Biembeddings of symmetric configurations and 3-homogeneous Latin trades

M.J. Grannell, T.S. Griggs, M. Knor

Abstract. Using results of Altshuler and Negami, we present a classification of biembeddings of symmetric configurations of triples in the torus or Klein bottle. We also give an alternative proof of the structure of 3-homogeneous Latin trades.

Keywords: topological embedding, torus, Klein bottle, 6-regular graph, symmetric configuration of triples, partial Latin square, 3-homogeneous Latin trade

Classification: 05B15, 05B30, 05C10

1. Introduction

This paper is concerned with three related problems in different areas of Combinatorial Mathematics. We first introduce the problems, and the appropriate terminology to describe them, and discuss their relationship.

Problem #1. Let $G = (V, E)$ be a regular graph of valency 6, where V is the vertex set and E is the edge set. It is **not** assumed that G is simple, i.e. G may contain loops and/or multiple edges. However we do require that G is connected. The first problem is from Topological Graph Theory: to determine all triangular embeddings of such 6-regular graphs in a closed connected 2-manifold. In such an embedding let F denote the set of triangular faces. Then if $|V| = n$, the Euler characteristic $|V| + |F| - |E| = n + 2n - 3n = 0$. Hence in the orientable case, the supporting surface is the torus, and in the nonorientable case, the Klein bottle.

Problem #2. An (n_r, b_k) *configuration* is an incidence structure of n points and b lines such that

1. each line contains k points,
2. each point lies on r lines,
3. two different points are connected by at most one line.

If $b = n$, and therefore $r = k$, the configuration is said to be *symmetric* and denoted by n_k. Our interest is in the case where $k = 3$ and the problem of biembedding a pair of symmetric configurations of triples in a closed surface, where each triple is realized as a triangular face. To explain precisely what we mean by this and to give some background and motivation for why it may be of interest, first note that when a graph is embedded in a surface, or indeed a pseudosurface, the faces of the embedding may be thought of as some kind of combinatorial design. In the case where the graph is the complete graph K_n and the embedding is a triangulation, the faces form a *twofold triple system*, TTS(n), and if the surface is orientable, also a *Mendelsohn triple system*, MTS(n). See [4] for precise definitions of these terms. If in addition the faces of the embedding can be properly 2-coloured, i.e. no two faces with a common edge have the same colour, then the triangles of each colour class, say black and white, form the blocks of a *Steiner triple system*, STS(n). We here recall that an STS(n) is an ordered pair (V, B), where V is an n-element set (the *points*) and B is a set of 3-element subsets of V (the *blocks*) such that every 2-element subset of V appears in precisely one block. Such systems are known to exist if and only if $n \equiv 1$ or 3 $(\mathrm{mod}\, 6)$ [11], see also [4].

The above description leads naturally to the question of which pairs of isomorphism classes of Steiner triple systems admit biembeddings in an orientable or nonorientable surface. The question seems to be a deep one. A survey of work in this area is [7] but a definitive answer to the question seems far away. In [7], the first two authors conjectured that **every** pair of isomorphism classes of STS(n)s, $n \equiv 1$ or 3 $(\mathrm{mod}\, 6)$ and $n \geq 9$ admit a biembedding in a nonorientable surface, but the situation with regard to orientable biembeddings is less clear.

A similar problem may be asked concerning the biembedding of pairs of Latin squares of side n. These correspond to face 2-colourable triangular embeddings of regular complete tripartite graphs $K_{n,n,n}$ which, as was shown in [8], can exist if and only if the surface is orientable. In a recent paper [13], Lefevre, Donovan, and the first two authors have given the first known infinite class of pairs of Latin squares which **cannot** be biembedded. No Latin square which is the Cayley table of an Abelian 2-group may be biembedded with any isomorphic copy of itself.

It seems natural therefore to investigate the biembedding of pairs of symmetric configurations of triples. In this case the embedded graph is the incidence graph of each of the two configurations, where two vertices are joined by an edge if they occur together in some triple. This is the second problem.

Problem #3. A *partial Latin square* is an $n \times n$ array in which each cell is either empty or contains precisely one entry, and no entry occurs more than once in any row or column. Let T be a partial Latin square. Then T is a *Latin trade* if there exist a partial Latin square T' with the properties that

1. a cell is filled in T' if and only if it is filled in T,
2. no entry occurs in the same cell in T and T',
3. in any given row or column, T and T' contain precisely the same entries.

The partial Latin square T' is called a *trade mate* of T and the unordered pair $\{T, T'\}$ a *Latin bitrade*.

A Latin trade T is *k-homogeneous* if each row and column contains precisely

k entries and each entry occurs precisely k times in T. An example of a 2-homogeneous Latin trade is the Latin square of order 2 and it is easy to see that every 2-homogeneous Latin trade is the disjoint union of such squares. But the situation for $k = 3$ is more complex as the example in Figure 1 shows. The third problem is to classify all 3-homogeneous Latin trades.

2	3	4	
3		1	4
	1	2	3
1	4		2

3	4	2	
1		4	3
	3	1	2
2	1		4

FIGURE 1. A 3-homogeneous Latin bitrade

The status of these problems is as follows. The solution to Problem #1 is known. All triangulations of 6-regular graphs on the torus were determined by Altshuler [1], and later by Negami [14]. In a further paper, Negami [15], extended this work to the Klein bottle. This latter paper is a preprint and seems never to have been published. This is a pity because the work deserves to be better known and possibly the present paper will help to correct the situation.

In contrast, the solution to Problem #2 on the biembedding of pairs of symmetric configurations of triples is unknown. However, observe that the embedded graph of such a biembedding is 6-regular. But not every triangular embedding of a 6-regular graph yields a biembedding of a pair of symmetric configurations of triples. We need the extra property that the graph embedding is face 2-colourable. So, by identifying those embeddings contained in the papers of Altshuler and Negami which have this additional property, a solution can be obtained. In short, Problem #2 is a subproblem of Problem #1 and it is an aim of the present paper to present a solution to Problem #2.

The solution to Problem #3 is also known. A construction for 3-homogeneous Latin trades, based on a hexagonal packing of circles in the plane, is given in [3]. Cavenagh [2], then classified all such trades, showing in fact that the construction in [3] gives **all** 3-homogeneous Latin trades. But in fact Problem #3 is a subproblem of Problem #2 and a further aim of the present paper is to present an alternative solution to the classification of 3-homogeneous Latin trades based on the work of Altshuler and Negami. To our mind this is simpler than the exposition given in [3] and also to that given in [2].

Let T be a 3-homogeneous Latin trade and let T' be a (not necessarily unique) trade mate of T. Then T' is also a 3-homogeneous Latin trade. For both T and T', let $R = \{r_1, r_2, \dots, r_n\}$, $C = \{c_1, c_2, \dots, c_n\}$, and $E = \{e_1, e_2, \dots, e_n\}$ be the sets of rows, columns and entries respectively. The Latin trade T can now be represented as a set S_T of triples $\{r_i, c_j, e_k\}$ where the entry e_k occurs in row r_i, column c_j of the trade. From the definition of 3-homogeneity the set S_T is a symmetric configuration of $3n$ points and $3n$ lines and the same is true of the set $S_{T'}$. Now take the sets of triples S_T and $S_{T'}$ as black and white triangular faces respectively and sew them together along common edges to obtain a "topological representation" of the Latin bitrade. This representation will not necessarily be connected but each component will itself be the representation of a 3-homogeneous Latin subtrade. However, by considering the rotation about each point, each component will be a surface, rather than a pseudosurface. So with this in mind, define a 3-homogeneous Latin trade to be *connected* if its representation as described above is a single connected closed surface. Then every 3-homogeneous Latin trade is the disjoint union of connected 3-homogeneous Latin trades and it suffices to classify the latter. As we have seen these can be obtained from biembeddings of symmetric configurations of triples. But not every such biembedding will be the representation of a 3-homogeneous Latin trade and a trade mate. For this to be the case the biembedding will have to have the additional property of being equitably 3-vertex colourable. The three vertex colour classes identify with the row, column, and entry sets of a Latin trade and every triangle of the biembedding has one vertex from each class. The set of black triangles forms a trade T with the trade mate T' given by the white triangles. We identify those biembeddings of symmetric configurations of triples which yield such trades in Section 4. But first we recall the solution to Problem #1: triangular embeddings of 6-regular graphs.

2. Triangular embeddings of 6-regular graphs

Our account and notation follows that of Negami [14], [15]. Considering first triangulations of the torus, we define *the standard* 6*-regular triangulation* $T(p,q,r)$ of the torus. To do this, consider the triangulation, shown in Figure 2, of the domain

$$\{(x,y) \in R^2 :\ 0 \le x \le r,\ 0 \le y \le p\},$$

where p and r are positive integers.

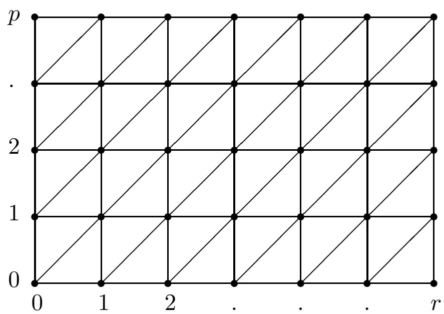

FIGURE 2. Triangulation of $\{(x,y) \in R^2 :\ 0 \le x \le r,\ 0 \le y \le p\}$

In order to convert this into a triangulation of the torus, first identify the upper and lower sides of the rectangle in the usual way to form an open-ended cylinder. The embedded graph of this triangulation we denote by H_r^p, and we make use of this later when considering embeddability in the Klein bottle. Now glue one of the boundaries of the cylinder to the other so that the point $(0,y)$, $0 \le y \le p$ coincides with the point (r,y'), $0 \le y' \le p$ if $y - y' \equiv q \pmod p$, where q is an integer satisfying $0 \le q < p$. Informally we make a "twist" in the cylinder before gluing the two boundaries. This procedure defines the standard triangulation $T(p,q,r)$. For our purposes, the main result in both [1] and [14] is the following theorem.

Theorem 2.1. *Every triangular embedding M of a 6-regular graph G in the torus, is isomorphic to some standard triangulation $T(p,q,r)$.*

We remark that different ordered triples (p,q,r) and (p',q',r') can lead to isomorphic triangular embeddings. For example, as shown in [14], $T(p,q,r)$ is isomorphic to $T(p,q',r)$ if $q' \equiv -(q+r) \pmod p$. Also, the embedded graph G of $T(p,q,r)$ need not be simple, although Negami identifies those which are not. He also goes on to prove that if G is a simple 6-regular graph which has a triangular embedding in the torus, then that triangular embedding is unique up to isomorphism.

Turning now to triangular embeddings of 6-regular graphs in the Klein bottle, we describe the relevant graphs beginning with H_r^p defined above. This has $p(r+1)$ vertices, those vertices with coordinates $(0,j)$ or (r,j) for $0 \le j \le p-1$ have degree 4, but all other vertices have degree 6. From the graph H_r^p and its cylindrical embedding, two families of triangulations of the Klein bottle may be constructed.

The first of these is achieved by identifying, for each y, $0 \le y \le p$, the points with coordinates $(0,y)$ and $(r,p-y)$. These embeddings are called *Klein bottle triangulations of handle type* and denoted by $\mathrm{Kh}(p,r)$.

The construction of the second family of triangulations depends on the parity of p. Again referring to H_r^p, if $p = 2m$ is even, identify the point $(0,y)$ with $(0,y+m)$ and the point (r,y) with $(r,y+m)$, $0 \le y \le m$. If $p = 2m+1$ is odd, use the graph H_{r-1}^p and **join** the point $(0,y)$ to $(0,y+m)$ and the point $(r-1,y)$ to $(r-1,y+m)$, $0 \le y \le p$, with arithmetic on the second coordinate modulo p. In this second case, when p is odd, the Klein bottle is formed by placing the additional joins across two crosscaps. The resulting triangulations, for p even or odd, are called *Klein bottle triangulations of crosscap type* and denoted by $\mathrm{Kc}(p,r)$.

In both $\mathrm{Kh}(p,r)$ and $\mathrm{Kc}(p,r)$ the number of vertices is pr and the two families of triangulations are distinct. The classification theorem given in [15] is now as follows.

Theorem 2.2. *Every triangular embedding M of a 6-regular graph G in the Klein bottle, is isomorphic to precisely one of $\mathrm{Kh}(p,r)$, $p \ge 3$, $r \ge 3$ or $\mathrm{Kc}(p,r)$, $p \ge 5$, $r \ge 2$.*

As with graphs which have triangular embeddings in the torus, Negami proves that the triangular embedding of any 6-regular graph in the Klein bottle is unique. It remains to consider the question of whether any 6-regular graph has a triangulation in both the torus and the Klein bottle. This is not so and follows from the fact that none of the embedded graphs of $T(p,q,r)$ triangulations are isomorphic to any of those of $\mathrm{Kh}(p,r)$ or $\mathrm{Kc}(p,r)$ triangulations. An alternative and perhaps simpler proof of this result, which does not rely on the above classification, is given in [12].

3. Symmetric configurations of triples

As stated in Section 1, biembeddings of pairs of symmetric configurations of triples correspond to face 2-colourable triangulations of 6-regular graphs. It is therefore necessary, and sufficient, to identify these. Clearly all triangulations $T(p,q,r)$ of the torus are face 2-colourable. Moreover the mapping $f : (x,y) \mapsto (r-x, p-y)$, $0 \le x \le r$, $0 \le y \le p$ maps triangles of one colour class to triangles of the other class, i.e. the two biembedded configurations are isomorphic. The same is true of the triangulations $\mathrm{Kh}(p,r)$ of the Klein bottle; they are face 2-colourable and the mapping f interchanges the colour classes. However the triangulations $\mathrm{Kc}(p,r)$ are **not** face 2-colourable. To see this, consider the two cases depending on the parity of p. Let $p = 2m$ be even and attempt to face 2-colour the triangulation. Suppose that the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ is coloured white (W). Then all triangles with vertices $(0,y),(1,y),(1,y+1)$, $0 \le y \le m-1$, must also be coloured white (W) and all triangles with vertices $(0,y),(0,y+1),(1,y+1)$, $0 \le y \le m-1$ must be coloured black (B). Now consider the colours of the triangles around any point $(0,y)$, $0 \le y \le m-1$. The colour scheme is BWBBWB. Thus the triangulation is not face 2-colourable. Now let $p = 2m+1$ be odd and again attempt to face 2-colour the triangulation. As before suppose that the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ is coloured white. Then all triangles with vertices $(0,y)$, $(1,y)$, $(1,y+1)$, $0 \le y \le 2m$, must also be coloured white and all triangles with vertices $(0,y)$, $(0,y+1)$, $(1,y+1)$, $0 \le y \le 2m$ must be coloured black. But now all the triangles formed by joining the point $(0,y)$ to the point $(0,y+m)$, $0 \le y \le m$ across one of the two crosscaps must be coloured white, i.e. the triangles around any point $(0,y)$, $0 \le y \le 2m$ have the colour scheme WWWBWB. Again the triangulation is not face 2-colourable.

The above discussion, together with the results stated in Section 2 that if G is a simple 6-regular graph then any triangular embedding is unique up to isomorphism, thus leads to the following classification theorem.

Theorem 3.1. *A symmetric configuration n_3 is biembeddable in the torus if and only if its incidence graph is isomorphic to the embedded graph of some $T(p,q,r)$. It is biembeddable in the Klein bottle if and only if its incidence graph is isomorphic to the embedded graph of some* $\mathrm{Kh}(p,r)$, $p \ge 3$, $r \ge 3$. *Any such biembedding is unique and the two n_3 configurations that appear in the* *biembedding are isomorphic. No n_3 configuration has a biembedding in both the torus and the Klein bottle.*

Also in [14], Negami lists the isomorphism classes for standard triangulations $T(p,q,r)$ on fewer than 15 vertices. For 11 vertices or less, those with simple embedded graphs comprise $T(n,2,1)$, $7 \le n \le 11$ together with $T(3,0,3)$. In general, $T(n,2,1)$ is the biembedding of the cyclic symmetric configuration on the base set $\{0,1,2,\dots,n-1\}$ generated from the triple $\{0,2,3\}$ under the action of the mapping $z \mapsto z+1 \pmod n$, and the two colour classes that result are isomorphic under $z \mapsto -z \pmod n$. The particular case $T(3,0,3)$ is the biembedding of the Pappus configuration with a copy of itself. It follows that the unique 7_3 and 8_3 configurations, two of the three 9_3 configurations and one of each of the ten 10_3 and 31 11_3 configurations are biembeddable in the torus, and that the remaining configurations on 11 vertices or less are not. Further analysis shows that for the 12_3, 13_3 and 14_3 configurations respectively, only four of 229, two of 2,036 and two of 21,399 are biembeddable in the torus; these are the biembeddings corresponding to the standard triangulations $T(12,2,1)$, $T(12,3,1)$, $T(12,4,1)$, $T(6,2,2)$, $T(13,2,1)$, $T(13,3,1)$, $T(14,2,1)$, and $T(14,3,1)$. The classification also implies that any connected cyclic symmetric configuration n_3 has a unique biembedding with an isomorphic copy of itself in the torus. This is because the incidence graph of such a configuration is isomorphic to the embedded graph of $T(p,q,r)$ for some values of p,q,r. Specifically, if the cyclic symmetric configuration n_3 is generated from the triple $\{0,a,b\}$ under the action of the mapping $z \mapsto z+1 \pmod n$, then we can take $r = \mathrm{GCD}(n, b-a)$, $p = n/r$, and q chosen modulo p such that $q(b-a) \equiv ra \pmod n$. The vertex (x,y) of Figure 2 is identified with $xa + y(b-a) \in Z_n$. An alternative and purely combinatorial proof that any connected cyclic symmetric configuration n_3 has a unique biembedding with an isomorphic copy of itself in the torus appears in [9].

Turning now to biembeddings of symmetric configurations n_3 in the Klein bottle there are just three examples on 14 vertices or less. One of these corresponds to the triangulation $\mathrm{Kh}(3,3)$ and thus gives a biembedding of the third 9_3 configuration. The other two biembeddings are of 12_3 configurations and correspond to the triangulations $\mathrm{Kh}(4,3)$ and $\mathrm{Kh}(3,4)$. Thus, at least for small values of n, the vast majority of n_3 configurations do **not** have minimum genus embeddings, a situation which is in sharp contrast to that for Latin squares and Steiner triple systems. In particular there is no biembedding of the Desargues configuration with itself. However an embedding of the Desargues configuration in the double torus does exist and is contained in [6] and [16].

4. Classification of 3-homogeneous Latin trades

In order to classify connected 3-homogeneous Latin trades, we must determine which of the triangulations $T(p,q,r)$ and $\mathrm{Kh}(p,r)$ can be equitably 3-vertex coloured. Consider first the triangulations $\mathrm{Kh}(p,r)$. Suppose that the vertices

$(0,0)$, $(1,0)$, $(1,1)$ receive colours 0, 1, 2 respectively corresponding to the row, column, and entry sets of the Latin trade. Then the vertex (x,y), $0 \le x \le r$, $0 \le y \le p$ must have colour $x+y \pmod 3$. But vertex $(0,0)$ identifies with vertex (r,p). So $r+p \equiv 0 \pmod 3$. Now vertex $(0,1)$ has colour 1 and vertex $(r,p-1)$ has colour 2. But these two points are also identified. Thus none of the triangulations $\mathrm{Kh}(p,r)$ have equitably 3-vertex colourings. At this point it is perhaps appropriate to recall the theorem proved by the present authors in [8], and already noted in Section 1, that biembeddings of pairs of Latin squares can exist if and only if the surface is orientable. The proof of this result can trivially be extended to any connected Latin bitrade, thus giving an alternative proof that no triangulation $\mathrm{Kh}(p,r)$ can represent a 3-homogeneous Latin trade.

Now consider the triangulations $T(p,q,r)$ and assume the same colouring method as in the previous paragraph. Then since the points $(x,0)$ and (x,p), $0 \le x \le r$ are identified we must first have that $p \equiv 0 \pmod 3$. Further the point $(0,y)$, $0 \le y \le p$ coincides with the point (r,y'), $0 \le y' \le p$, if $y \equiv y'+q \pmod p$ and, since $p \equiv 0 \pmod 3$, this implies that $y \equiv y'+q \pmod 3$. But $y \equiv y'+r \pmod 3$ so $r \equiv q \pmod 3$.

We therefore have the following theorem.

Theorem 4.1. *There is a one-one correspondence between connected 3-homogeneous Latin bitrades and triangulations $T(p,q,r)$ of the torus with $p \equiv 0 \pmod 3$ and $q \equiv r \pmod 3$.*

Further, by Theorem 3.1, the two 3-homogeneous Latin trades which form the connected 3-homogeneous Latin bitrade are isomorphic.

Finally in [2], Cavenagh proved the additional result that every 3-homogeneous Latin trade may be partitioned into three disjoint transversals. Further independent proofs were then given by Donovan, Drápal and Lefevre [5] and by Hämäläinen [10], see http://arxiv.org/abs/0710.0938. The same result also follows immediately from our classification; indeed we can give explicit formulae for the vertices of the triangles corresponding to each transversal.

Transversal α
$(0+3x, 0+3y)$, $(1+3x, 0+3y)$, $(1+3x, 1+3y)$.
$(1+3x, 2+3y)$, $(2+3x, 2+3y)$, $(2+3x, 3+3y)$.
$(2+3x, 1+3y)$, $(3+3x, 1+3y)$, $(3+3x, 2+3y)$.

Transversal β
$(1+3x, 0+3y)$, $(2+3x, 0+3y)$, $(2+3x, 1+3y)$.
$(0+3x, 1+3y)$, $(1+3x, 1+3y)$, $(1+3x, 2+3y)$.
$(2+3x, 2+3y)$, $(3+3x, 2+3y)$, $(3+3x, 3+3y)$.

Transversal γ
$(1+3x, 1+3y)$, $(2+3x, 1+3y)$, $(2+3x, 2+3y)$.
$(2+3x, 0+3y)$, $(3+3x, 0+3y)$, $(3+3x, 1+3y)$.
$(0+3x, 2+3y)$, $(1+3x, 2+3y)$, $(1+3x, 3+3y)$.

In each case $0 \le x \le \lfloor (r-1)/3 \rfloor$ and $0 \le y \le (p-3)/3$.

We conclude the paper with an example. Figure 3 below shows a 3-homogeneous Latin trade with three disjoint transversals denoted by α, β, and γ, together with a trade mate, again with three disjoint transversals denoted by α', β', and γ'. Figure 4 shows the triangulation $T(3,1,7)$ corresponding to this Latin bitrade, where R, C and E denote vertex points in the row, column and entry sets respectively.

	1	2	3	4	5	6	7
1			2β	6γ		7α	
2	4β				1α		5γ
3	5α	7γ				3β	
4			6α		4γ		1β
5		3α		7β		2γ	
6			1γ	2α	6β		
7	3γ	5β					4α

	1	2	3	4	5	6	7
1			$6\beta'$	$7\alpha'$		$2\gamma'$	
2	$5\beta'$				$4\gamma'$		$1\alpha'$
3	$3\gamma'$	$5\alpha'$				$7\beta'$	
4			$1\gamma'$		$6\alpha'$		$4\beta'$
5		$7\gamma'$		$2\beta'$		$3\alpha'$	
6			$2\alpha'$	$6\gamma'$	$1\beta'$		
7	$4\alpha'$	$3\beta'$					$5\gamma'$

FIGURE 3. A 3-homogeneous Latin bitrade with disjoint transversals

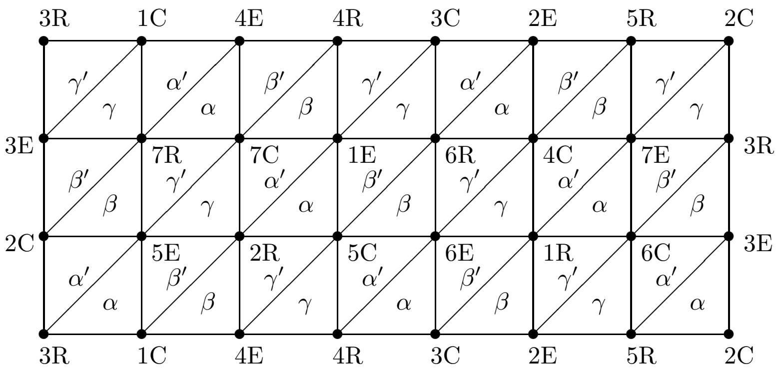

FIGURE 4. The triangulation $T(3,1,7)$

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M.J. Grannell, T.S. Griggs:
Department of Mathematics and Statistics, The Open University, Walton Hall, Milton Keynes MK7 6AA, United Kingdom

M. Knor:
Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia

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