|\mathbf{u}\| &= |u|. \end{aligned} \tag{11}$$

Accordingly, the translation of Möbius addition from its complex form in (3) into a real form is obtained by the following chain of equations. For all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$ we have [13]

$$\begin{aligned} \mathbb{D} \ni u \oplus v &= \frac{u+v}{1+\bar{u}v} \\ &= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\ &= \frac{(1+\bar{u}v+u\bar{v}+|v|^2)u + (1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\ &= \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u} + (1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \mathbf{u}\oplus\mathbf{v} \in \mathbb{B}^2 \end{aligned} \tag{12}$$

thus translating Möbius addition, $u \oplus v$, in the complex open unit disc $\mathbb{D}$ into Möbius addition, $\mathbf{u}\oplus\mathbf{v}$, in the real open unit disc $\mathbb{B}^2$ of the Euclidean plane $\mathbb{R}^2$.

The first equation in (12) is a complex number equation. It is therefore restricted to the two dimensions of the disc since complex numbers do not exist in higher dimensions. The last equation in (12) is a vector equation, so that its restriction to the disc $\mathbb{B}^2$ of the Euclidean two-dimensional space $\mathbb{R}^2$ is a mere artifact. Indeed, it survives unimpaired in higher dimensional balls suggesting the following definition of Möbius addition in the ball $\mathbb{B}$ of any real inner product space $\mathbb{V}$.

Definition 3 (Möbius addition in the ball)**.** Let $\mathbb{V}$ be a real inner product space [17], and let $\mathbb{B}$ be the open unit ball of $\mathbb{V}$,

$$\mathbb{B} = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < 1\}. \tag{13}$$

Möbius addition $\oplus$ in the ball $\mathbb{B}$ is a binary operation in $\mathbb{B}$ given by the equation

$$\mathbf{u}{\oplus}\mathbf{v} = \frac{(1 + 2\mathbf{u}{\cdot}\mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u}{\cdot}\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}\,, \tag{14}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{B}$, where $\cdot$ and $\|{\cdot}\|$ are the inner product and norm that the ball $\mathbb{B}$ inherits from its space $\mathbb{V}$.

The binary operation $\oplus$ in (14) is well-defined in the ball since, following Cauchy-Schwartz inequality, we have

$$1 + 2\mathbf{u}{\cdot}\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \geq (1 + \mathbf{u}{\cdot}\mathbf{v})^2 > 0 \tag{15}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$.

Remarkably, like the complex Möbius disc groupoid $(\mathbb{D}, \oplus)$, also the real Möbius ball groupoid $(\mathbb{B}, \oplus)$ forms a gyrocommutative gyrogroup [23], [24], [25], [26], [27], called a Möbius gyrogroup.

We thus see from (12) and Definition 3 that the decomplexification of the complex Möbius addition $u{\oplus}v$ into its real counterpart $\mathbf{u}{\oplus}\mathbf{v}$ is rewarding. Unlike the former, the latter admits a natural generalization into higher dimensions. In Section 5 we will encounter a decomplexification of a complex density matrix, which will prove rewarding as well, allowing an extension to higher dimensions.

Möbius addition in the ball $\mathbb{B}$ of any real inner product space $\mathbb{V}$ satisfies the *gamma identity*

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1 + 2\mathbf{u}{\cdot}\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{16}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$, where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \|\mathbf{v}\|^2}} \tag{17}$$

in the ball $\mathbb{B}$. For any $\mathbf{v} \in \mathbb{V}$ we have the result that $\mathbf{v} \in \mathbb{B}$ if and only if its gamma factor $\gamma_{\mathbf{v}}$ is real. The gamma identity (16), along with inequality (15), demonstrates that $\mathbf{u}, \mathbf{v} \in \mathbb{B} \Rightarrow \mathbf{u}{\oplus}\mathbf{v} \in \mathbb{B}$ as anticipated in Definition 3.

3. Möbius scalar multiplication in the ball

Möbius addition in the ball admits scalar multiplication, turning itself into a gyrovector space according to the following definition.

Definition 4 (Möbius scalar multiplication)**.** Let $(\mathbb{B}, \oplus)$ be a Möbius gyrogroup. Then its corresponding Möbius gyrovector space $(\mathbb{B}, \oplus, \otimes)$ involves the Möbius scalar multiplication $r\otimes\mathbf{v} = \mathbf{v}\otimes r$ in $\mathbb{B}$, given by the equation

$$\begin{aligned} r\otimes\mathbf{v} &= \frac{(1+\|\mathbf{v}\|)^r - (1-\|\mathbf{v}\|)^r}{(1+\|\mathbf{v}\|)^r + (1-\|\mathbf{v}\|)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \tanh(r\ \tanh^{-1}\|\mathbf{v}\|)\frac{\mathbf{v}}{\|\mathbf{v}\|}\,, \end{aligned} \tag{18}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{B}$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes\mathbf{0} = \mathbf{0}$.

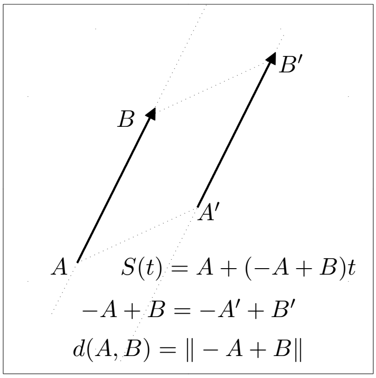

Figure 1. Vector space approach to the standard model of Euclidean geometry in the Euclidean plane $\mathbb{R}^2$. Here, the common vector addition, +, and scalar multiplication are employed with a vector $-A+B$ from point A to point B.

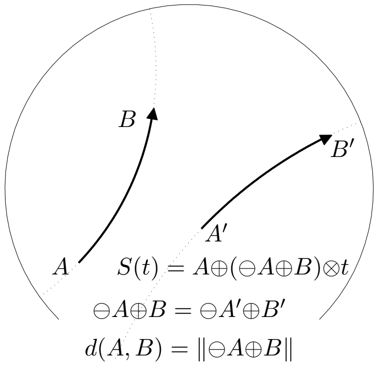

Figure 2. Gyrovector space approach to the Poincaré model of hyperbolic geometry. Here, Möbius gyrovector addition, $\oplus$, and scalar multiplication, $\otimes$, in the open unit disk $\mathbb{B}^2$ of $\mathbb{R}^2$ are employed with a gyrovector $\ominus A\oplus B$ from point A to point B.

Gyrovector spaces are studied in [20], [23], [26], where it is shown that they form the setting for hyperbolic geometry just as vector spaces form the setting for Euclidean geometry. A gyrovector space approach to hyperbolic geometry, fully analogous to the common vector space approach to Euclidean geometry [11], is accordingly presented in [20], [23], [26]. As a striking example we note that the segment between two points $A, B \in \mathbb{R}^2$, $A \neq B$, of the Euclidean plane consists of all the points

$$S(t) = A + (-A+B)t \tag{19}$$

with $0 \le t \le 1$, Figure 1. The directed segment from point A to point B represents a vector with value $-A + B$. Two vectors, $-A + B$ and $-A' + B'$ are equivalent if they have equal values, that is, if $-A + B = -A' + B'$, as shown in Figure 1. Vectors are thus equivalence classes that add according to the parallelogram law. Two equivalent vectors, like the vectors $-A + B$ and $-A' + B'$ in Figure 1, have the same Euclidean length, $\| - A + B\| = \| - A' + B'\|$, and they are parallel.

In full analogy, the hyperbolic segment between two points $A, B \in \mathbb{B}^2$, $A \neq B$ of the two-dimensional Poincaré disc model of hyperbolic geometry consists of all the points

$$S(t) = A \oplus (\ominus A \oplus B) \otimes t \tag{20}$$

with $0 \le t \le 1$, Figure 2, where $\oplus$ and $\otimes$ are Möbius addition and scalar multiplication in the disc $\mathbb{B}^2$. The directed hyperbolic segment from point A to point B represents a hyperbolic vector, called a *gyrovector*, with value $\ominus A \oplus B$. Two gyrovectors, $\ominus A \oplus B$ and $\ominus A' \oplus B'$ are equivalent if they have equal values, that is, if $\ominus A \oplus B = \ominus A' \oplus B'$, as shown in Figure 2. Gyrovectors are thus equivalence classes that add according to the gyroparallelogram law as explained, for instance, in [24] and shown graphically in [24, Figure 8]. Clearly, two equivalent gyrovectors, like the gyrovectors $\ominus A \oplus B$ and $\ominus A' \oplus B'$ in Figure 2, have the same hyperbolic length, $\|\ominus A \oplus B\| = \|\ominus A' \oplus B'\|$.

4. The Bloch gyrovector of QIC

Is the "Bloch vector" of QIC a vector, illustrated in Figure 1, or a gyrovector, illustrated in Figure 2? We will find in this paper that the "Bloch vector" of QIC is actually a gyrovector rather than a vector.

The Bloch vector is well known in the theory of quantum information and computation (QIC). We will find that, in fact, the Bloch vector is not a vector but, rather, a gyrovector which is regulated by Möbius addition [4], [21], [22], [24].

A *qubit* is a two state quantum system completely described by the *qubit density matrix* $\rho_{\mathbf{v}}$ [19],

$$\rho_{\mathbf{v}} = \frac{1}{2} \begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} \tag{21}$$

parametrized by the vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$ in the open unit ball $\mathbb{B}^3$ of the Euclidean 3-space $\mathbb{R}^3$. The vector $\mathbf{v}$ in the ball is known in QIC as the Bloch vector. However, various identities that we present below suggest that it would be more appropriate to consider it as a gyrovector regulated by Möbius addition rather than a vector. Gyrovectors are the hyperbolic vectors illustrated in Figure 2 and studied in [20], [23], [26].

The qubit density matrix $\rho_{\mathbf{v}}$, $\mathbf{v} \in \mathbb{B}^3$, possesses useful properties which are important in QIC. For instance, its trace is 1 and its determinant is positive,

$$\begin{aligned} \operatorname{tr}(\rho_{\mathbf{v}}) &= 1, \\ \det(\rho_{\mathbf{v}}) &= \frac{1}{4\gamma_{\mathbf{v}}^2} = \frac{1}{2^2}(1 - \|\mathbf{v}\|^2) > 0. \end{aligned} \tag{22}$$

The list of properties of the qubit density matrix $\rho_{\mathbf{v}}$ which make it useful in QIC is presented in Section 5.

The product of two qubit density matrices is equivalent to a single qubit density matrix preceded by a PSU(2) matrix,

$$\rho_{\mathbf{u}}\rho_{\mathbf{v}} = \rho_{\mathbf{u}\oplus\mathbf{v}} R(\mathbf{u}, \mathbf{v}), \tag{23}$$

where

$$R(\mathbf{u}, \mathbf{v}) = \tfrac{1}{2} \begin{pmatrix} 1 + \mathbf{u}\cdot\mathbf{v} + i(\mathbf{u} \times \mathbf{v})_3 & (\mathbf{u} \times \mathbf{v})_2 + i(\mathbf{u} \times \mathbf{v})_1 \\ -(\mathbf{u} \times \mathbf{v})_2 + i(\mathbf{u} \times \mathbf{v})_1 & 1 + \mathbf{u}\cdot\mathbf{v} - i(\mathbf{u} \times \mathbf{v})_3 \end{pmatrix}. \tag{24}$$

The matrix $R(\mathbf{u}, \mathbf{v}) \in PSU(2)$ is an elegant matrix with parameters $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$ and with a positive determinant,

$$4 \det(R(\mathbf{u}, \mathbf{v})) = (1 + \mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2. \tag{25}$$

Some decompositions give rise to gyrogroups [8], [9] as, for instance, the polar decomposition (2) that gives rise in Section 2 to the Möbius gyrocommutative gyrogroup $(\mathbb{D}, \oplus)$. Similarly, it follows from the decomposition (23) that the set

$$D = \{\rho_{\mathbf{v}} : \ \mathbf{v} \in \mathbb{B}^3\} \tag{26}$$

of all mixed state qubit density matrices forms a gyrocommutative gyrogroup, with gyrogroup operation given by

$$\rho_{\mathbf{u}} \odot \rho_{\mathbf{v}} = \rho_{\mathbf{u}\oplus\mathbf{v}} \tag{27}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$. Clearly, the resulting gyrocommutative gyrogroup $(D, \odot)$ of qubit density matrices is isomorphic with Möbius gyrogroup $(\mathbb{B}^3, \oplus)$.

The qubit density matrix "symmetric product" of the four qubit density matrices in the following equation, (28), which are parametrized by two distinct Bloch vectors $\mathbf{u}$ and $\mathbf{v}$, is free of PSU(2) matrices. It can be written as a single qubit density matrix parametrized by the Bloch vector $\mathbf{w}$, multiplied by the trace of the matrix product,

$$\rho_{\mathbf{u}}\rho_{\mathbf{v}}\rho_{\mathbf{v}}\rho_{\mathbf{u}} = \operatorname{tr}[\rho_{\mathbf{u}}\rho_{\mathbf{v}}\rho_{\mathbf{v}}\rho_{\mathbf{u}}]\rho_{\mathbf{w}}, \tag{28}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$. Here tr$[m]$ is the trace of a square matrix m, and $\mathbf{w}$,

$$\mathbf{w} := \mathbf{u}\oplus(2\otimes\mathbf{v}\oplus\mathbf{u}) = 2\otimes(\mathbf{u}\oplus\mathbf{v}) \tag{29}$$

is a Bloch gyrovector which is determined by the two Bloch gyrovectors $\mathbf{u}$ and $\mathbf{v}$ in terms of Möbius addition $\oplus$ and Möbius scalar multiplication $\otimes$. Identity (29) is verified in [23, Theorem 6.7]. It clearly demonstrates the compatibility of some qubit density matrix manipulations and the Möbius gyrovector space operations $\oplus$ and $\otimes$ in the open unit ball $\mathbb{B}^3$ of the Euclidean 3-space $\mathbb{R}^3$. The reason why the qubit density product in (28) is free of PSU(2) matrices, unlike the qubit density product in (23), will become clear following the definition of a "symmetric sum" and a "symmetric product" in Definition 5 and its related result in Theorem 6.

The Poincaré ball model of hyperbolic geometry is algebraically regulated by the Möbius gyrovector space structure of the ball [23], [26]. The link between the qubit density matrix and the Möbius gyrovector plane thus exposes the link with the Poincaré ball model of hyperbolic geometry, emphasized by the author in [21], [22]. Following the author, this link was further emphasized and exploited by Péter Lévay in [15], [16].

We now wish to identify the features of the qubit density matrix that link it to the Poincaré ball model of hyperbolic geometry by means of Möbius addition and scalar multiplication in the ball. For the identification of these features we need the following definition and theorem, which are accompanied by illustrative examples.

Definition 5. A symmetric sum of $1+n$ elements $\mathbf{v}_k \in G$, $k = 0, 1, \ldots, n$, of a gyrogroup $(G, \oplus)$ or a gyrovector space $(G, \oplus, \otimes)$, is given by the equation

$$\sum_{k=0}^{n}{}^{\oplus}\, \mathbf{v}_k = \mathbf{v}_n\oplus((\ldots\mathbf{v}_3\oplus((\mathbf{v}_2\oplus((\mathbf{v}_1\oplus(\mathbf{v}_0\oplus\mathbf{v}_1))\oplus\mathbf{v}_2))\oplus\mathbf{v}_3))\ldots\oplus\mathbf{v}_n). \tag{30}$$

Note that in this nonassociative symmetric sum, (30), one starts the gyrosummation with the central element $\mathbf{v}_0$, which is the only term that appears in the symmetric sum once (naturally, the central element $\mathbf{v}_0$ could be the neutral element of G and, hence, be unseen). Then, one gyro-adds $\mathbf{v}_1$ to $\mathbf{v}_0$ on both right (first) and left (second). Then, similarly, one gyro-adds $\mathbf{v}_2$ to the result on both right (first) and left (second), etc., as in (30).

A symmetric product of $2n+1$ qubit density matrices parametrized by the $n+1$ Bloch gyrovectors $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{B}^3$, and raised to the respective powers $r_0, r_1, r_2, \ldots, r_n \in \mathbb{R}$ is the matrix product

$$\rho_s = \prod_{k=0}^{n}{}^{\otimes}\, \rho_{\mathbf{v}_k}^{r_k} = \rho_{\mathbf{v}_n}^{r_n}\rho_{\mathbf{v}_{n-1}}^{r_{n-1}}\ldots\rho_{\mathbf{v}_2}^{r_2}\rho_{\mathbf{v}_1}^{r_1}\rho_{\mathbf{v}_0}^{r_0}\rho_{\mathbf{v}_1}^{r_1}\rho_{\mathbf{v}_2}^{r_2}\ldots\rho_{\mathbf{v}_{n-1}}^{r_{n-1}}\rho_{\mathbf{v}_n}^{r_n}. \tag{31}$$

(Naturally, the central qubit density matrix $\rho_{\mathbf{v}_0}^{r_0}$ could be the identity matrix and, hence, be unseen.)

Let $\rho_s = \boxtimes_{k=1}^{n} \rho_{\mathbf{v}_k}^{r_k}$ be a symmetric matrix product of qubit density matrices. Its Bloch gyrovector $\mathbf{w}$ (or, equivalently, the Bloch gyrovector $\mathbf{w}$ that it possesses) is given by the equation

$$\mathbf{w} = \sum_{k=0}^{n}{}^{\oplus} (r_k \otimes \mathbf{v}_k) \tag{32}$$

in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$.

Examples illustrating Definition 5 follow. The matrix product

$$\rho_s = \rho_{\mathbf{u}} \rho_{\mathbf{v}} \rho_{\mathbf{u}} \tag{33}$$

is symmetric, possessing the Bloch gyrovector

$$\mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{u}). \tag{34}$$

Similarly, the matrix product

$$\rho_s = \rho_{\mathbf{u}} \rho_{\mathbf{v}} \rho_{\mathbf{v}} \rho_{\mathbf{u}} \tag{35}$$

is symmetric, possessing the Bloch gyrovector

$$\mathbf{w} = \mathbf{u} \oplus (2 \otimes \mathbf{v} \oplus \mathbf{u}) = 2 \otimes (\mathbf{u} \oplus \mathbf{v}). \tag{36}$$

The second identity in (36) results from the Two-Sum Identity in [23, Theorem 6.7, p. 140].

Theorem 6. *Let ρ_s be a symmetric matrix product of qubit density matrices,* (31), *possessing the Bloch gyrovector* $\mathbf{w} \in \mathbb{B}^3$, (32). *Then*

$$\rho_s = \mathrm{tr}(\rho_s) \rho_{\mathbf{w}}. \tag{37}$$

Theorem 6 follows from [26, Theorem 2.44]. It states that up to a positive coefficient, $\mathrm{tr}(\rho_s)$, a symmetric matrix product of qubit density matrices, ρ_s, is equivalent to a single qubit density matrix $\rho_{\mathbf{w}}$ parametrized by the Bloch gyrovector $\mathbf{w}$ that it possesses. The latter, in turn, is a gyrovector in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$ generated by the operations $\oplus$ and $\otimes$ in $\mathbb{B}^3$, as (32) indicates.

An illustrative example of a result that follows from Theorem 6 is presented below.

Example. Let

$$\rho_s = \rho_{\mathbf{v}}^n, \tag{38}$$

where n is a positive integer. By Definition 5, ρ_s is a symmetric product, possessing the Bloch gyrovector, (32),

$$\mathbf{w} = \overbrace{\mathbf{v} \oplus \cdots \oplus \mathbf{v}}^{n \text{ terms}} = n \otimes \mathbf{v}. \tag{39}$$

Hence, by Theorem 6,

$$\rho_{\mathbf{v}}^n = \mathrm{tr}(\rho_{\mathbf{v}}^n)\rho_{n\otimes\mathbf{v}} \tag{40}$$

for all $\mathbf{v} \in \mathbb{B}^3$ and $n \in \mathbb{R}$. Remarkably, identity (40) remains valid for any real n as well, expressing any real power $n \in \mathbb{R}$ of a qubit density matrix as a qubit density matrix with a positive coefficient. We may note that this remarkable result follows readily from the spectral theorem [14].

5. Properties of the complex qubit density matrix

The complex 2×2 qubit density matrix $\rho_{\mathbf{v}}$ in (21), $\mathbf{v} \in \mathbb{B}^3$, proves useful in QIC owing to the following five properties that it possesses.

(1) Hermiticity: The matrix $\rho_{\mathbf{v}}$ is Hermitian, that is,

$$\rho_{\mathbf{v}}^\dagger = \rho_{\mathbf{v}}. \tag{41}$$

(2) Unit trace: The matrix $\rho_{\mathbf{v}}$ has a unit trace,

$$\mathrm{tr}(\rho_{\mathbf{v}}) = 1. \tag{42}$$

(3) Positivity: The matrix $\rho_{\mathbf{v}}$ has a positive determinant,

$$\det(\rho_{\mathbf{v}}) > 0. \tag{43}$$

(4) Idempotency: If $\mathbf{v} \in \partial\mathbb{B}^3$ lies on the boundary $\partial\mathbb{B}^3$ of the ball $\mathbb{B}^3$, then $\rho_{\mathbf{v}}$ is idempotent, that is,

$$\rho_{\mathbf{v}}^2 = \rho_{\mathbf{v}}, \qquad (\|\mathbf{v}\| = 1). \tag{44}$$

(5) Symmetric Product Property: If ρ_s is a symmetric matrix product of qubit density matrices, (31), whose Bloch gyrovector is $\mathbf{w}$, (32), then, by (37) in Theorem 6,

$$\rho_s = \mathrm{tr}(\rho_s)\rho_{\mathbf{w}}. \tag{45}$$

Unlike property (5), properties (1)–(4) are well known in the literature. Property (5) of the qubit density matrix $\rho_{\mathbf{v}}$, discovered in [21], [22], demonstrates that the natural home of the Bloch vector $\mathbf{v} \in \mathbb{B}^3$ that parametrizes $\rho_{\mathbf{v}}$ is the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$ where, if parallel transported, it experiences a geometric phase [5], [16], which is the angular defect in hyperbolic geometry as studied in [20], [23], [26].

The set of all qubit density matrices $\rho_{\mathbf{v}}$, $\mathbf{v} \in \mathbb{B}^3$, which is a set of 2×2 Hermitian matrices, (21), is initially a four-dimensional real vector space. The condition of unit trace reduces to a three dimensional submanifold in which one has to locate the domain of positivity. The qubit density matrix $\rho_{\mathbf{v}}$ is parametrized by the Bloch vector $\mathbf{v} \in \mathbb{B}^3$, and it represents two-level quantum systems.

Progress in the quantum spin 1/2 case was triggered by the use of the Bloch vector $\mathbf{v} \in \mathbb{B}^3$ as the parameter of the qubit density matrix $\rho_{\mathbf{v}}$. Naturally, explorers search for generalized Bloch vectors that can illuminate the study of arbitrary spins. Owing to its importance, there is an intensive work in the extension of the qubit density matrices $\rho_{\mathbf{v}}$, and their Bloch vectors $\mathbf{v} \in \mathbb{B}^3$, to higher-level quantum mechanical systems; see, for instance, [2], [3], [12]. Naturally, attempts to generalize the qubit density matrix $\rho_{\mathbf{v}}$ in (21) are guided by its properties. Thus, for instance, in the extension from two-level quantum mechanical systems, called qubits, to three-level quantum mechanical systems, called qutrits, Arvind, Mallesh and Mukunda [3] employed the 3×3 eight Gelmann Hermitian matrices. Further extension is proposed by James, Kwiat, Munro and White [12], who replace the eight Gelmann matrices by a set of 16 Hermitian 4×4 matrices.

Being complex, and having no geometric interpretation [28], it is difficult to extend the qubit density matrix $\rho_{\mathbf{v}}$ into a qubit density matrix that is parametrized by a generalized Bloch vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{B}^n$, $n > 3$, and that represents higher-level quantum mechanical systems. We therefore propose in (46) below a real counterpart, $\mu_{\mathbf{v}}$, $\mathbf{v} \in \mathbb{B}^3$, of $\rho_{\mathbf{v}}$, as a candidate of a real qubit density matrix. The advantage of replacing the complex qubit density matrix $\rho_{\mathbf{v}}$, if possible, by its real counterpart $\mu_{\mathbf{v}}$ lies on the result that the latter admits a natural extension from the three-dimensional open unit ball $\mathbb{B}^3$ to the n-dimensional open unit ball $\mathbb{B}^n$ of the Euclidean n-space $\mathbb{R}^n$ for any $n \geq 2$.

For $n = 3$, let $\mu_{\mathbf{v}} = \mu_{3,\mathbf{v}}$ be the 4×4 real, symmetric matrix

$$\mu_{\mathbf{v}} = \mu_{3,\mathbf{v}} = \frac{1}{2}\begin{pmatrix} 1-\frac{1}{2\gamma_{\mathbf{v}}^2} & v_1 & v_2 & v_3 \\ v_1 & \frac{1}{2\gamma_{\mathbf{v}}^2}+v_1^2 & v_1v_2 & v_1v_3 \\ v_2 & v_1v_2 & \frac{1}{2\gamma_{\mathbf{v}}^2}+v_2^2 & v_2v_3 \\ v_3 & v_1v_3 & v_2v_3 & \frac{1}{2\gamma_{\mathbf{v}}^2}+v_3^2 \end{pmatrix} \tag{46}$$

parametrized by the 3-dimensional vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$. We will find that this vector $\mathbf{v}$ is a 3-dimensional gyrovector that possesses all the useful properties of the Bloch 3-dimensional gyrovector in (21).

Clearly, the matrix $\mu_{\mathbf{v}} = \mu_{3,\mathbf{v}}$ has trace 1 and positive determinant in the ball $\mathbb{B}^3$,

$$\begin{aligned} \operatorname{tr}(\mu_{\mathbf{v}}) &= 1, \\ \det(\mu_{\mathbf{v}}) &= \frac{1}{2^8}(1 - \|\mathbf{v}\|^2)^4 > 0. \end{aligned} \tag{47}$$

The similarity between (47) and (22) is remarkable. Furthermore, it can be shown straightforwardly that the matrix $\mu_{\mathbf{v}} = \mu_{3,\mathbf{v}}$ possesses properties (1)–(5) of the qubit density matrix $\rho_{\mathbf{v}}$, listed in Section 5, (41)–(45), where $\mathbf{v} \in \mathbb{B}^3$ in (41)–(45), except for property (4), where $\mathbf{v} \in \partial\mathbb{B}^3$; and where Hermiticity reduces to symmetry in property (1).

The properties that the complex qubit density matrix $\rho_{\mathbf{v}}$, (21), for mixed state qubits shares with its real counterpart $\mu_{\mathbf{v}}$, (46)–(47), are enhanced by their trace equations

$$\operatorname{tr}(\rho_{\mathbf{u}}\rho_{\mathbf{v}}) = \frac{1}{2}(1 + \mathbf{u}\cdot\mathbf{v}) \quad \text{and} \quad \operatorname{tr}(\mu_{\mathbf{u}}\mu_{\mathbf{v}}) = \frac{1}{4}(1 + \mathbf{u}\cdot\mathbf{v})^2 \tag{48}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$. The identities in (48) are useful, allowing the inner product of $\mathbf{u}$ and $\mathbf{v}$ to be extracted from the trace of the product of their corresponding qubit density matrices.

Hence, in the study of two-level quantum mechanical systems, one may explore the possibility of replacing the Hermitian 2×2 qubit density matrix $\rho_{\mathbf{v}}$ in (21) by its real counterpart, the symmetric 4×4 matrix $\mu_{\mathbf{v}}$ in (46). The latter, in turn, has the advantage of being susceptible of obvious generalization into higher dimensions, as we will see in Section 6.

6. Extending the real density matrix

Let $\mu_{n,\mathbf{v}}$ be the $(n+1)\times(n+1)$ symmetric matrix,

$$\mu_{n,\mathbf{v}} = \frac{2\gamma_{\mathbf{v}}^2}{(n-3)+4\gamma_{\mathbf{v}}^2} \begin{pmatrix} 1 - \frac{1}{2\gamma_{\mathbf{v}}^2} & v_1 & v_2 & v_3 & \cdots & v_n \\ v_1 & \frac{1}{2\gamma_{\mathbf{v}}^2} + v_1^2 & v_1v_2 & v_1v_3 & \cdots & v_1v_n \\ v_2 & v_1v_2 & \frac{1}{2\gamma_{\mathbf{v}}^2} + v_2^2 & v_2v_3 & \cdots & v_2v_n \\ v_3 & v_1v_3 & v_2v_3 & \frac{1}{2\gamma_{\mathbf{v}}^2} + v_3^2 & \cdots & v_3v_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & v_1v_n & v_2v_n & v_3v_n & \cdots & \frac{1}{2\gamma_{\mathbf{v}}^2} + v_n^2 \end{pmatrix} \tag{49}$$

for any $n \geq 2$, parametrized by the gyrovector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{B}^n$. The matrix $\mu_{n,\mathbf{v}}$ has trace 1 and positive determinant in the ball $\mathbb{B}^n$,

$$\begin{aligned} &\operatorname{tr}(\mu_{n,\mathbf{v}}) = 1, \\ &\det(\mu_{n,\mathbf{v}}) = \left(\frac{1 - \|\mathbf{v}\|^2}{(n+1) - (n-3)\|\mathbf{v}\|^2}\right)^{n+1} > 0. \end{aligned} \tag{50}$$

Clearly, the real qubit density matrix $\mu_{n,\mathbf{v}}$ of (49), $n \geq 2$, generalizes the real qubit density matrix $\mu_{\mathbf{v}} = \mu_{3,\mathbf{v}}$ of (46), to which it reduces when $n = 3$. Furthermore, for any $n \geq 2$ the matrix $\mu_{n,\mathbf{v}}$ in (49) possesses properties (1)–(5) in (41)–(45), where $\mathbf{v} \in \mathbb{B}^n$, except for property (4), where $\mathbf{v} \in \partial\mathbb{B}^3$; and where, in property (1), Hermiticity reduces to symmetry. Indeed, if $\mathbf{v} \in \partial\mathbb{B}^n$ lies on the boundary $\partial\mathbb{B}^n$ of the ball $\mathbb{B}^n$ then $\|\mathbf{v}\| = 1$. In this special case the qubit density matrix $\rho_{\mathbf{v}}$ and the real density matrices $\mu_{n,\mathbf{v}}$ represent pure states (as opposed to mixed states that $\rho_{\mathbf{v}}$, $\mathbf{v} \in \mathbb{B}^n$ represents) and are idempotent,

$$\rho_{\mathbf{v}}^2 = \rho_{\mathbf{v}}, \qquad \mu_{n,\mathbf{v}}^2 = \mu_{n,\mathbf{v}}\,, \qquad\qquad (\|\mathbf{v}\| = 1) \tag{51}$$

for all $n \geq 2$. Hence, suggestively, one may explore whether the matrices $\mu_{n,\mathbf{v}}$ with $n > 3$ remain useful as generalized qubit density matrices in the study of higher-level quantum states.

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