Weak selections and flows in networks

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Abstract. We demonstrate that every Vietoris continuous selection for the hyperspace of at most 3-point subsets implies the existence of a continuous selection for the hyperspace of at most 4-point subsets. However, in general, we do not know if such "extensions" are possible for hyperspaces of sets of other cardinalities. In particular, we do not know if the hyperspace of at most 3-point subsets has a continuous selection provided the hyperspace of at most 2-point subsets has a continuous selection.

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1. Introduction

Let X be a topological space, and let $\mathscr{F}(X)$ be the set of all nonempty closed subsets of X. Also, let $\mathscr{D} \subset \mathscr{F}(X)$. A map $f : \mathscr{D} \to X$ is a *selection* for $\mathscr{D}$ if $f(S) \in S$ for every $S \in \mathscr{D}$. A selection $f : \mathscr{D} \to X$ is *continuous* if it is continuous with respect to the relative Vietoris topology τ_V on $\mathscr{D}$. Let us recall that a base for τ_V is given by all collections of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{F}(X) : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X. Sometimes, for reasons of convenience, we will refer to any such neighbourhood as a basic τ_V-neighbourhood.

In the sequel, all spaces are assumed to be at least Hausdorff. Here, we are interested in continuous selections for $\mathscr{D}$ when $\mathscr{D}$ is a family of finite subsets of X. To this end, for every $n < \omega$, with $n \geq 1$, we let

$$\mathscr{F}_n(X) = \{S \in \mathscr{F}(X) : |S| \leq n\}, \quad \text{and}$$
$$[X]^n = \{S \in \mathscr{F}(X) : |S| = n\}.$$

Every selection $f : \mathscr{F}_2(X) \to X$ defines a natural order-like relation $\preceq$ on X [6] by letting that $x \preceq y$ if and only if $f(\{x, y\}) = x$. For convenience, we write that $x \prec y$ if $x \preceq y$ and $x \neq y$. This relation is very similar to a linear order on X in that it is both total and antisymmetric, but, unfortunately, it may fail to be transitive. In this regard, one of the fundamental questions in the theory of continuous selections for at most 2-point subsets is the following.

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Question 1 (van Mill and Wattel, [7]). Let X be a space which has a continuous selection for $\mathscr{F}_2(X)$. Does there exist a linear order $\preceq$ on X such that, for each $y \in X$, the sets $\{x \in X : x \preceq y\}$ and $\{x \in X : y \preceq x\}$ are both closed?

Recall that a space X is *orderable* (or, *linearly orderable*) if the topology of X coincides with the open interval topology on X generated by a linear ordering on X. Following [7], we say that a space X is *weakly orderable* if there exists a coarser orderable topology on X. In this terminology, Question 1 states the hypothesis if a space X is weakly orderable provided it has a continuous selection for $\mathscr{F}_2(X)$. In view of that, a selection $f : \mathscr{F}_2(X) \to X$ is often called a *weak selection* for X. For a detailed discussion on Question 1, we refer the interested reader to [5].

Whenever X is an weakly orderable space, by [6, Lemma 7.5.1], there exists a continuous selection for $\mathscr{F}_n(X)$ for every $n \geq 2$. Hence, if the answer to Question 1 is in the affirmative, then a space X must have a continuous selection for $\mathscr{F}_n(X)$ for every $n \geq 3$ provided it has a continuous weak selection. Thus, the following question was posed in [4, Problem 2.8] and [5, Question 383].

Question 2 ([4], [5]). Does there exist a space X which has a continuous weak selection, but it has no continuous selection for $\mathscr{F}_n(X)$ for some $n \geq 3$?

It should be remarked that there are spaces X, for instance the real line, which have a continuous selection for $\mathscr{F}_n(X)$ for every $n \geq 2$, but they have no continuous selection for $\mathscr{F}(X)$, see [1]. Also, it is clear that a positive solution to Question 2 will imply a negative solution to Question 1.

Question 2 is open even when $n = 3$. Related to this, it was obtained in [2, Corollary 4.1] that $\mathscr{F}_3(X)$ has a continuous selection provided both $\mathscr{F}_2(X)$ and $[X]^3$ have continuous selections. One of the main obstacles in this particular case is that a selection $f : \mathscr{F}_2(X) \to X$ may generate a triple of distinct points $x, y, z \in X$ such that

$$\cdots \prec x \prec y \prec z \prec x \prec \cdots$$

This may explain the appearance of "$[X]^3$" in the mentioned result of [2]. On the other hand, $\mathscr{F}_2(X)$ has a continuous selection if and only if $[X]^2$ has a continuous selection, which is an implication of the fact that every selection is continuous on the singletons (see, for instance, [4, Proposition 1.4]). Again, we do not know if, in general, $\mathscr{F}_{n+1}(X)$ has a continuous selection provided both $\mathscr{F}_n(X)$ and $[X]^{n+1}$ have continuous selections, see [2, Question 1] and [5, Question 384].

We are now ready to state the main goal of this paper. Namely, in this paper we demonstrate that $\mathscr{F}_4(X)$ has a continuous selection provided $\mathscr{F}_3(X)$ has a continuous selection, see Theorem 4.1. While a possible pure topological proof may work in this case, our arguments are simplified being based in part on Graph Theory and flows in networks. A preparation for that proof is done in Sections 2 and 3, while the proof itself will be finally accomplished in the last Section 4 of the paper.

2. Some properties of continuous selections

Suppose that $f : \mathscr{F}_2(X) \to X$ is a selection, and $\preceq$ is the order-like relation on X generated by f. Following [2], we consider a natural extension of this relation to subsets of X. Namely, if B and C are subsets of X (not necessarily nonempty), then we will write that $B \preceq C$ (respectively, $B \prec C$) if $y \preceq z$ (respectively, $y \prec z$) for every $y \in B$ and $z \in C$. Obviously, $B \prec C$ implies $B \cap C = \emptyset$.

In terms of this relation, we have the following simple criterion for continuity in $\mathscr{F}_2(X)$, see [3, Theorem 3.1].

Proposition 2.1 ([2], [3])**.** *Let X be a space, $f : \mathscr{F}_2(X) \to X$ be a selection, and let $\preceq$ be the order-like relation generated by f. Also, let $x, y \in X$ be such that $x \prec y$. Then, f is continuous at $\{x, y\}$ if and only if there are open sets V and W such that $x \in V$, $y \in W$, and $V \prec W$.*

Motivated by this, we shall say that a basic τ_V-neighbourhood $\langle \mathscr{W} \rangle$ is f-*decisive*, where $f : \mathscr{F}_2(X) \to X$ is a selection, if $V \prec W$ or $W \prec V$ for every two distinct members $V, W \in \mathscr{W}$. Clearly, in this case, $\mathscr{W}$ is a finite pairwise disjoint family of open subsets of X. The following is an immediate consequence of Proposition 2.1.

Corollary 2.2. *Let X be a space, and let $f : \mathscr{F}_2(X) \to X$ be a selection. Then, f is continuous if and only if every nonempty finite subset $S \subset X$ has an f-decisive τ_V-neighbourhood $\langle \mathscr{W} \rangle$, with $|\mathscr{W}| = |S|$.*

We conclude this section with the following technical observation that suggests a possible way to extend continuous selections for hyperspaces from given ones.

Proposition 2.3. *Let $\mathscr{D} \subset \mathscr{F}(X)$, $f : \mathscr{D} \to X$ be a continuous selection, and let $\mathscr{W}$ be a finite pairwise disjoint family of nonempty open subsets of X. Set*

$$\mathscr{M} = \big\{ S \in \langle \mathscr{W} \rangle : S \cap W \in \mathscr{D}, \quad \textit{whenever} \quad W \in \mathscr{W} \big\},$$

and then define a map $\Phi : \mathscr{M} \to \mathscr{F}(X)$ by

$$\Phi(S) = \big\{ f(S \cap W) : W \in \mathscr{W} \big\}, \quad S \in \mathscr{M}.$$

Then, Φ is continuous when both $\mathscr{M}$ and $\mathscr{F}(X)$ are endowed with the Vietoris topology τ_V.

Proof: Take an $S \in \mathscr{M}$, and a basic τ_V-neighbourhood $\big\langle \{U_W : W \in \mathscr{W}\} \big\rangle$ of $\Phi(S)$ such that $U_W \subset W$ for every $W \in \mathscr{W}$. Next, whenever $W \in \mathscr{W}$, take a basic τ_V-neighbourhood $\langle \mathscr{O}_W \rangle$ of $S \cap W$, with $\langle \mathscr{O}_W \rangle \subset f^{-1}(U_W) \cap \big\langle \{W\} \big\rangle$. Then,

$$\begin{aligned}\mathscr{O} &= \big\{T \in \mathscr{M} : T \cap W \in \langle \mathscr{O}_W \rangle, \quad \text{whenever} \quad W \in \mathscr{W}\big\}\\ &= \mathscr{M} \cap \Big\langle \bigcup \{\mathscr{O}_W : W \in \mathscr{W}\}\Big\rangle\end{aligned}$$

is a τ_V-neighbourhood of S, with $\Phi(\mathscr{O}) \subset \big\langle \{U_W : W \in \mathscr{W}\}\big\rangle$. $\square$

3. Selection flows in networks

Let us recall that a *graph* is a pair $G = (V, E)$ of sets satisfying $E \subset [V]^2$; thus the elements of E are 2-element subsets of V. To avoid notational ambiguities, we shall always assume tacitly that $V \cap E = \emptyset$. The elements of V are the *vertices* (or *nodes*, or *points*) of the graph G, while the elements of E are its *edges* (or *lines*). A graph with a vertex set V is said to be a graph on V. The vertex set of a graph G is referred to as $\mathsf{V}(G)$, its edge set as $\mathsf{E}(G)$. The number of vertices of a graph G is its *order*, written as $|G|$. Finally, for $a \in \mathsf{V}(G)$, we let $\mathsf{N}(a) = \big\{b \in \mathsf{V}(G) \setminus \{a\} : \{a, b\} \in \mathsf{E}(G)\big\}$. In what follows, we will consider only finite graphs.

Two vertices $x, y \in \mathsf{V}(G)$ of a graph G are *adjacent*, or *neighbours*, if the set $\{x, y\}$ is an edge of G, i.e. $\{x, y\} \in \mathsf{E}(G)$. If all vertices of G are pairwise adjacent, then G is *complete*. A complete graph of n vertices is denoted by K_n.

In this section, we are mainly interested to view a graph as a *network*, when its edges carry some kind of a flow. Towards this end, for a graph G, we let

$$\vec{\mathsf{E}}(G) = \big\{(x, y), (y, x) : \{x, y\} \in \mathsf{E}(G)\big\}.$$

A function $\xi : \vec{\mathsf{E}}(G) \to \mathbb{Z}$ in the integers $\mathbb{Z}$ will be called a *flow* on the graph G if $\xi(x, y) = -\xi(y, x)$, $(x, y) \in \vec{\mathsf{E}}(G)$. Thus, $\xi(x, y)$ expresses that a flow of $\xi(x, y)$-units passes through the edge $e = \{x, y\}$ from x to y, while $\xi(y, x) = -\xi(x, y)$ are the units of flow that passes through e the other way, from y to x. Finally, for a flow ξ on G and $a \in \mathsf{V}(G)$, we let $\varphi_\xi(a)$ be the *total flow* through this node, i.e.

$$\varphi_\xi(a) = \sum_{b \in \mathsf{N}(a)} \xi(a, b).$$

The following is a very simple observation based on the property of a flow ξ that $\xi(x, y) + \xi(y, x) = 0$.

Proposition 3.1. *If $\xi : \vec{\mathsf{E}}(G) \to \mathbb{Z}$ is a flow on G, then $\sum_{a \in \mathsf{V}(G)} \varphi_\xi(a) = 0$.*

Here is also a particular example of a flow on graphs generated by selections for hyperspaces.

Example 3.2. Let X be a set, $f : \mathscr{F}_2(X) \to X$ be a selection, $\preceq$ be the order-like relation generated by f, and let $S \in [X]^n$ for some $n \geq 2$. Consider the complete graph $K_n = K(S)$, with $S = \mathsf{V}(K_n)$, and define a flow $\xi : \vec{\mathsf{E}}(K_n) \to \mathbb{Z}$ by letting for distinct points $x, y \in S$ that $\xi(x, y) = 1$ if $x \prec y$ and $\xi(x, y) = -1$ if $y \prec x$. In the sequel, we will refer to this flow as a *selection flow* generated by f, or an *f-flow*. Also, for convenience, we will sometimes denote this flow by ξ_S to emphasize that ξ is the selection flow generated on S.

In terms of the selection flow, Corollary 2.2 implies the following alternative characterization of continuity of weak selections.

Proposition 3.3. *Let X be a space, f be a selection for $\mathscr{F}_2(X)$, $S \in [X]^n$ for some $n \geq 2$, and let $\langle \mathscr{W} \rangle$ be an f-decisive τ_V-neighbourhood of S, with $|\mathscr{W}| = n$. Then, for every $T \in \langle \mathscr{W} \rangle \cap [X]^n$, $\{W_1, W_2\} \in [\mathscr{W}]^2$, $t_i \in T \cap W_i$, $i = 1, 2$, and $s_i \in S \cap W_i$, $i = 1, 2$, we have that $\xi_T(t_1, t_2) = \xi_S(s_1, s_2)$.*

Proof: Follows from the fact that $t_1 \prec t_2$ if and only if $s_1 \prec s_2$, where $\preceq$ is the order-like relation generated by f. $\square$

In what follows, let $\mathbb{S}^0 = \{-1, 1\}$ be the 0-dimensional sphere.

Proposition 3.4. *Let $\xi : \vec{\mathsf{E}}(K_4) \to \mathbb{S}^0$ be a flow on K_4. Then, one of the following holds.*

(i) *There exists a point $a \in \mathsf{V}(K_4)$, with $|\varphi_\xi(a)| = 3$.*
(ii) *$|\varphi_\xi(a)| = 1$ for every $a \in \mathsf{V}(K_4)$.*

Proof: By hypothesis, $|\xi(x, y)| = 1$ for every $(x, y) \in \vec{\mathsf{E}}(K_4)$, while $|\mathsf{N}(a)| = 3$ for every $a \in \mathsf{V}(K_4)$. Hence, we have that $|\varphi_\xi(a)| = 1$ or $|\varphi_\xi(a)| = 3$ for every $a \in \mathsf{V}(K_4)$. $\square$

Here is a natural example of a selection flow realizing (i) of Proposition 3.4. To this end, let $f : \mathscr{F}_2(X) \to X$ be a selection, and let $\preceq$ be the order-like relation generated by f. Also, let $S \in \mathscr{F}(X)$. Following [2], we shall say that a subset $B \subset S$, $B \in \mathscr{F}(X)$, is an *f-minimum* of S if

(a) $B \preceq S \setminus B$,
(b) $B \subset C$, whenever $C \subset S$, $C \in \mathscr{F}(X)$, and $C \preceq S \setminus C$.

According to [2, Lemma 2.4], every compact $S \in \mathscr{F}(X)$ has an unique f-minimum B, which we will denote by $B = \min_f S$.

Proposition 3.5. *Let X be a space, $f : \mathscr{F}_2(X) \to X$ be a selection, and let $S \in [X]^4$. Consider the complete graph $K_4 = K(S)$, with $S = \mathsf{V}(K_4)$, and the corresponding f-flow $\xi : \vec{\mathsf{E}}(K_4) \to \mathbb{S}^0$. Then, $|\min_f S| < |S|$ if and only if $|\varphi_\xi(a)| = 3$ for some $a \in S$.*

Proof: First of all, observe that $|\min_f S| = 2$ is impossible. Hence, in this case, $|\min_f S| < |S|$ if and only if $|\min_f S| = 1$ or $|\min_f S| = 3$. Now, for a point

$a \in S$, we have that $\{a\} = \min_f S$ if and only if $\varphi_\xi(a) = 3$. In the same way, we have that $\varphi_\xi(a) = -3$ if and only if $S \setminus \{a\} \prec \{a\}$. $\square$

In our next considerations, we shall say that a flow $\xi : \vec{\mathsf{E}}(K_n) \to \mathbb{S}^0$ on the complete graph K_n, $n \geq 2$, is *bi-conservative* if there are points $e, \ell \in \mathsf{V}(K_n)$ such that $\xi(e, x) = 1 = \xi(x, \ell)$ for every $x \in \mathsf{V}(K_n) \setminus \{e, \ell\}$.

Clearly, every flow $\xi : \vec{\mathsf{E}}(K_2) \to \mathbb{S}^0$ on K_2 is bi-conservative. Also, every flow $\xi : \vec{\mathsf{E}}(K_3) \to \mathbb{S}^0$ on K_3 is bi-conservative as well. Indeed, by Proposition 3.1, there exists a point $x \in \mathsf{V}(K_3)$, with $\varphi_\xi(x) = 0$, because $|\mathsf{V}(K_3)| = 3$. In this case, $\xi(e, x) = 1 = \xi(x, \ell)$ for some $e, \ell \in \mathsf{V}(K_3) \setminus \{x\}$. In particular, if $\varphi_\xi(x) = 0$ for every $x \in \mathsf{V}(K_3)$, then we will get three different pairs of vertices as those in the definition of a bi-conservative flow. In contrast to this, we have the following observation for the special case when $n = 4$.

Lemma 3.6. *Let $\xi : \vec{\mathsf{E}}(K_4) \to \mathbb{S}^0$ be a flow, with $|\varphi_\xi(a)| = 1$ for every $a \in \mathsf{V}(K_4)$. Then, there is at most one pair of vertices $\{e, \ell\} \subset \mathsf{V}(K_4)$ such that*

$$\xi(e, x) = 1 = \xi(x, \ell) \quad \textit{for every} \ \ x \in \mathsf{V}(K_4) \setminus \{e, \ell\}.$$

Proof: Let $e_i, \ell_i \in \mathsf{V}(K_4)$, $i = 1, 2$, be vertices such that

$$\text{(3.1)} \quad \xi(e_i, x) = 1 = \xi(x, \ell_i) \ \text{ for every } \ x \in M_i = \mathsf{V}(K_4) \setminus \{e_i, \ell_i\} \ \text{ and } \ i = 1, 2.$$

It now suffices to show that $M_1 = M_2$. Suppose if possible that $M_1 \neq M_2$. We distinguish the following two possibilities. If $M_1 \cap M_2 \neq \emptyset$, then $|M_1 \cup M_2| = 3$, and therefore there are vertices $a \in \mathsf{V}(K_4) \setminus (M_1 \cup M_2)$ and $b \in M_1 \cap M_2$. Hence, by (3.1), $\xi(a, b) = \xi(a, x)$ for every $x \in M_1 \cup M_2 = \mathsf{V}(K_4) \setminus \{a\}$. However, this implies that $|\varphi_\xi(a)| = 3$, which contradicts the hypothesis that $|\varphi_\xi(a)| = 1$. So, it remains the other possibility that $M_1 \cap M_2 = \emptyset$. In this case we have that $M_1 = \{e_2, \ell_2\}$ and $M_2 = \{e_1, \ell_1\}$. According to (3.1), this now implies that, for instance, $\xi(\ell_2, \ell_1) = 1 = \xi(\ell_1, \ell_2)$, which is clearly impossible. The contradiction so obtained completes the proof. $\square$

Concerning selections, we will rely on bi-conservative flows in the following situation.

Proposition 3.7. *Let X be a space, $f : \mathscr{F}_2(X) \to X$ be a selection, $\preceq$ be the order like relation generated by f, and let $x, y, z \in X$ be such that*

$$\cdots \prec x \prec y \prec z \prec x \prec \cdots$$

Also, let $\langle \mathscr{W} \rangle$ be an f-decisive τ_V-neighbourhood of $\{x, y, z\}$, with $|\mathscr{W}| = 3$, and let $S \in \langle \mathscr{W} \rangle \cap [X]^4$. Consider the complete graph $K_4 = K(S)$, with $S = \mathsf{V}(K_4)$, *and the corresponding f-flow $\xi : \vec{\mathsf{E}}(K_4) \to \mathbb{S}^0$. Then, ξ is a bi-conservative flow such that $|\varphi_\xi(a)| = 1$ for every $a \in S$.*

PROOF: First of all, let us observe that $\min_f S = S$ because $\langle \mathscr{W} \rangle$ is an f-decisive τ_V-neighbourhood of $\{x, y, z\}$, with $|\mathscr{W}| = 3$, while $\min_f\{x, y, z\} = \{x, y, z\}$. Hence, by Proposition 3.5, $|\varphi_\xi(a)| = 1$ for every $a \in S$. On the other hand, $|S| = 4$ and $|\mathscr{W}| = 3$, so $|S \cap W_S| = 2$ for some $W_S \in \mathscr{W}$. Therefore, the complement $S \setminus W_S$ consists of two points e and ℓ such that

$$\{e\} \prec S \cap W_S \prec \{\ell\}.$$

Thus, by definition, ξ is bi-conservative. □

4. Extensions of 3-point selections

Theorem 4.1. *Let X be a space, and let $f : \mathscr{F}_3(X) \to X$ be a continuous selection. Then, there exists a continuous selection for $\mathscr{F}_4(X)$.*

PROOF: For convenience, let $\mathscr{D} = \mathscr{F}_4(X) \setminus \mathscr{F}_1(X)$. Since any selection for $\mathscr{F}_4(X)$ is continuous on the singletons of X (see [4, Proposition 1.4]), it now suffices to construct a continuous selection for $\mathscr{D}$. Towards this end, let $\preceq$ be the order-like relation generated by f, and let

$$\mathscr{P} = \big\{S \in \mathscr{D} : |\min_f S| < |S|\big\}, \tag{4.1}$$

and

$$\mathscr{Q} = \big\{S \in \mathscr{D} : |\min_f S| = |S|\big\}. \tag{4.2}$$

It should be mentioned that always $\mathscr{P} \neq \emptyset$, while $\mathscr{Q} = \emptyset$ is allowed. On the other hand, $\mathscr{P}$ is open, see [2, Lemma 3.4].

In what follows, with every $S \in \mathscr{Q}$ we associate the corresponding f-flow $\xi_S : \vec{\mathsf{E}}\big(K_{|S|}\big) \to \mathbb{S}^0$ on the complete graph $K_{|S|} = K(S)$ generated by S. Next, we define

$$\mathscr{Q}_0 = \big\{S \in \mathscr{Q} : \xi_S \text{ is bi-conservative}\big\},$$

and

$$\mathscr{Q}_1 = \big\{S \in \mathscr{Q} : \xi_S \text{ is not bi-conservative}\big\}.$$

It is clear that $\mathscr{Q}_0 \cap \mathscr{Q}_1 = \emptyset$ and $\mathscr{Q}_0 \cup \mathscr{Q}_1 = \mathscr{Q}$. Let us see that both $\mathscr{Q}_0$ and $\mathscr{Q}_1$ are τ_V-open in $\mathscr{D}$. Indeed, take an $S \in \mathscr{Q}$, and let $\langle \mathscr{W} \rangle$ be an f-decisive τ_V-neighbourhood of S, with $|\mathscr{W}| = |S|$, which exists by Corollary 2.2. Also, take $T \in \langle \mathscr{W} \rangle \cap \mathscr{D}$. Then, $|T \setminus \min_f T| \leq 1$ because, by [2, Lemma 3.4],

$$3 \leq |S| = |\min_f S| \leq |\min_f T| \leq |T| \leq 4.$$

On the other hand, for every $W \in \mathscr{W}$ there exists some $V \in \mathscr{W}$, with $W \prec V$, because $\min_f S = S$. Hence, for every $t \in T$ there exists an $s \in T$, with $t \prec s$, and therefore $|T \setminus \min_f T| = 0$. That is, $T \in \mathscr{Q}$. Now, we have the following two possibilities. If $|S| = 3$, then $S \in \mathscr{Q}_0$ because, as it was mentioned before, any selection flow $\xi : \vec{\mathsf{E}}(K_3) \to \mathbb{S}^0$ is bi-conservative. Hence, in particular, $T \in \mathscr{Q}_0$ provided $|T| = 3$. If $|T| = 4$, then, by Proposition 3.7, $T \in \mathscr{Q}_0$ because $|S| = 3$ and $\min_f S = S$. Thus, in this case, $\langle \mathscr{W} \rangle \cap \mathscr{D} \subset \mathscr{Q}_0$. Finally, suppose that $|S| = 4$. Then, $|T| = 4$ and, by Proposition 3.3, ξ_T is bi-conservative if and only if ξ_S is bi-conservative. That is, now $\langle \mathscr{W} \rangle \cap \mathscr{D} \subset \mathscr{Q}_i$ provided $S \in \mathscr{Q}_i$, $i = 0, 1$.

Thus, we get a clopen partition $\{\mathscr{P}, \mathscr{Q}_0, \mathscr{Q}_1\}$ of $\mathscr{D}$. Hence, to define a continuous selection for $\mathscr{D}$, it suffices to define continuous selections for $\mathscr{P}$, $\mathscr{Q}_0$ and $\mathscr{Q}_1$. This is what we will do till the end of this proof.

Concerning $\mathscr{P}$, we have that $\{\min_f S : S \in \mathscr{P}\} \subset \mathscr{F}_3(X)$. Hence, by [2, Theorem 3.2], $\mathscr{P}$ has a continuous selection.

In order to define a continuous selection for $\mathscr{Q}_0$, let us observe that every $S \in \mathscr{Q}_0$ has an unique partition $\{S_e, S_m, S_\ell\}$ (i.e., $\{S_e, S_m, S_\ell\}$ is pairwise disjoint and $S = S_e \cup S_m \cup S_\ell$) such that

$$|S_e| = 1 = |S_\ell|, \ |S_m| \le 2, \tag{4.3}$$

and

$$\cdots \prec S_e \prec S_m \prec S_\ell \prec S_e \prec \cdots \tag{4.4}$$

Indeed, if $|S| = 3$, this follows from the fact that $\min_f S = S$, so we can take S_e, S_m and S_ℓ to be singletons. If $|S| = 4$, then, by Proposition 3.5, $\left|\varphi_{\xi_S}(a)\right| = 1$ for every $a \in S$, while, by the definition of $\mathscr{Q}_0$, ξ_S is bi-conservative. Hence, by Lemma 3.6, there exists only one pair of points $\{e, \ell\} \subset S$ such that

$$\xi_S(e, x) = 1 = \xi_S(x, \ell) \ \text{ for every } \ x \in S \setminus \{e, \ell\}.$$

So, in this case, we can take $S_e = \{e\}$, $S_m = S \setminus \{e, \ell\}$ and $S_\ell = \{\ell\}$.

We can now define a map $\Phi : \mathscr{Q}_0 \to [X]^3$ by letting that

$$\Phi(S) = \big\{f(S_e), f(S_m), f(S_\ell)\big\}, \quad S \in \mathscr{Q}_0,$$

which is possible because of (4.3). By (4.4) and Corollary 2.2, every $S \in \mathscr{Q}_0$ has a basic τ_V-neighbourhood $\big\langle\{W_e, W_m, W_\ell\}\big\rangle$ such that $T_e \subset W_e$, $T_m \subset W_m$ and $T_\ell \subset W_\ell$ for every $T \in \big\langle\{W_e, W_m, W_\ell\}\big\rangle \cap \mathscr{Q}_0$. Hence, by Proposition 2.3, Φ is continuous. Then, $g = f \circ \Phi$ is a continuous selection for $\mathscr{Q}_0$.

We finish the proof by defining a continuous selection for $\mathscr{Q}_1$. To this end, let us observe that every $S \in \mathscr{Q}_1$ has a unique partition $S = A_S \cup B_S$ such that

$$A_S \ne \emptyset \ne B_S, \quad A_S \cap B_S = \emptyset, \tag{4.5}$$

and

$$\sum_{a\in A_S} \varphi_{\xi_S}(a) = 2 = -\sum_{b\in B_S} \varphi_{\xi_S}(b). \tag{4.6}$$

Indeed, by Proposition 3.5, $|\varphi_{\xi_S}(a)| = 1$ for every $a \in S$ because $\min_f S = S$. On the other hand, by Proposition 3.1,

$$\sum_{a\in S} \varphi_{\xi_S}(a) = 0.$$

Hence, we can set $A_S = \{a \in S : \varphi_{\xi_S}(a) = 1\}$ and $B_S = \{b \in S : \varphi_{\xi_S}(b) = -1\}$. Clearly, $|A_S| = |B_S|$, which completes the verification.

We may now define a map $\Psi : \mathscr{Q}_1 \to [X]^2$ by letting for every $S \in \mathscr{Q}_1$

$$\Psi(S) = \big\{x \in S : \varphi_{\xi_S}(x) = 1\big\} = \big\{f(x) : x \in S \ \text{ and } \ \varphi_{\xi_S}(x) = 1\big\},$$

which is possible because of (4.5) and (4.6). According to Corollary 2.2 and Proposition 3.3, every $S \in \mathscr{Q}_1$ has a basic τ_V-neighbourhood $\langle \mathscr{W}_A \cup \mathscr{W}_B\rangle$ such that $|\mathscr{W}_A| = 2 = |\mathscr{W}_B|$ and $\big\{x \in T : \varphi_{\xi_T}(x) = 1\big\} \in \langle \mathscr{W}_A\rangle$ for every $T \in \langle \mathscr{W}_A \cup \mathscr{W}_B\rangle \cap \mathscr{Q}_1$. Hence, by Proposition 2.3, Ψ is continuous. Then, in this case, $h = f \circ \Psi$ is a continuous selection for $\mathscr{Q}_1$, which completes the proof. $\square$

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