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Affine regular decagons in GS–quasigroup

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Abstract. In this article the "geometric" concept of the affine regular decagon in a general GS–quasigroup is introduced. The relationships between affine regular decagon and some other geometric concepts in a general GS–quasigroup are explored. The geometrical presentation of all proved statements is given in the GS–quasigroup $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$.

Keywords: GS–quasigroup, affine regular decagon, affine regular pentagon
Classification: 20N05

1. Introduction

In [1] the concept of GS–quasigroup is defined. A quasigroup $(Q,\cdot)$ is said to be golden section quasigroup or shortly GS–quasigroup if it satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c = b, \tag{1}$$

$$a\cdot(a\cdot bc)c = b \tag{1'}$$

and the identity of idempotency

$$aa = a. \tag{2}$$

The considered GS–quasigroup $(Q,\cdot)$ satisfies the identities of mediality, elasticity, left and right distributivity i.e. we have the identities

$$ab\cdot cd = ac\cdot bd, \tag{3}$$

$$a\cdot ba = ab\cdot a, \tag{4}$$

$$a\cdot bc = ab\cdot ac, \tag{5}$$

$$ab\cdot c = ac\cdot bc. \tag{5'}$$

Further, the identities

$$a(ab\cdot b) = b, \tag{6}$$

$$(b\cdot ba)a = b, \tag{6'}$$

$$a(ab\cdot c) = b\cdot bc, \tag{7}$$

$$(c\cdot ba)a = cb\cdot b, \tag{7'}$$

$$a(a\cdot bc) = b(b\cdot ac), \tag{8}$$

$$(cb\cdot a)a = (ca\cdot b)b \tag{8'}$$

and equivalencies

$$ab = c \Leftrightarrow a = c \cdot cb, \tag{9}$$

$$ab = c \Leftrightarrow b = ac \cdot c \tag{9'}$$

also hold.

Let $\mathbb{C}$ be the set of points of the Euclidean plane. For any two different points a, b we define $ab = c$ if the point b or a divides the pair a, c or the pair b, c respectively, in ratio of the golden section.

In [1] it is proved that $(\mathbb{C}, \cdot)$ is a GS–quasigroup in both cases. We shall denote these two quasigroups by $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$ and $\mathbb{C}(\frac{1}{2}(1-\sqrt{5}))$ because we have $c = \frac{1}{2}(1+\sqrt{5})$ or $c = \frac{1}{2}(1-\sqrt{5})$ if $a = 0$ and $b = 1$. These quasigroups can give a motivation for the definition of "geometric" notions and proving of "geometric" properties of a general GS–quasigroup. In the quasigroup $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$ we shall illustrate (by figures) the properties of a general GS–quasigroup. If we interchange the roles of both factors in all products we will get the presentation in the same figure for the quasigroup $\mathbb{C}(\frac{1}{2}(1-\sqrt{5}))$.

These two mentioned quasigroups are mutually equivalent since the following statement is obviously valid.

Lemma 1.1. *If the operation $\bullet$ on the set Q is defined by the equivalency $a \bullet b = c \Leftrightarrow ba = c$, i.e. by the identity $a \bullet b = ba$, then $(Q, \bullet)$ is a GS–quasigroup if and only if $(Q, \cdot)$ is a GS–quasigroup.*

From now on let $(Q, \cdot)$ be any GS–quasigroup. The elements of the set Q are said to be *points*.

The following statements are proved in [1] and they will be used later.

Lemma 1.2. *Any three of four equalities $ab = d$, $ae = f$, $dc = e$, $fc = b$ imply the remaining equality.*

Lemma 1.3. *Any two of four equalities $ab = c$, $dc = b$, $ac = d$, $db = a$ imply the remaining two equalities.*

The points a, b, c, d are said to be the vertices of a *parallelogram* and we write $\mathrm{Par}(a, b, c, d)$ iff the identity $a \cdot b(ca \cdot a) = d$ holds. In [1] numerous properties of the quaternary relation Par on the set Q are proved. Let us mention just the following statements which we shall use afterwards.

Lemma 1.4. *From $\mathrm{Par}(a, b, c, d)$ and $\mathrm{Par}(c, d, e, f)$ it follows $\mathrm{Par}(a, b, f, e)$.*

Lemma 1.5. *Let a, b, c be any three points and $d = ac$, $e = ab$, $f = ec$, $g = df$. Then the statements $\mathrm{Par}(a, b, d, f)$, $\mathrm{Par}(b, e, f, g)$, $\mathrm{Par}(a, e, d, g)$ are valid.*

We shall say that b is the *midpoint* of the pair of points a, c and write $\mathrm{M}(a, b, c)$ if $\mathrm{Par}(a, b, c, b)$. In [1] the following statements, by means of the properties of the quaternary relation Par, are proved.

Lemma 1.6. *For any points a, b there is only one point c such that $M(a,b,c)$. The statement $M(a,b,c)$ implies the statement $M(c,b,a)$. For any point a it is valid $M(a,a,a)$.*

Lemma 1.7. *The statement $M(a,b,c)$ holds if and only if $c = ba \cdot b$.*

Lemma 1.8. *For any point p the statements $M(a,b,c)$, $M(pa,pb,pc)$, $M(ap,bp,cp)$ are mutually equivalent.*

In [2] the concept of the GS–trapezoid is defined. The points a, b, c, d successively are said to be the vertices of the *golden section trapezoid* and it is denoted by $\mathrm{GST}(a,b,c,d)$ if the identity $a \cdot ab = d \cdot dc$ holds. Because of (9) this identity is equivalent to the identity $d = (a \cdot ab)c$.

In [4] the concept of affine regular pentagon is defined. The points a, b, c, d, e successively are said to be the vertices of the *affine regular pentagon* and it is denoted by $\mathrm{ARP}(a,b,c,d,e)$ if any two (and then all five) of the five statements $\mathrm{GST}(a,b,c,d)$, $\mathrm{GST}(b,c,d,e)$, $\mathrm{GST}(c,d,e,a)$, $\mathrm{GST}(d,e,a,b)$, $\mathrm{GST}(e,a,b,c)$ are valid.

The concept of the DGS–trapezoid is introduced in [3]. Points a, b, c, d are said to be the vertices of the *double golden section trapezoid* or shorter DGS–trapezoid and we write $\mathrm{DGST}(a,b,c,d)$ if the equality $ab = dc$ holds.

The points o, a, b, c are said to be the vertices of a *golden section deltoid* and we write $\mathrm{GSD}(o,a,b,c)$ if and only if the identity $c = oa \cdot b$ is valid ([5]). In [5] the following statements are proved.

Lemma 1.9. *For any point p the statements $GSD(o,a,b,c)$, $GSD(po,pa,pb,pc)$ and $GSD(op,ap,bp,cp)$ are mutually equivalent.*

Lemma 1.10. *If the statements $GSD(o,a,b,c)$, $GSD(o,b,c,d)$ hold then $ab = dc = e$ i.e. $DGST(a,b,c,d)$ and $Par(o,a,e,d)$ hold.*

2. Affine regular decagon in GS–quasigroup

Now we are going to introduce the concept of the affine regular decagon in a general GS–quasigroup. Firstly, we will prove the theorem which will lead to the definition of the mentioned concept.

Theorem 2.1. *From the equations*

$$oa_i \cdot a_{i+1} = a_{i+2} \tag{10}$$

for $i = 0,1,2,3,4,5,6,7$ the equations (10) *for $i = 8,9$ follow, where indexes are taken modulo* 10 (*Figure* 1).

Proof: If we denote by k the equality (10) for $i = k$ then we get

$$\begin{aligned} oa_1 \cdot a_0 &\overset{(1)}{=} o[o(oa_1 \cdot a_0) \cdot o] \cdot o \overset{(5)}{=} o[(o \cdot oa_1)(oa_0) \cdot o] \cdot o \\ &\overset{(6')}{=} o[(o \cdot oa_1)(oa_0) \cdot (o \cdot oa_1)a_1] \cdot o \overset{(5)}{=} [o \cdot (o \cdot oa_1)(oa_0 \cdot a_1)]o \\ &\overset{0}{=} [o \cdot (o \cdot oa_1)a_2]o \overset{(5')}{=} [o \cdot (oa_2)(oa_1 \cdot a_2)]o \overset{1}{=} o(oa_2 \cdot a_3) \cdot o \overset{2}{=} oa_4 \cdot o \\ &\overset{(6')}{=} oa_4 \cdot (o \cdot oa_4)a_4 \overset{(5')}{=} o(o \cdot oa_4) \cdot a_4 \overset{(1)}{=} o(o \cdot oa_4) \cdot [o(oa_4 \cdot a_5) \cdot a_5] \\ &\overset{(3)}{=} [o \cdot o(oa_4 \cdot a_5)] \cdot (o \cdot oa_4)a_5 \overset{(5')}{=} [o \cdot o(oa_4 \cdot a_5)][oa_5 \cdot (oa_4 \cdot a_5)] \\ &\overset{4}{=} (o \cdot oa_6)(oa_5 \cdot a_6) \overset{5}{=} (o \cdot oa_6)a_7 \overset{(5')}{=} oa_7 \cdot (oa_6 \cdot a_7) \overset{6}{=} oa_7 \cdot a_8 \overset{7}{=} a_9, \end{aligned}$$

wherefrom it follows

$$oa_9 \cdot a_0 = o(oa_1 \cdot a_0) \cdot a_0 \overset{(1)}{=} a_1,$$

which means that from equality (10) where $i = 0, 1, 2, 4, 5, 6, 7$ the equality (10) for $i = 9$ follows, and similarly from equalities (10) for $i = 9, 0, 1, 3, 4, 5, 6$ the equality (10) for $i = 8$ follows. □

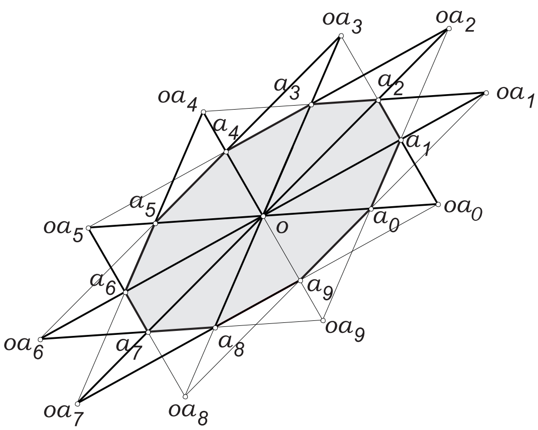

Figure 1

We shall say that $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ is the *affine regular decagon with the center o* and we shall write $Aff_o(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ if in a cycle of the equalities (10) for $i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, any eight adjacent (and then all ten) equalities are valid.

From Theorem 2.1 it follows immediately

Corollary 2.2. *For any points o, a_0, a_1 there is a unique octuple of the points $a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$ so that $Aff_o(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$.*

Theorem 2.3. *If* $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9)$ *is any cyclic permutation of* $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$ *or of* $(9, 8, 7, 6, 5, 4, 3, 2, 1, 0)$ *then* $Aff_o(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ *implies* $Aff_o(a_{i_0}, a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}, a_{i_7}, a_{i_8}, a_{i_9})$.

PROOF: It is enough to prove the identity

$$oa_i \cdot a_{i-1} = a_{i-2}. \tag{11}$$

However, we get

$$oa_i \cdot a_{i-1} \overset{(10)}{=} o(oa_{i-2} \cdot a_{i-1}) \cdot a_{i-1} \overset{(1)}{=} a_{i-2}.$$

□

Further, let $Aff_o(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ where $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

If we denote

$$oa_i = b_i, \tag{12}$$

then from (10) and (11) immediately follows

$$b_i a_{i\pm1} = a_{i\pm2}. \tag{13}$$

Theorem 2.4. *The statements* $Par(o, b_i, b_{i\pm1}, a_{i\pm3})$, $Par(a_i, b_i, a_{i\pm2}, a_{i\pm3})$, $Par(o, a_i, b_{i\pm1}, a_{i\pm2})$ *are valid.*

PROOF: On the bases of (12) and (13) we get equalities $oa_i = b_i$, $oa_{i\pm1} = b_{i\pm1}$, $b_i a_{i\pm1} = a_{i\pm2}$, $b_{i\pm1} a_{i\pm2} = a_{i\pm3}$ wherefrom according to Lemma 1.5 the statements of theorem follow. □

Because of (13) we get equalities $b_i a_{i\pm1} = a_{i\pm2}$ and $b_{i\pm3} a_{i\pm2} = a_{i\pm1}$ wherefrom owing to Lemma 1.3 it follows

$$b_i a_{i\pm2} = b_{i\pm3}. \tag{14}$$

Theorem 2.5. *The statements* $M(a_i, o, a_{i+5})$, $M(b_i, o, b_{i+5})$ *are valid.*

PROOF: Owing to Theorem 2.4 $\mathrm{Par}(a_i, o, b_{i+3}, b_{i+2})$ and $\mathrm{Par}(b_{i+3}, b_{i+2}, o, a_{i+5})$ are valid, wherefrom according to Lemma 1.4 it follows $\mathrm{Par}(a_i, o, a_{i+5}, o)$, i.e. $\mathrm{M}(a_i, o, a_{i+5})$. Hence, according to Lemma 1.8 it follows $\mathrm{M}(oa_i, oo, oa_{i+5})$ i.e. $\mathrm{M}(b_i, o, b_{i+5})$. □

The property $\mathrm{M}(a_i, o, a_{i+5})$ justifies the fact that the point o is called the center of the considered affine–regular decagon.

If we apply Lemma 1.7 we get $\mathrm{M}(a_i, o, oa_i \cdot o)$, which together with $\mathrm{M}(a_i, o, a_{i+5})$, according to Lemma 1.6, gives the equality $a_{i+5} = oa_i \cdot o$. Whence, on the basis of (12), we get

$$b_i o = a_{i+5}. \tag{15}$$

If the statement $Aff_o(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ is valid then we shall say that $(a_0, a_3, a_6, a_9, a_2, a_5, a_8, a_1, a_4, a_7)$ is the *affine regular star–shaped decagon with the center o* and we shall write $\overline{Aff_o}(a_0, a_3, a_6, a_9, a_2, a_5, a_8, a_1, a_4, a_7)$ (Figure 2).

It is obvious that the implication $\overline{Aff_o}(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \Rightarrow Aff_o(a_0,\, a_3, a_6, a_9, a_2, a_5, a_8, a_1, a_4, a_7)$ is valid.

If $(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9)$ is any cyclic permutation of $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$ or of $(9, 8, 7, 6, 5, 4, 3, 2, 1, 0)$ then the implication $\overline{Aff_o}(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \Rightarrow \overline{Aff_o}(a_{i_0}, a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}, a_{i_7}, a_{i_8}, a_{i_9})$ is also valid.

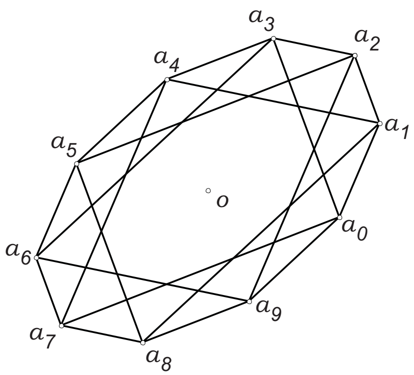

FIGURE 2

Let us take (Figure 3)

$$a_i o = c_{i+5}. \tag{16}$$

Then we get successively

$$a_{i+5} \overset{(15)}{=} b_i o \overset{(12)}{=} oa_i \cdot o \overset{(4)}{=} o \cdot a_i o \overset{(16)}{=} oc_{i+5},$$

i.e. the equation

$$oc_i = a_i \tag{17}$$

is valid. Further, we get

$$o \cdot a_i c_{i\pm 2} \overset{(5)}{=} oa_i \cdot oc_{i\pm 2} \overset{(12),(17)}{=} b_i a_{i\pm 2} \overset{(14)}{=} b_{i\pm 3} \overset{(12)}{=} oa_{i\pm 3},$$

wherefrom it follows

$$a_i c_{i\pm 2} = a_{i\pm 3}. \tag{18}$$

Now, we obtain

$$a_{i+3} \cdot a_i o \overset{(16)}{=} a_{i+3} \cdot c_{i+5} \overset{(18)}{=} a_{i+6}.$$

If we interchange the roles of both factors in all products in the above equality, then we get the equality $oa_i \cdot a_{i+3} = a_{i+6}$, which means that in the quasigroup $\mathbb{C}(\frac{1}{2}(1-\sqrt{5}))$ the statement $Aff_o(a_0, a_3, a_6, a_9, a_2, a_5, a_8, a_1, a_4, a_7)$ holds when in the quasigroup $\mathbb{C}(\frac{1}{2}(1+\sqrt{5}))$ the statement $Aff_o(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ i.e. $\overline{Aff_o}(a_0, a_3, a_6, a_9, a_2, a_5, a_8, a_1, a_4, a_7)$ holds.

Therefore, the affine regular decagon in the first quasigroup is the affine regular star–shaped decagon in the second quasigroup, and conversely the affine regular decagon in the second quasigroup is the affine regular star–shaped decagon in the first quasigroup.

According to Lemma 1.1 these two quasigroups are equivalent, so it means that it is a matter of convention which of these two decagons $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ and $(a_0, a_3, a_6, a_9, a_2, a_5, a_8, a_1, a_4, a_7)$ will be called affine regular decagon and which one affine regular–star shaped decagon, since we cannot differ them in a general GS–quasigroup. Besides that, it means that each statement about affine regular decagons which is proved in a general GS–quasigroup, it is also valid for affine regular star–shaped decagon and vice versa (with above mentioned interchange of both factors in all products).

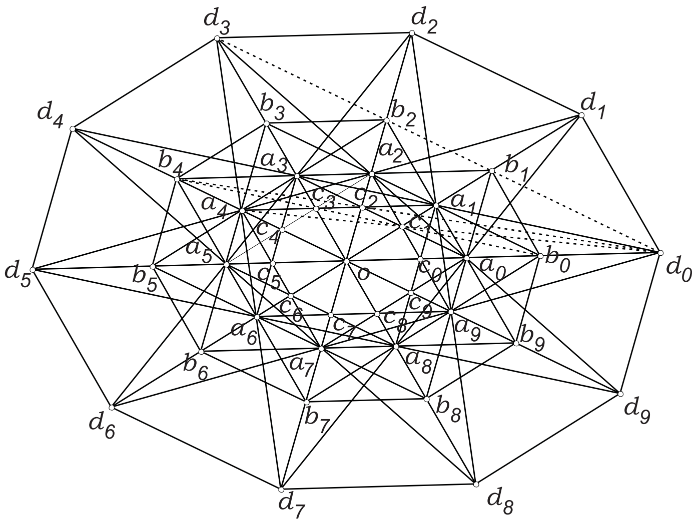

FIGURE 3

In Figure 3 dashed lines present the formulas (20), (21), (30), (31), (36) and (37) for $i=0$ and if the sign on top is considered.

On the base of (18) we get equalities $a_i c_{i\pm2} = a_{i\pm3}$, $a_{i\pm3}c_{i\pm1} = a_i$, wherefrom owing to Lemma 1.3 it follows

$$a_i c_{i\pm1} = c_{i\pm2}. \tag{19}$$

Because of (17) and (19) we get equalities $oc_i = a_i$, $oc_{i\pm1} = a_{i\pm1}$, $a_i c_{i\pm1} = c_{i\pm2}$, $a_{i\pm1}c_{i\pm2} = c_{i\pm3}$, wherefrom owing to Lemma 1.5 we have the statements analogous to the statements of Theorem 2.4 i.e. we get the following:

Theorem 2.6. *The statements* $Par(o, a_i, a_{i\pm1}, c_{i\pm3})$, $Par(c_i, a_i, c_{i\pm2}, c_{i\pm3})$, $Par(o, c_i, a_{i\pm1}, c_{i\pm2})$ *are valid.*

Further we get

$$oc_i \cdot c_{i+1} \overset{(17)}{=} a_i c_{i+1} \overset{(19)}{=} c_{i+2},$$

so it follows:

Theorem 2.7. *The statement* $Aff_o(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9)$ *is valid* (*Figure* 3).

Owing to (17), (12) and (18) we get equalities $oc_{i\pm3} = a_{i\pm3}$, $oa_i = b_i$, $a_{i\pm3}c_{i\pm1} = a_i$ wherefrom according to Lemma 1.2 it follows

$$b_i c_{i\pm1} = c_{i\pm3}. \tag{20}$$

According to (20) we get equalities $b_i c_{i\pm1} = c_{i\pm3}$, $b_{i\pm4}c_{i\pm3} = c_{i\pm1}$, wherefrom because of Lemma 1.3 it follows

$$b_i c_{i\pm3} = b_{i\pm4}. \tag{21}$$

From (12) and (17) owing to Lemma 1.3 we also get

$$b_i a_i = c_i, \tag{22}$$

$$b_i c_i = o. \tag{23}$$

Further, we get because of Lemma 1.3 and $oa_i = b_i$

$$\begin{aligned} a_i c_{i+5} &\overset{(17),(16)}{=} oc_i \cdot a_i o \overset{(3)}{=} oa_i \cdot c_i o \overset{(12)}{=} b_i \cdot c_i o \overset{(5)}{=} b_i c_i \cdot b_i o \\ &\overset{(23),(15)}{=} oa_{i+5} \overset{(12)}{=} b_{i+5}, \end{aligned}$$

i.e.

$$a_i c_{i+5} = b_{i+5}. \tag{24}$$

Now, introducing the notation (Figure 3)

$$ob_i = d_i \tag{25}$$

we get

$$d_ib_i = a_i, \tag{26}$$
$$d_ia_i = o. \tag{27}$$

Because of (27), (18) and (17) we obtain the equalities

$$d_ia_i = o, \quad a_{i\pm3}c_{i\pm1} = a_i, \quad oc_{i\pm1} = a_{i\pm1},$$

wherefrom on the basis of Lemma 1.2 it follows

$$d_ia_{i\pm1} = a_{i\pm3}. \tag{28}$$

Using (28) we get $d_ia_{i\pm1} = a_{i\pm3}$, $d_{i\pm4}a_{i\pm3} = a_{i\pm1}$ wherefrom according to Lemma 1.3 it follows

$$d_ia_{i\pm3} = d_{i\pm4}. \tag{29}$$

Theorem 2.8. *The statements $ARP(a_0, a_2, a_4, a_6, a_8)$ and $ARP(a_1, a_3, a_5, a_7, a_9)$ are valid.*

Proof: The first statement, for example, follows on the base of equalities

$$d_3a_2 = a_0, \quad d_3a_4 = a_6, \quad d_5a_4 = a_2, \quad d_5a_6 = a_8$$

which are presented in (28). □

Using (27), (12) and (13) we obtain equalities

$$d_ia_i = o, \quad oa_{i\pm1} = b_{i\pm1}, \quad b_{i\pm2}a_{i\pm1} = a_i,$$

which, because of Lemma 1.2, imply the following

$$d_ib_{i\pm1} = b_{i\pm2}. \tag{30}$$

From (30) we get

$$d_ib_{i\pm1} = b_{i\pm2}, \quad d_{i\pm3}b_{i\pm2} = b_{i\pm1},$$

whence owing to Lemma 1.3 we obtain

$$d_ib_{i\pm2} = d_{i\pm3}. \tag{31}$$

Now, we have

$$ob_i \cdot b_{i+1} \overset{(25)}{=} d_ib_{i+1} \overset{(30)}{=} b_{i+2},$$

so it immediately follows

Theorem 2.9. *The statement* $Aff_o(b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9)$ *is valid* (*Figure* 3).

Further we get

$$d_i o \overset{(25)}{=} ob_i \cdot o \overset{(4)}{=} o \cdot b_i o \overset{(15)}{=} oa_{i+5} \overset{(12)}{=} b_{i+5},$$
$$d_i c_i \overset{(25)}{=} ob_i \cdot c_i \overset{(5')}{=} oc_i \cdot b_i c_i \overset{(17),(23)}{=} a_i o \overset{(16)}{=} c_{i+5},$$
$$b_i a_{i+5} \overset{(12),(17)}{=} oa_i \cdot oc_{i+5} \overset{(5)}{=} o \cdot a_i c_{i+5} \overset{(24)}{=} ob_{i+5} \overset{(25)}{=} d_{i+5},$$

i.e.

$$d_i o = b_{i+5}, \tag{32}$$
$$d_i c_i = c_{i+5}, \tag{33}$$
$$b_i a_{i+5} = d_{i+5} \tag{34}$$

and

$$d_i c_{i+5} \overset{(16)}{=} d_i \cdot a_i o \overset{(5)}{=} d_i a_i \cdot d_i o \overset{(27),(32)}{=} ob_{i+5} \overset{(25)}{=} d_{i+5},$$

i.e.

$$d_i c_{i+5} = d_{i+5}. \tag{35}$$

On the basis of (25), (12) and (21) we get

$$ob_i = d_i, \quad oa_{i\pm4} = b_{i\pm4}, \quad b_{i\pm4} c_{i\pm1} = b_i,$$

wherefrom according to Lemma 1.2 we get

$$d_i c_{i\pm1} = a_{i\pm4}. \tag{36}$$

Analogously, because of (36), (19) and (20) we get equalities

$$d_i c_{i\pm1} = a_{i\pm4}, \quad a_{i\pm4} c_{i\pm3} = c_{i\pm2}, \quad b_{i\pm4} c_{i\pm3} = c_{i\pm1}$$

whence it follows

$$d_i c_{i\pm2} = b_{i\pm4}. \tag{37}$$

Theorem 2.10. *The statements* $M(c_i, b_i, d_i)$, $M(d_i, a_i, c_{i+5})$ *are valid.*

Proof: The statement follows on the base of Lemma 1.7, the first one from

$$d_i \overset{(25)}{=} ob_i \overset{(23)}{=} b_i c_i \cdot b_i,$$

and the second one from the equalities (27) and (16). □

Theorem 2.11. *The statement* $Aff_o(d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9)$ *is valid* (*Figure* 3).

Proof:

$$od_i \cdot d_{i+1} \overset{(25)}{=} od_i \cdot ob_{i+1} \overset{(5)}{=} o \cdot d_i b_{i+1} \overset{(30)}{=} ob_{i+2} \overset{(25)}{=} d_{i+2}.$$

□

Now, we will introduce some new points which will also be the vertices of new affine regular decagons.

Owing to (10) the statements $\mathrm{GSD}(o, a_{i-1}, a_i, a_{i+1})$ and $\mathrm{GSD}(o, a_i, a_{i+1}, a_{i+2})$ are valid, wherefrom according to Lemma 1.10 the statement $\mathrm{DGST}(a_{i-1}, a_i, a_{i+1}, a_{i+2})$ holds i.e. $a_{i-1}a_i = a_{i+2}a_{i+1}$. Let us denote now (Figure 4)

$$a_{i,i+1} = a_{i-1}a_i = a_{i+2}a_{i+1}, \tag{38}$$

then we have

$$\begin{aligned} oa_{i,i+1} \cdot a_{i+1,i+2} &\overset{(38)}{=} (o \cdot a_{i-1}a_i) \cdot a_i a_{i+1} \overset{(5)}{=} (oa_{i-1} \cdot oa_i) \cdot a_i a_{i+1} \\ &\overset{(3)}{=} (oa_{i-1} \cdot a_i)(oa_i \cdot a_{i+1}) \overset{(12)}{=} b_{i-1}a_i \cdot b_i a_{i+1} \overset{(13)}{=} a_{i+1}a_{i+2} \\ &\overset{(38)}{=} a_{i+2,i+3} \end{aligned}$$

i.e.

$$oa_{i,i+1} \cdot a_{i+1,i+2} = a_{i+2,i+3}, \tag{39}$$

so it immediately follows:

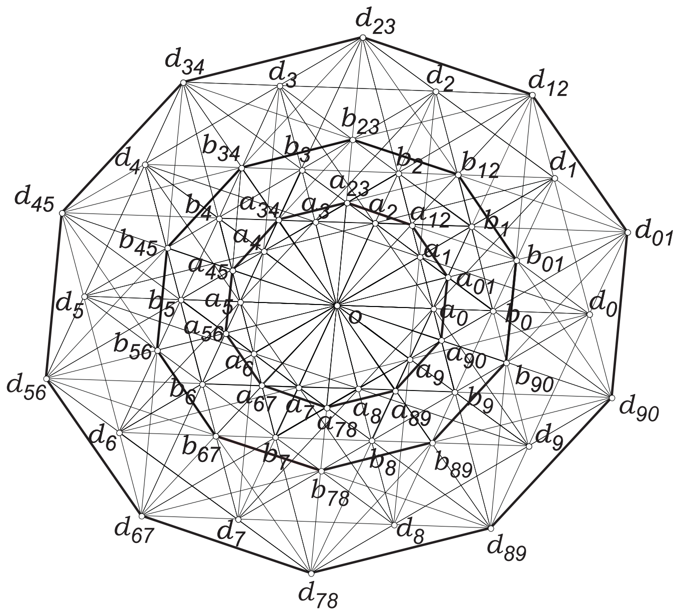

Figure 4

Theorem 2.12. *The statement* $Aff_o(a_{01}, a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{67}, a_{78}, a_{89}, a_{90})$ *holds* (*Figure* 4).

Because of (12) and (2) according to Lemma 1.9 the statements $\mathrm{GSD}(o, b_{i-1}, b_i, b_{i+1})$ and $\mathrm{GSD}(o, b_i, b_{i+1}, b_{i+2})$ hold, wherefrom according to Lemma 1.10 the statement $\mathrm{DGST}(b_{i-1}, b_i, b_{i+1}, b_{i+2})$ follows i.e. $b_{i-1}b_i = b_{i+2}b_{i+1}$. Let us denote

$$b_{i,i+1} = b_{i-1}b_i = b_{i+2}b_{i+1}. \tag{40}$$

Because of (12) and (5) it is obviously valid

$$oa_{i,i+1} = b_{i,i+1}. \tag{41}$$

As we have

$$\begin{aligned} ob_{i,i+1} \cdot b_{i+1,i+2} &\overset{(41)}{=} (o \cdot oa_{i,i+1}) \cdot oa_{i+1,i+2} \overset{(5)}{=} o(oa_{i,i+1} \cdot a_{i+1,i+2}) \\ &\overset{(39)}{=} oa_{i+2,i+3} \overset{(41)}{=} b_{i+2,i+3} \end{aligned}$$

we immediately get:

Theorem 2.13. *The statement* $Aff_o(b_{01}, b_{12}, b_{23}, b_{34}, b_{45}, b_{56}, b_{67}, b_{78}, b_{89}, b_{90})$ *is valid* (*Figure* 4).

Analogously, because of (25) and (2) and according to Lemma 1.9 $\mathrm{GSD}(o, d_{i-1}, d_i, d_{i+1})$ and $\mathrm{GSD}(o, d_i, d_{i+1}, d_{i+2})$ are valid whence applying Lemma 1.10 the statement $\mathrm{DGST}(d_{i-1}, d_i, d_{i+1}, d_{i+2})$ is valid i.e. $d_{i-1}d_i = d_{i+2}d_{i+1}$. Let us take

$$d_{i,i+1} = d_{i-1}d_i = d_{i+2}d_{i+1}. \tag{42}$$

Because of (25) and (5) we obtain

$$ob_{i,i+1} = d_{i,i+1}. \tag{43}$$

If we apply (43) and Theorem 2.13 we can get

$$od_{i,i+1} \cdot d_{i+1,i+2} = d_{i+2,i+3},$$

so it follows

Theorem 2.14. *The statement* $Aff_o(d_{01}, d_{12}, d_{23}, d_{34}, d_{45}, d_{56}, d_{67}, d_{78}, d_{89}, d_{90})$ (*Figure* 4) *is valid.*

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