Spectrum of twisted Dirac operators on the complex projective space $\mathbb{P}^{2q+1}(\mathbb{C})$

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Abstract. In this paper, we explicitly determine the spectrum of Dirac operators acting on smooth sections of twisted spinor bundles over the complex projective space $\mathbb{P}^{2q+1}(\mathbb{C})$ for $q \geq 1$.

Keywords: complex projective space, Dirac operator, spectral theory

Classification: 53C35, 53C27, 58C40

1. Introduction

For spin Riemannian symmetric spaces, the spectrum of the Dirac operator is explicitly known only in few cases (see, e.g., [6], [7], [9], [10]).

Consider the complex projective space $\mathbb{P}^n(\mathbb{C}) = SU(n+1)/S(U(n)\times U(1))$ with $n = 2q+1$, equipped with the metric induced by the negative of the Killing form of $SU(n+1)$. As a compact simply connected Riemannian symmetric space, the manifold $M = \mathbb{P}^n(\mathbb{C})$ admits a unique homogeneous spin structure (see, e.g., [1]). Let $L = \{([z], w) \in M \times \mathbb{C}^{n+1};\ w \in [z]\}$ be the tautological bundle over M. Recall that each complex line bundle over M is (up to isomorphism) of the form $L^m := L^{\otimes m}$ for some $m \in \mathbb{Z}$. Let $S \longrightarrow M$ be the spinor bundle, and let ∇^S be the spinor connection. We fix $m \in \mathbb{Z}$ and consider the vector bundle $S \otimes L^m$ endowed with the connection $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^{L^m}$, where ∇^{L^m} denotes the Levi-Civita connection on L^m. Let $\Gamma^\infty(S \otimes L^m)$ be the space of smooth sections of the bundle $S \otimes L^m$. The twisted Dirac operator D_m acting on $\Gamma^\infty(S \otimes L^m)$ is defined by $D_m = \widetilde{\mu} \circ \nabla$, where $\widetilde{\mu} : T^*M \otimes S \otimes L^m \longrightarrow S \otimes L^m$ is the bundle homomorphism induced by the Clifford multiplication. Being an elliptic operator, D_m has discrete (real) eigenvalues with finite multiplicities. The aim of this paper is to establish the following result:

Theorem 1. *On the complex projective space of dimension $n = 2q+1$ $(q \geq 1)$, the spectrum of the twisted Dirac operator D_m acting on smooth sections of the spinor bundle tensored with L^m $(m \in \mathbb{Z})$ is the union of the following sets:*

(1) $\Big\{ \pm \sqrt{a_{k,m}(0,0)};\ k \geq \max\big\{0, -\frac{n+1}{2} - m\big\}\Big\}$;

(2) $\Big\{ \pm \sqrt{a_{k,m}(\varepsilon,l)};\ \varepsilon \in \{0,1\},\ 1 \leq l \leq n-1,\ k \geq \max\big\{\varepsilon,\ l - m - \frac{n-1}{2}\big\}\Big\}$;

(3) $\Big\{ \pm \sqrt{a_{k,m}(1,n)};\ k \geq \max\big\{0,\ \frac{n+1}{2} - m\big\}\Big\}$,

where

$$a_{k,m}(\varepsilon,l) = \frac{1}{2(n+1)}\left\{(2k+2m-2\varepsilon+n+1)(k+n-l)\right\}$$

$$\text{for } \ \varepsilon \in \{0,1\} \ \text{ and } \ 0 \leq l \leq n.$$

For $m = 0$, the above theorem gives the spectrum of the classical Dirac operator on $\mathbb{P}^{2q+1}(\mathbb{C})$ which has already been computed by Seifarth and Semmelmann [9]. Observing that each $\mathrm{Spin}^{\mathbb{C}}$-bundle over $\mathbb{P}^{2q+1}(\mathbb{C})$ is (up to isomorphism) of the form $S \otimes L^m$, $m \in \mathbb{Z}$, with S and L being as above (see [4] for generalities on $\mathrm{Spin}^{\mathbb{C}}$ Riemannian manifolds), one obtains from Theorem 1 the spectrum of each $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator on $\mathbb{P}^{2q+1}(\mathbb{C})$, $q \geq 1$.

2. Preliminaries

2.1 Twisted Dirac operator on a Riemannian symmetric space

Let $M = G/K$ be a compact, simply connected n-dimensional irreducible Riemannian symmetric space with G compact and simply connected. We assume that G and K have the same rank and that M has a spin structure (which is necessarily unique). Let $\mathfrak{g}$ and $\mathfrak{k}$ be the respective Lie algebras of G and K. We denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. We identify canonically $\mathfrak{m}$ with the tangent space to M at the point eK, e being the neutral element of G. The fixed Riemannian metric on M is the metric induced by the restriction to $\mathfrak{m}$ of the scalar product $\langle\cdot,\cdot\rangle := -B$ on $\mathfrak{g}$.

The isotropy representation $\mathrm{Ad} : K \longrightarrow SO(\mathfrak{m})$ lifts to a homomorphism $\widetilde{\mathrm{Ad}} : K \longrightarrow \mathrm{Spin}(\mathfrak{m})$ satisfying $\widetilde{\mathrm{Ad}} \circ \lambda = \mathrm{Ad}$, where $\lambda : \mathrm{Spin}(\mathfrak{m}) \longrightarrow SO(\mathfrak{m})$ is the usual 2-fold covering. Let $\kappa : \mathrm{Spin}(\mathfrak{m}) \longrightarrow \mathrm{Aut}(\Delta)$ be the spin representation. The spinor bundle S over M is naturally identified with the homogeneous vector bundle $G \times_{K,\chi} \Delta$, where $\chi := \kappa \circ \widetilde{\mathrm{Ad}}$.

Let τ be an irreducible unitary representation of K on a finite dimensional complex vector space V. Let us set $E = G \times_{K,\tau} V$, the associated homogeneous vector bundle over M, and denote by ∇^E the Levi-Civita connection on E. We consider the spinor bundle twisted by E, i.e. the vector bundle $S \otimes E$, endowed with the connection $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^E$, where ∇^S denotes the spinor connection. The Clifford multiplication $\mu : \mathrm{Cliff}(\mathfrak{m}) \longrightarrow \mathrm{End}(\Delta)$ induces a bundle homomorphism $\widetilde{\mu} : T^*M \otimes S \otimes E \longrightarrow S \otimes E$. Let $\Gamma^\infty(S \otimes E)$ be the space of smooth sections of $S \otimes E$. The twisted Dirac operator acting on $\Gamma^\infty(S \otimes E)$ is defined by $D_E = \widetilde{\mu} \circ \nabla$.

The main goal of this section is to describe the spectrum of D_E. To this end, we next recall a remarkable formula of its square D_E^2 which is essentially due to Parthasarathy [8].

2.2 A formula for the square of D_E

Let us introduce the following vector space:

$$\big(C^\infty(G)\otimes\Delta\otimes V\big)^K := \big\{\varphi\in C^\infty(G)\otimes\Delta\otimes V;\ (r\otimes\chi\otimes\tau)(k)\varphi=\varphi,\ \ \forall\, k\in K\big\},$$

where r denotes the right regular representation of G on $C^\infty(G)$, the space of complex valued smooth functions on G. Obviously, $(C^\infty(G)\otimes\Delta\otimes V)^K$ is isomorphic to $\Gamma^\infty(S\otimes E)$. In what follows, we will identify these two spaces in the canonical fashion.

Fix an orthonormal basis $\{X_1,\ldots,X_d,Y_1,\ldots,Y_{d'}\}$ of $\mathfrak{g}$ with respect to the scalar product $\langle\cdot,\cdot\rangle$ in such a way that $\{X_1,\ldots,X_d\}$ forms a basis of $\mathfrak{m}$ and $\{Y_1,\ldots,Y_{d'}\}$ a basis of $\mathfrak{k}$. The Casimir operators of G and K (relative to $\langle\cdot,\cdot\rangle$) are respectively:

$$\Omega_G=-\sum_{i=1}^{d}X_i^2-\sum_{p=1}^{d'}Y_p^2 \quad\text{and}\quad \Omega_K=-\sum_{p=1}^{d'}Y_p^2.$$

Denote by ν the representation of $\mathfrak{g}$ on $C^\infty(G)$ given by

$$\nu(X)f=Xf \quad (X\in\mathfrak{g}: f\in C^\infty(G)).$$

We have ([8, Proposition 1.1])

$$D_E=\sum_{i=1}^{d}\nu(X_i)\otimes\mu(X_i)\otimes 1.$$

Thus the square of D_E satisfies

$$\begin{aligned} D_E^2&=\Big(\sum_i \nu(X_i)\otimes\mu(X_i)\otimes 1\Big)^2\\ &=-\sum_i\nu(X_i)^2\otimes 1\otimes 1+\frac{1}{2}\sum_{i,j}\nu([X_i,\,X_j])\otimes\mu(X_i)\mu(X_j)\otimes 1.\end{aligned}$$

Using the facts that $[X_i,\,X_j]=\sum_p\langle[X_i,\,X_j],\,Y_p\rangle Y_p$ for $1\le i,j\le d$, and that the action of $Y_p\in\mathfrak{k}$ on Δ is given by $\chi(Y_p)=\frac{1}{4}\sum_{i,j}\langle[Y_p,\,X_i],\,X_j\rangle\,\mu(X_i)\mu(X_j)$, we obtain

$$D_E^2=-\sum_i\nu(X_i)^2\otimes 1\otimes 1+2\sum_p\nu(Y_p)\otimes\chi(Y_p)\otimes 1.$$

By a direct computation, one has

$$\sum_p (\nu\otimes\chi)(Y_p)^2 \otimes 1 - \sum_p \nu(Y_p)^2 \otimes 1\otimes 1 - \sum_p 1\otimes\chi(Y_p)^2 \otimes 1 = 2\sum_p \nu(Y_p)\otimes\chi(Y_p)\otimes 1.$$

Since $(\nu\otimes\chi)(\Omega_K)\otimes 1 = 1\otimes 1\otimes\tau(\Omega_K)$, it follows that

$$D_E^2 = \nu(\Omega_G)\otimes 1\otimes 1 + 1\otimes\chi(\Omega_K)\otimes 1 - 1\otimes 1\otimes\tau(\Omega_K).$$

Let T be a common maximal torus of G and K, and let $\mathfrak{h}$ be its Lie algebra. The Killing form of $\mathfrak{g}$ induces a natural scalar product on the dual $(\mathfrak{h}_{\mathbb{R}})^* := (i\mathfrak{h})^*$ which we also denote by $\langle\cdot,\cdot\rangle$. Let R_G be the root system of G with respect to T. Let R_G^+ be the system of positive roots of G, R_K^+ be the system of positive roots of K, with respect to a fixed lexicographic ordering in R_G. Let δ_G (resp. δ_K) be the half-sum of the positive roots of G (resp. K).

Note that $\chi(\Omega_K) = \frac{R}{8}\,\mathrm{Id}$ with R the (constant) scalar curvature of the Riemannian space M (see [4, Chapter 3] for details). Furthermore, if η is the highest weight of the representation τ, then we have $\tau(\Omega_K) = \langle\eta + 2\delta_K, \eta\rangle\,\mathrm{Id}$ (see, e.g., [11, Lemma 5.6.4]). Consequently, we get

$$D_E^2 = \nu(\Omega_G)\otimes 1\otimes 1 + \left(\frac{R}{8} - \langle\eta + 2\delta_K, \eta\rangle\right).1.$$

For simplicity, we write

$$D_E^2 = \Omega_G + \frac{R}{8} - \langle\eta + 2\delta_K, \eta\rangle.$$

2.3 The spectrum of D_E

Let us fix a K-invariant Hermitian scalar product $(\cdot,\cdot)$ on the space $\Delta\otimes V$. Let $L^2(G,\Delta\otimes V)^K$ denote the completion of the vector space

$$C^\infty(G,\Delta\otimes V)^K := \left\{\psi\in C^\infty(G,\Delta\otimes V);\ \psi(gk) = (\chi\otimes\tau)(k^{-1})\psi(g),\ \forall\, g\in G,\ k\in K\right\}$$

with respect to the scalar product $(\cdot,\cdot)_{L^2}$ defined by

$$(\psi,\psi')_{L^2} = \int_G (\psi(g),\psi'(g))\,dg,$$

where dg is a Haar measure on G.

Let $\widehat{G}$ be the unitary dual of G. For $\gamma \in \widehat{G}$, let $(\pi_\gamma,\, V_\gamma)$ be a fixed representative of the equivalence class γ. Applying the Peter-Weyl theorem, we obtain

$$L^2(G,\, \Delta \otimes V)^K \cong \widehat{\bigoplus_{\gamma\in\widehat{G}}} V_\gamma \otimes \mathrm{Hom}_K(V_\gamma,\, \Delta \otimes V),$$

where $\mathrm{Hom}_K(V_\gamma,\, \Delta \otimes V)$ is the vector space of K-equivariant linear homomorphisms from V_γ to $\Delta \otimes V$. Let $\Delta \otimes V = \bigoplus_{\delta\in\Lambda} m_\delta\, W_\delta$ be the irreducible decomposition of the K-module $\Delta \otimes V$ into irreducible K-submodules, where $\Lambda \subset \widehat{K}$ and $m_\delta \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ for each $\delta \in \Lambda$. Then

$$\begin{aligned} L^2(G,\, \Delta \otimes V)^K &\cong \widehat{\bigoplus_{\gamma\in\widehat{G}}} \bigoplus_{\delta\in\Lambda} m_\delta\, (V_\gamma \otimes \mathrm{Hom}_K(V_\gamma,\, W_\delta)) \\ &\cong \widehat{\bigoplus_{\gamma\in\widehat{G}}} \bigoplus_{\delta\in\Lambda} m_\delta\, m_{\gamma|_K}(\delta)\, V_\gamma, \end{aligned}$$

where $m_{\gamma|_K}(\delta) := \dim_{\mathbb{C}} \mathrm{Hom}_K(V_\gamma,\, W_\delta)$.

The Casimir operator Ω_G acts on the G-module V_γ as a scalar multiple of the identity:

$$\Omega_G\big|_{V_\gamma} = \langle \lambda_\gamma + 2\delta_G,\, \lambda_\gamma\rangle\ \mathrm{Id},$$

where λ_γ is the highest weight of the representation π_γ. Using the identifications $\Gamma^\infty(S\otimes E) = (C^\infty(G) \otimes \Delta \otimes V)^K = C^\infty(G,\, \Delta\otimes V)^K$, one can deduce that the spectrum of the operator D_E^2 is given by

$$\mathrm{Spec}(D_E^2,\, M) = \Big\{ \langle \lambda_\gamma + 2\delta_G,\, \lambda_\gamma\rangle - \langle \eta + 2\delta_K,\, \eta\rangle + \frac{R}{8}\,; \\ \gamma \in \widehat{G},\, \delta \in \Lambda,\, m_{\gamma|_K}(\delta) \neq 0 \Big\}.$$

Since $M = G/K$ is a Riemannian symmetric space, the spectrum of the twisted Dirac operator D_E is symmetric with respect to the origin (see, e.g., [3]). Hence it is completely determined by the spectrum of D_E^2.

3. Spectrum of twisted Dirac operators on $\mathbb{P}^{2q+1}(\mathbb{C})$

Throughout this section, let $G = SU(n+1)$ and $K = S(U(n) \times U(1))$ with $n \geq 2$. The Riemannian symmetric space $G/K = \mathbb{P}^n(\mathbb{C})$ admits a spin structure (which is, of course, unique) if and only if n is odd. In the sequel, we assume that $n = 2q+1$ with $q \geq 1$. Let S be the spinor bundle associated to the homogeneous spin structure of $M = G/K$. Since M is a spin Kähler manifold, one has ([4]) the following isomorphism:

$$S \cong S_0 \oplus \cdots \oplus S_n,$$

where S_n is a complex line bundle satisfying $S_n^2 = K_M := \bigwedge^{n,0}(T^*M)^{\mathbb{C}}$, and $S_{n-l} = \bigwedge^{0,l}(T^*M)^{\mathbb{C}} \otimes S_n$ for $l = 0, \ldots, n$.

Fix the maximal torus

$$T = \{A = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\sum_{j=1}^{n}\theta_j}); \theta_j \in \mathbb{R} \ \text{ for } \ j = 1, \ldots, n\}$$

of G and K, and denote by $\mathfrak{h}$ its Lie algebra. For $j \in \{1, \ldots, n+1\}$, we define the linear functional

$$e_j : \mathfrak{h}^{\mathbb{C}} \longrightarrow \mathbb{C}, \quad H = \mathrm{diag}(h_1, \ldots, h_{n+1}) \longmapsto h_j.$$

Considering the root spaces decomposition of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of G under the action of T, we can choose the following systems of positive roots

$$\begin{aligned} &R_G^+ = \{e_i - e_j;\ 1 \le i < j \le n\} \cup \{e_1 + \cdots + 2\,e_i + \cdots + e_n;\ 1 \le i \le n\} \ \text{ and} \\ &R_K^+ = \{e_i - e_j;\ 1 \le i < j \le n\}, \end{aligned}$$

respectively of G and K relative to T. Then

$$\delta_G = \sum_{j=1}^{n} (n + 1 - j)e_j \quad \text{and} \quad \delta_K = \frac{1}{2}\sum_{j=1}^{n}(n - 2j + 1)e_j.$$

Recall that finite-dimensional irreducible representations of K are classified by their highest weights which are of the form $\eta = \eta_1 e_1 + \cdots + \eta_n e_n$ with $\eta_j \in \mathbb{Z}$ for all $1 \le j \le n$, and $\eta_1 \ge \cdots \ge \eta_n$. Observe that $(S_n)_{eK}$ is an irreducible (one-dimensional!) K-module with highest weight

$$\xi = \tfrac{n+1}{2}\left(e_1 + \cdots + e_n\right).$$

It follows that $(S_{n-l})_{eK}$ is also an irreducible K-module whose highest weight ϑ_l is easily calculated from the observation that $(TM)_{eK} \cong (\mathbb{C}^n)^* \otimes \mathbb{C}$ as a $U(n) \otimes U(1)$-module:

$$\vartheta_l = \begin{cases} \tfrac{n+1}{2}(e_1 + \cdots + e_n) & \text{if } \ l = 0, \\ (\tfrac{n+1}{2} - l)(e_1 + \cdots + e_{n-l}) & \\ +(\tfrac{n-1}{2} - l)(e_{n-l+1} + \cdots + e_n) & \text{if } \ 1 \le l \le n-1, \\ -\tfrac{n+1}{2}(e_1 + \cdots + e_n) & \text{if } \ l = n. \end{cases}$$

Let $L = \{([z], w) \in \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1};\, w \in [z]\}$ be the tautological bundle over $\mathbb{P}^n(\mathbb{C})$. For $m \in \mathbb{Z}$, we set $L^m := L^{\otimes m}$. As a homogeneous vector bundle, L^m is associated to the irreducible K-representation τ^m defined by

$$\tau^m\left(\begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}\right) = (\det A)^m \quad (A \in U(n)),$$

whose highest weight is $\eta^m = m(e_1 + \cdots + e_n)$. Consequently, $(S_{n-l} \otimes L^m)_{eK}$ is an irreducible K-module with highest weight

$$\sigma_{l,m} = \begin{cases} (\frac{n+1}{2} + m)(e_1 + \cdots + e_n) & \text{if } \ l = 0, \\ (\frac{n+1}{2} - l + m)(e_1 + \cdots + e_{n-l}) & \\ +(\frac{n-1}{2} - l + m)(e_{n-l+1} + \cdots + e_n) & \text{if } \ 1 \le l \le n-1, \\ (-\frac{n+1}{2} + m)(e_1 + \cdots + e_n) & \text{if } \ l = n. \end{cases}$$

We shall denote by $\tau_{\sigma_{l,m}}$ the unique (up to equivalence) irreducible representation of K with highest weight $\sigma_{l,m}$.

To compute the spectrum of the twisted Dirac operator $D_m := D_{L^m}$, we need the following well known result (compare [2], [5]):

Proposition 1. *Let ρ_λ (resp. τ_σ) be an irreducible representation of $SU(n+1)$ (resp. $S(U(n)\times U(1))$) with highest weight $\lambda = \sum_{j=1}^n \lambda_j e_j$ (resp. $\sigma = \sum_{j=1}^n \sigma_j e_j$). Then the multiplicity $m(\lambda, \sigma) := m_{\rho_\lambda|_{S(U(n)\times U(1))}}(\tau_\sigma)$ is either 0 or 1, and is equal to 1 if and only if σ is of the form*

$$\sigma = \sum_{j=1}^n (\xi_j - a)e_j$$

with $\lambda_1 \ge \xi_1 \ge \lambda_2 \ge \xi_2 \ge \cdots \ge \lambda_n \ge \xi_n \ge 0$ and $a = \sum_{j=1}^n (\lambda_j - \xi_j)$.

Applying directly this proposition, we get:

Proposition 2. *Let ρ_λ be an irreducible representation of $SU(n+1)$ with highest weight $\lambda = \sum_{j=1}^n \lambda_j e_j$, and let $\tau_{\sigma_{l,m}}$ be as before with $0 \le l \le n$ and $m \in \mathbb{Z}$. Then the multiplicity $m(\lambda, \sigma_{l,m})$ is non-zero (and hence equal to 1) if and only if,*

(i) *for $l = 0$, λ is of the form*

$$\lambda = \left(\frac{n+1}{2} + 2\,k + m\right) e_1 + \left(\frac{n+1}{2} + k + m\right) (e_2 + \cdots + e_n)$$

with $k \ge \max\{0, -\frac{n+1}{2} - m\}$;

(ii) *for* $1 \leq l \leq n-1$, λ *is of the form*

$$\begin{aligned}\lambda = &\left(\frac{n+1}{2} + 2\,k + m - l - \varepsilon\right) e_1 + \left(\frac{n+1}{2} + k + m - l\right) (e_2 + \cdots + e_{n-l}) \\ &+ \left(\frac{n-1}{2} + k + m - l + \varepsilon\right) e_{n-l+1} \\ &+ \left(\frac{n-1}{2} + k + m - l\right) (e_{n-l+2} + \cdots + e_n)\end{aligned}$$

with $\varepsilon \in \{0, 1\}$ *and* $k \geq \max\{\varepsilon, l - m - \frac{n-1}{2}\}$*;*

(iii) *for* $l = n$, λ *is of the form*

$$\lambda = \left(-\frac{n+1}{2} + 2\,k + m\right) e_1 + \left(-\frac{n+1}{2} + k + m\right) (e_2 + \cdots + e_n)$$

with $k \geq \max\{0, \frac{n+1}{2} - m\}$*.*

On the Lie algebra $\mathfrak{g} = \mathfrak{su}(n+1)$, we fix the scalar product $\langle \cdot, \cdot \rangle$ defined by

$$\langle X, Y \rangle = -B(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g},$$

where B is the Killing form of $\mathfrak{g}$. As usual, we extend $\langle \cdot, \cdot \rangle$ to a scalar product on the vector space of real valued linear forms on $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$. Observe that

$$\langle e_i, e_i \rangle = \frac{n}{2\,(n+1)^2} \quad \text{and} \quad \langle e_i, e_j \rangle = \frac{-1}{2\,(n+1)^2}$$

for $1 \leq i \neq j \leq n$.

The scalar curvature of the Riemannian space $\mathbb{P}^n(\mathbb{C}) = G/K$ is $R = n$. Thus the spectrum of the twisted Dirac operator D_m is given by

$$\begin{gathered}\mathrm{Spec}(D_m, \mathbb{P}^n(\mathbb{C})) = \left\{ \pm \sqrt{c(\lambda)};\ \ \exists\, l \in \{0, \ldots, n\}, : m(\lambda, \sigma_{l,m}) \neq 0 \right\} \ \text{ with} \\ c(\lambda) := \langle \lambda + 2\delta_G, \lambda \rangle - \langle \eta^m + 2\delta_K, \eta^m \rangle + \frac{n}{8}\,.\end{gathered}$$

Computing the eigenvalues of D_m, we can easily derive Theorem 1.

Acknowledgment. The author would like to thank Tilmann Wurzbacher for valuable discussions. This paper is based on the author's Ph.D. Thesis.

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(*Received* August 1, 2007, *revised* December 10, 2007)