Comment.Math.Univ.Carolin. 49,3 (2008)397–410

Skewsquares in quadratical quasigroups

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Abstract. The concept of pseudosquare in a general quadratical quasigroup is introduced and connections to some other geometrical concepts are studied. The geometrical presentations of some proved statements are given in the quadratical quasigroup $\mathbb{C}(\frac{1+i}{2})$.

Keywords: quadratical quasigroup, skewsquare

Classification: 20N05

1. Introduction

The “geometrical” concept of skewsquare is defined and investigated in any quadratical quasigroup.

A groupoid $(Q,\cdot)$ is said to be *quadratical* if the identity

$$ab \cdot a = ca \cdot bc \tag{1}$$

holds and the equation $ax = b$ has a unique solution $x \in Q$ for any $a, b \in Q$ (cf. [12] and [2]). Every quadratical groupoid $(Q,\cdot)$ is a quasigroup, i.e. the equations $xa = b$ and $ay = b$ have unique solutions for any $a, b \in Q$. In a quadratical quasigroup $(Q,\cdot)$ the identities

$$aa = a, \tag{2}$$

$$ab \cdot cd = ac \cdot bd, \tag{3}$$

$$a \cdot ba = ab \cdot a, \tag{4}$$

$$ab \cdot c = ac \cdot bc, \tag{5}$$

$$a \cdot bc = ab \cdot ac, \tag{6}$$

and the equivalencies

$$ab = cd \Leftrightarrow bc = da, \tag{7}$$

$$ab = c \Leftrightarrow bc = ca \tag{8}$$

hold (cf. [12]).

If $\mathbb{C}$ is the set of all points of a Euclidean plane and if a groupoid $(\mathbb{C},\cdot)$ is defined so that $aa = a$ for any $a \in \mathbb{C}$ and for any two different points $a, b \in \mathbb{C}$ the point ab is the centre of the positively oriented square with two adjacent vertices a and b, then $(\mathbb{C},\cdot)$ is a quadratical quasigroup (cf. [12]). This quasigroup will be denoted by $\mathbb{C}(\frac{1+i}{2})$ because if $a = 0$ and $b = 1$ then $ab = \frac{1+i}{2}$. The figures in this quasigroup illustrate the "geometrical" relations in any quadratical quasigroup $(Q,\cdot)$.

From now on let $(Q,\cdot)$ be any quadratical quasigroup. The elements of Q are said to be *points*, the pairs of points are *segments*, the quadruples of points are *quadrangles* and an ordered quadruple of points is said to be an *oriented quadrangle*.

If an operation $\bullet$ is defined on the set Q by

$$a \bullet b = a \cdot ba = ab \cdot a = ca \cdot bc, \tag{9}$$

then $(Q, \bullet)$ is an idempotent medial commutative quasigroup (cf. [12]), i.e. the identities

$$a \bullet a = a, \tag{10}$$

$$(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d), \tag{11}$$

$$a \bullet b = b \bullet a \tag{12}$$

hold and the operations $\cdot$ and $\bullet$ are mutually medial, i.e. the identity

$$ab \bullet cd = (a \bullet c)(b \bullet d) \tag{13}$$

holds. The point $a \bullet b$ is said to be the *midpoint* of two points a and b.
Because of

$$g(a,b,c,d) = (a \bullet c) \bullet (b \bullet d) \overset{(11)}{=} (a \bullet b) \bullet (c \bullet d) \overset{(12)}{=} (a \bullet b) \bullet (d \bullet c) \overset{(11)}{=} (a \bullet d) \bullet (b \bullet c)$$

the point $g(a,b,c,d)$ is said to be the *centroid* of the quadrangle $\{a,b,c,d\}$.

An oriented quadrangle (a,b,c,d) is said to be a *parallelogram* and we write $\mathrm{Par}(a,b,c,d)$ if $a \bullet c = b \bullet d$. If $a \bullet c = b \bullet d = o$, then we say that the point o is the *centre* of this parallelogram and we write $\mathrm{Par}_o(a,b,c,d)$. In [14] it is proved that (Q, Par) is a parallelogram space (cf. [8] and [11]) and the following statement which will be used later.

Lemma 1. *For any points a,b,c,d the statement $Par(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$ is valid.*

An oriented quadrangle (a,b,c,d) is said to be a *square* and we write $\mathrm{S}(a,b,c,d)$ if $ab = bc = cd = da$. If $ab = bc = cd = da = o$, then we say that the point o is the *centre* of this square and we write $\mathrm{S}_o(a,b,c,d)$. Obviously $\mathrm{S}_o(a,b,c,d) \Rightarrow \mathrm{S}_o(e,f,g,h)$, where (e,f,g,h) is any cyclical permutation of (a,b,c,d).

In [13] it is proved:

Lemma 2. *The statement $S(a,b,c,d)$ is equivalent to any two of four (and then all four) equalities $ac=d$, $bd=a$, $ca=b$, $db=c$.*

In [14] the following statements are proved and they will be used later.

Lemma 3. *The statement $S_o(a,b,c,d)$ implies $\mathrm{Par}_o(a,b,c,d)$.*

Lemma 4. $\mathrm{Par}_o(a,b,c,d) \Leftrightarrow S_o(ba,cb,dc,ad)$.

2. The concept of skewsquare in quadratical quasigroup

In the set Q^2 a binary relation $\sim$ is defined by

$$(a,b) \sim (c,d) \Leftrightarrow \mathrm{Par}(a,b,d,c).$$

In [8] it is proved that $\sim$ is a relation of equivalence. The elements of the set $Q^2/_\sim$ are said to be the *vectors*. A vector with a representative (a,b) is denoted by $[a,b]$. Therefore, we have

$$[a,b]=[c,d] \Leftrightarrow \mathrm{Par}(a,b,d,c),$$

i.e.

$$[a,b]=[c,d] \Leftrightarrow a \bullet d = b \bullet c. \tag{14}$$

For any point a and any vector $\mathbf{v}$ there is one and only one point b such that $\mathbf{v}=[a,b]$.

A vector $\mathbf{u}$ is said to be *orthogonally equal* to a vector $\mathbf{v}$ and we write $\mathbf{u}\underline{\perp}\mathbf{v}$ if there are four points p,q,r,s such that

$$\mathbf{u}=[p,r], \quad \mathbf{v}=[q,s], \quad S(p,q,r,s)$$

(Figure 1).

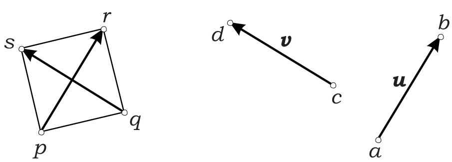

FIGURE 1

The properties of squares imply at once:

Theorem 1. *The statements $[a,b]\underline{\perp}[c,d]$, $[c,d]\underline{\perp}[b,a]$, $[b,a]\underline{\perp}[d,c]$ and $[d,c]\underline{\perp}[a,b]$ are mutually equivalent (Figure 1).*

The following theorem gives a simple characterization for orthogonally equal vectors.

Theorem 2. $[a,b]\underline{\perp}[c,d] \Leftrightarrow ac = bd.$

PROOF: Let $[a,b] = [p,r]$, $[c,d] = [q,s]$, $S_o(p,q,r,s)$ (Figure 1), i.e. $[a,b]\underline{\perp}[c,d]$. Then we have the equalities $pq = rs = o$ and by (14) the equalities $a \bullet r = b \bullet p$, $c \bullet s = d \bullet q$. Hence

$$ac \bullet o = ac \bullet rs \overset{(13)}{=} (a \bullet r)(c \bullet s) = (b \bullet p)(d \bullet q) \overset{(13)}{=} bd \bullet pq = bd \bullet o,$$

wherefrom $ac = bd$ follows. Conversely, let $ac = bd$ and let p be any point. There is a point r such that $[a,b] = [p,r]$. Let $q = rp$, $s = pr$, i.e. let $S(p,q,r,s)$ hold. There is a point d' such that $[q,s] = [c,d']$. Now we have $[a,b]\underline{\perp}[c,d']$ and the proved part of our theorem implies $ac = bd'$. Therefore we have $bd' = bd$, i.e. $d' = d$ and hence $[a,b]\underline{\perp}[c,d]$. □

Theorem 2 and the equivalence (7) give an alternative proof of Theorem 1.

The proof of Theorem 2 implies:

Corollary 1. *For any vector* $\mathbf{v}$ *and any point* c *there is one and only one point* d *such that* $\mathbf{v}\underline{\perp}[c,d]$ *holds.*

Because of Theorem 2 the equality (1) can be interpreted as the statement $[ca,ab]\underline{\perp}[bc,a]$.

Theorem 3. (i) $[a,b]\underline{\perp}[c,d]$, $[c,d] = [e,f] \Rightarrow [a,b]\underline{\perp}[e,f]$.
(ii) $[a,b] = [c,d]$, $[c,d]\underline{\perp}[e,f] \Rightarrow [a,b]\underline{\perp}[e,f]$.
(iii) $[a,b]\underline{\perp}[c,d]$, $[c,d]\underline{\perp}[e,f] \Rightarrow [a,b] = [f,e]$.
(iv) $[a,b]\underline{\perp}[d,e]$, $[b,c]\underline{\perp}[e,f] \Rightarrow [a,c]\underline{\perp}[d,f]$.

PROOF: (i) By Theorem 2 and by (14) we have the equalities $ac = bd$ and $c \bullet f = d \bullet e$. Therefore

$$ac \bullet ae = bd \bullet ae \overset{(13)}{=} (b \bullet a)(d \bullet e) \overset{(12)}{=} (a \bullet b)(c \bullet f) \overset{(13)}{=} ac \bullet bf,$$

wherefrom $ae = bf$ follows and by Theorem 2 we have the statement $[a,b]\underline{\perp}[e,f]$.

(ii) Now we have the equalities $a \bullet d = b \bullet c$ and $ce = df$ and we obtain

$$ae \bullet df \overset{(13)}{=} (a \bullet d)(e \bullet f) \overset{(12)}{=} (b \bullet c)(f \bullet e) \overset{(13)}{=} bf \bullet ce = bf \bullet df.$$

Therefore $ae = bf$, i.e. again $[a,b]\underline{\perp}[e,f]$.

(iii) We have the equalities $ac = bd$, $ce = df$, which imply

$$a \bullet e \overset{(12)}{=} e \bullet a \overset{(9)}{=} ce \cdot ac = df \cdot bd \overset{(9)}{=} f \bullet b \overset{(12)}{=} b \bullet f,$$

i.e. $[a,b] = [f,e]$ by (14).

(iv) By Theorem 2 we must prove the implication $ad = be$, $be = cf \Rightarrow ad = cf$. It is obvious. $\square$

Because of (7) the following definition has a sense.

An oriented quadrangle (a, b, c, d) is a *skewsquare* and we write $SS(a, b, c, d)$ if $ab = cd$ and $bc = da$. It is sufficient to have only one of these two equalities (cf. [7] and [4]). If we have the equalities $ab = cd = p$ and $bc = da = q$, then the points p and q are said to be the *skewcenters* of the considered skewsquare and we write $SS_{p,q}(a, b, c, d)$ (Figure 2) (cf. [4], where p and q are said to be the *foci* of the skewsquare).

Obviously we get:

Theorem 4. *The statements* $SS_{p,q}(a, b, c, d)$, $SS_{q,p}(b, c, d, a)$, $SS_{p,q}(c, d, a, b)$ *and* $SS_{q,p}(d, a, b, c)$ *are mutually equivalent.*

According to Theorem 2 it follows.

Corollary 2. $SS(a, b, c, d) \Leftrightarrow [a, c] \perp [b, d]$ (*Figure* 2).

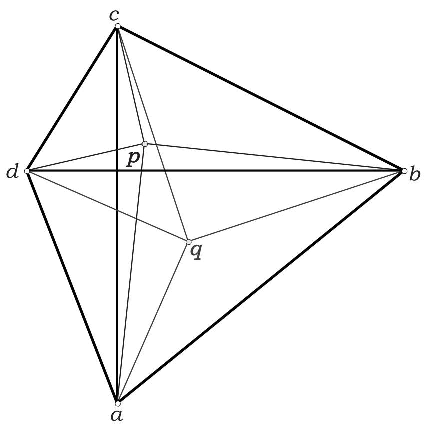

FIGURE 2

By Corollaries 1 and 2 we obtain the following statement.

Corollary 3. *For any points* a, b, c *there is one and only one point* d *such that* $SS(a, b, c, d)$ *holds.*

The equation $ax = b$ has a unique solution $x = (b \cdot ba) \cdot (b \cdot ba)(ba \cdot a)$ (cf. [12, Corollary]). Therefore the equality $ab = cd$ is equivalent to the equality

(15) $$d = (ab)(ab \cdot c) \cdot [(ab)(ab \cdot c) \cdot (ab \cdot c)c],$$

i.e. we have the following theorem, which expresses the statement of Corollary 3 precisely.

Theorem 5. *The statement* $SS(a,b,c,d)$ *is equivalent to the equality* (15) (*Figure* 3).

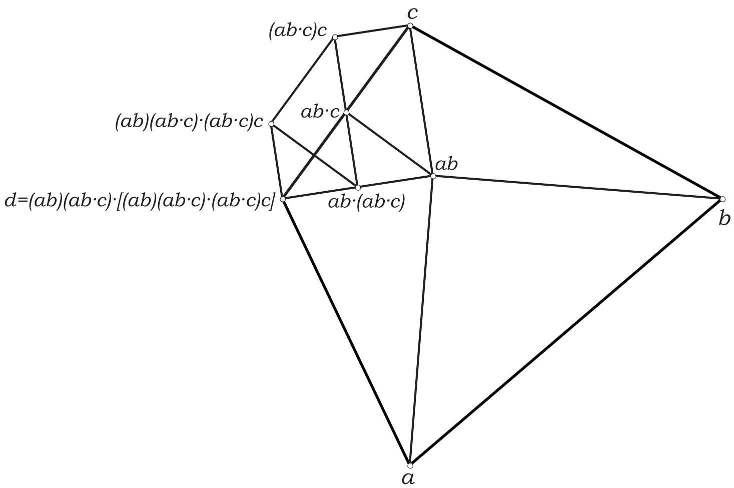

FIGURE 3

Obviously we obtain.

Theorem 6. $S_o(a,b,c,d) \Leftrightarrow SS_{o,o}(a,b,c,d)$.

Let us prove the following statement now.

Theorem 7. *The statement* $SS_{p,q}(a,b,c,d)$ *implies* $S_o(p, a\bullet c, q, b\bullet d)$ (*Figure* 4) *where*

$$o = a\bullet db = b\bullet ac = c\bullet bd = d\bullet ca = p\bullet q = g(a,b,c,d).$$

PROOF: Let $o = p\bullet q$. Because of $ab = p$, $da = q$ we get

$$o = p\bullet q = ab\bullet da \overset{(13)}{=} (a\bullet d)(b\bullet a) \overset{(12)}{=} (a\bullet d)(a\bullet b) \overset{(13)}{=} aa\bullet db \overset{(2)}{=} a\bullet db,$$

and similarly it can be obtained $o = b\bullet ac = c\bullet bd = d\bullet ca$. Further, we get

$$p(a\bullet c) = ab\cdot(a\bullet c) \overset{(9)}{=} ab\cdot(ac\cdot a) \overset{(1)}{=} (b\cdot ac)b \overset{(9)}{=} b\bullet ac = o,$$

$$(a\bullet c)q = (a\bullet c)\cdot da \overset{(9)}{=} (ac\cdot a)\cdot da \overset{(4)}{=} (a\cdot ca)\cdot da \overset{(1)}{=} (ca\cdot d)\cdot ca \overset{(9)}{=} ca\bullet d \overset{(12)}{=} d\bullet ca = o,$$

and similarly the following equalities $q(b\bullet d) = o$, $(b\bullet d)p = o$ can be proved, so it is valid $S_o(p, a\bullet c, q, b\bullet d)$, and then $\mathrm{Par}_o(p, a\bullet c, q, b\bullet d)$. Because of that we also get the equalities

$$p\bullet q = o = (a\bullet c)\bullet(b\bullet d) = g(a,b,c,d).$$

□

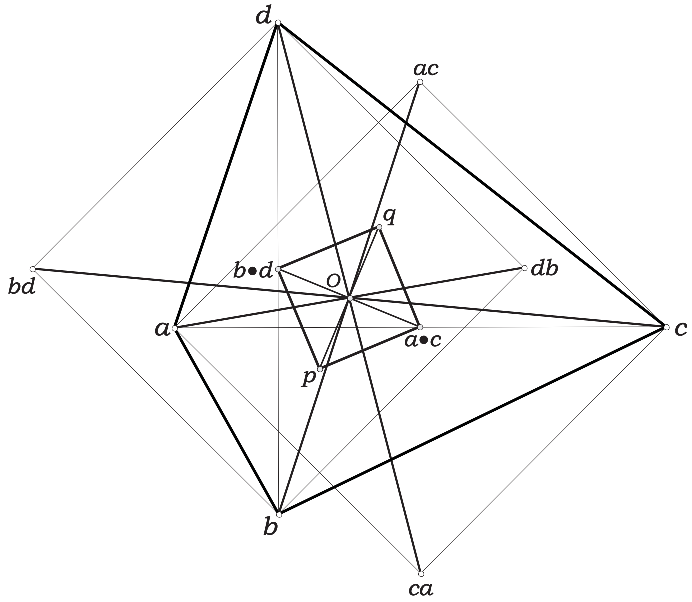

FIGURE 4

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 7 proves one statement from [4] and [7].

The point o from Theorem 7 will be called *centre* of the skewsquare (a, b, c, d).

Theorem 8. $SS(a, b, c, d) \Leftrightarrow S(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$ (*Figure* 5).

PROOF: As we have

$$(a \bullet b)(b \bullet c) \overset{(13)}{=} ab \bullet bc,$$
$$(b \bullet c)(c \bullet d) \overset{(13)}{=} bc \bullet cd \overset{(12)}{=} cd \bullet bc,$$

the equalities $ab = cd$ and $(a \bullet b)(b \bullet c) = (b \bullet c)(c \bullet d)$ are equivalent. The equivalence of the remaining equalities can be proved in a similar way. □

One part of Theorem 8 can be stated more precisely in the form:

Theorem 9. $SS_{p,q}(a, b, c, d) \Rightarrow S_{p \bullet q}(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$ (*Figure* 5).

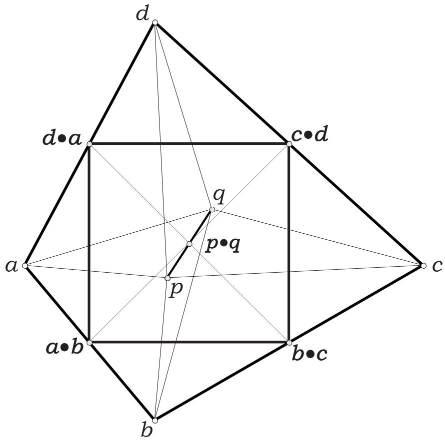

FIGURE 5

PROOF: We have for example

$$(a \bullet b)(b \bullet c) \overset{(13)}{=} ab \bullet bc = p \bullet q,$$

$$(b \bullet c)(c \bullet d) \overset{(13)}{=} bc \bullet cd = q \bullet p \overset{(12)}{=} p \bullet q$$

because of $p = ab = cd$, $q = bc$. □

In a case of the quasigroup $\mathbb{C}(\frac{i+1}{2})$ Theorem 9 proves one statement from [4].

Corollary 4. *The statement $SS(a,b,c,d)$ implies the equalities $(a \bullet b)(c \bullet d) = d \bullet a$, $(b \bullet c)(d \bullet a) = a \bullet b$, $(c \bullet d)(a \bullet b) = b \bullet c$, $(d \bullet a)(b \bullet c) = c \bullet d$ (Figure 5).*

Because of Lemma 3 and Theorem 6 the statement $S_o(a,b,c,d)$ implies $\mathrm{Par}_o(a,b,c,d)$ and $SS(a,b,c,d)$. However, the converse is also valid.

Theorem 10. *$\mathrm{Par}_o(a,b,c,d)$, $SS(a,b,c,d) \Rightarrow S_o(a,b,c,d)$.*

PROOF: Let $SS_{p,q}(a,b,c,d)$. Then according to Theorem 7 we get $S(p,o,q,o)$, since $\mathrm{Par}_o(a,b,c,d)$ implies $a \bullet c = b \bullet d = o$. Because of that we get $p = oo$, $q = oo$, i.e. because of (2) we obtain $p = q = o$, and then $SS_{o,o}(a,b,c,d)$, i.e. owing to Theorem 6 it follows $S_o(a,b,c,d)$. □

Theorem 11. *From statement $\mathrm{Par}_o(a,b,c,d)$ the statements $SS_{o,p}(ac,a,bd,b)$, $SS_{q,o}(ac,d,bd,c)$ follow where p and q are some points such that $qp = o$ (Figure 6).*

PROOF: Owing to (9) we have

$$ac \cdot a = a \bullet c = o, \quad bd \cdot b = b \bullet d = o, \quad d \cdot bd = d \bullet b = o, \quad c \cdot ac = c \bullet a = o,$$

and equalities $ac \cdot a = bd \cdot b$ and $d \cdot bd = c \cdot ac$ prove the first two statements of theorem. Because of that there are points p and q such that $a \cdot bd = b \cdot ac = p$ and $ac \cdot d = bd \cdot c = q$. Finally, we get

$$qp = (bd \cdot c)(a \cdot bd) \overset{(9)}{=} c \bullet a = o.$$

□

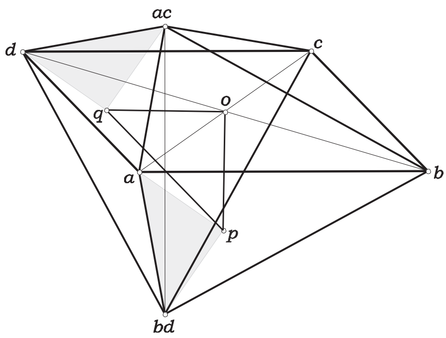

FIGURE 6

Theorem 12. *The validity of the statements $S(a,b,p,q)$, $S(c,a,s,r)$ and $o = cb$ imply the statements $SS_{o,a}(c,b,q,s)$ and $o = qs = p \bullet r$ (Figure 7).*

PROOF: On the basis of Lemma 2 we get equalities $pa = b, bq = a, ar = c, sc = a$. So we get $bq = sc$, wherefrom due to (7) it follows $qs = cb = o$. Besides that owing to (9) and (12) we obtain

$$p \bullet r = r \bullet p = ar \cdot pa = cb = o.$$

□

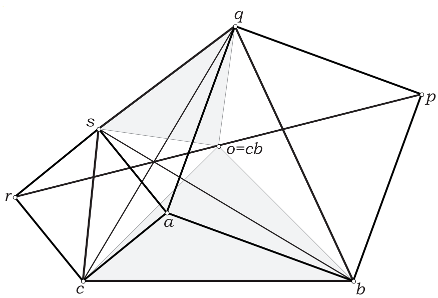

FIGURE 7

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 12 proves some results from [1].

Theorem 13. *The statements $S_m(a,b,p,q)$ and $S_n(c,a,s,r)$ imply $S(m, q \bullet s, n, c \bullet b)$.*

Proof: According to Theorem 12 it follows $SS(b,q,s,c)$, and owing to Theorem 8 we get $S(b \bullet q, q \bullet s, s \bullet c, c \bullet b)$. However, because of Lemma 3 it follows $b \bullet q = m$, $s \bullet c = n$, so the statement we are looking for follows. □

Theorem 14. *The statements $S(a,b,p,q)$, $S(b,a,q',p')$, $S(c,a,s,r)$, $S(a,c,r',s')$ imply $SS_{a,o}(p',p,r,r')$, where o is some point (Figure 8).*

Proof: According to Lemma 2 we get equalities $pa = b$, $ap' = b$, $ar = c$, $r'a = c$, so we have $ap' = pa$, $ar = r'a$, wherefrom owing to (8) the equalities $p'p = a$ and $rr' = a$ follow. □

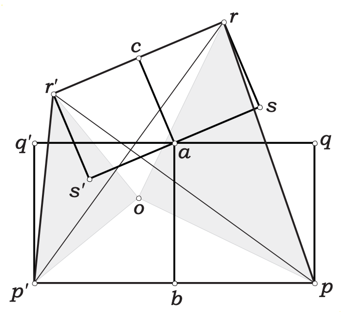

Figure 8

Theorem 15. *For any points a,b,c,d the statement $SS_{a\bullet c, b\bullet d}(ba, cb, dc, ad)$ is valid (Figure 10) (van Aubel's theorem).*

Proof: Based on (9) and (12) we get

$$ba \cdot cb = a \bullet c = c \bullet a = dc \cdot ad,$$
$$cb \cdot dc = b \bullet d = d \bullet b = ad \cdot ba.$$

□

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 15 proves the well known statement from (cf. [3], [5], [10]).

With $d = a$ from Theorem 15 we obtain:

Corollary 5. *For any points a,b,c the statement $SS_{a\bullet c, b\bullet a}(ba, cb, ac, a)$ is valid.*

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Corollary 5 proves the known Belatti's result.

If we denote by $\cdot$ the mapping which maps the quadrangle (a,b,c,d) to the quadrangle (ba, cb, dc, ad), and if $\bullet$ denote the mapping which maps quadrangle

(a, b, c, d) to the quadrangle $(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$, then on the basis of Lemma 1, Lemma 4, Theorem 8 and Theorem 15, we get following diagram (*Figure* 9).

In this diagram the operators $\cdot$ and $\bullet$ commute, it means: starting from the same quadrangle in two ways we get the same square. Really, on the basis of (13) we get for example

$$(b \bullet c)(a \bullet b) = ba \bullet cb.$$

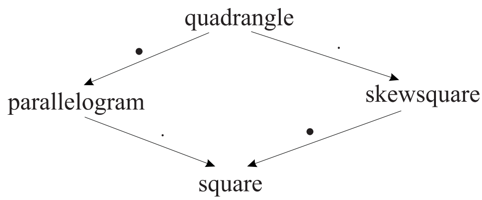

FIGURE 9

Theorem 16. *For any points* a, b, c, d *it is valid* $S(a \bullet c, ba \bullet dc, b \bullet d, cb \bullet ad)$ (*Figure* 10).

PROOF: On the basis of (12) and (13) we get

$$(b \bullet d)(a \bullet c) = ba \bullet dc,$$
$$(a \bullet c)(b \bullet d) = (c \bullet a)(b \bullet d) = cb \bullet ad,$$

so the statement follows according to Lemma 2. □

Theorem 17. *With the labels* $e_1 = ba \cdot ad$, $e_2 = cb \cdot ba$, $e_3 = dc \cdot cb$, $e_4 = ad \cdot dc$ *the statements* $SS_{cb\bullet ad,\, ba\bullet dc}(e_1, e_2, e_3, e_4)$, $e_1 \bullet e_3 = a \bullet c$, $e_2 \bullet e_4 = b \bullet d$ *hold* (*Figure* 10).

PROOF: If we apply Theorem 15 on the points ba, cb, dc, ad we will obtain the first statement. Since owing to Theorem 16 the equality $(ba \bullet dc)(cb \bullet ad) = a \bullet c$ holds, we get

$$e_1 \bullet e_3 = (ba \cdot ad) \bullet (dc \cdot cb) \overset{(13)}{=} (ba \bullet dc) \cdot (ad \bullet cb) = a \bullet c,$$

and similarly $e_2 \bullet e_4 = b \bullet d$. □

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorems 15, 16 and 17 prove results from [10] and [9].

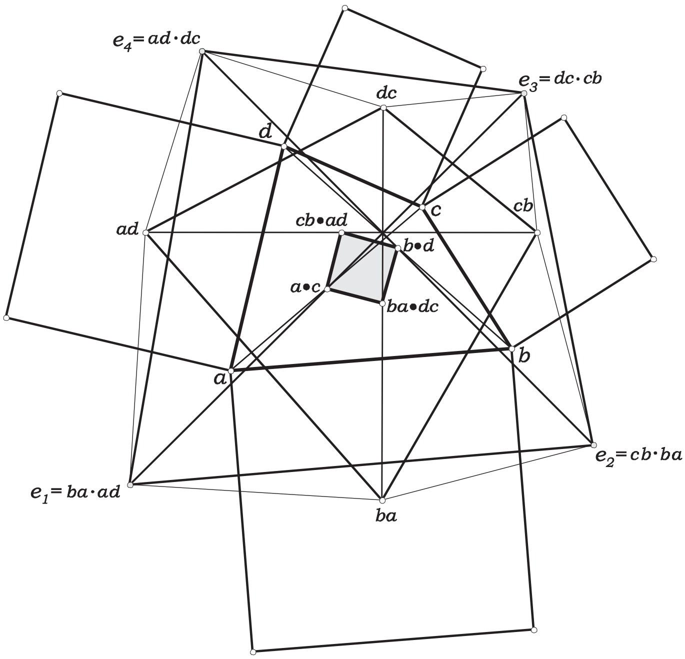

Figure 10

Theorem 18. *For any points* $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ *let us denote* $a_{i,i+1} = a_i a_{i+1}$, $m_{i,i+1,i+4,i+5} = a_{i,i+1} \bullet a_{i+4,i+5}$, *where indexes are taken modulo* 8 *from the set* $\{1,2,3,4,5,6,7,8\}$. *If* $p = g(a_2, a_4, a_6, a_8)$, $q = g(a_1, a_3, a_5, a_7)$, *then we get* $SS_{p,q}(m_{1256}, m_{4581}, m_{7834}, m_{2367})$ (*Figure* 11).

Proof: On the bases of (9), (12) and (13) we get for example

$$\begin{aligned} m_{1256} m_{4581} &= (a_{12} \bullet a_{56})(a_{45} \bullet a_{81}) = (a_{12} \bullet a_{56})(a_{81} \bullet a_{45}) \\ &= a_{12} a_{81} \bullet a_{56} a_{45} = (a_1 a_2 \cdot a_8 a_1) \bullet (a_5 a_6 \cdot a_4 a_5) \\ &= (a_2 \bullet a_8) \bullet (a_6 \bullet a_4) = g(a_2, a_4, a_6, a_8) = p. \end{aligned}$$

□

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 18 proves the result stated in [3], [6] and [9]:

The centres of squares constructed on the sides of the given octagon determine new octagon, and the midpoints of the main diagonals of the obtained octagon determine an skewsquare (*Figure* 11).

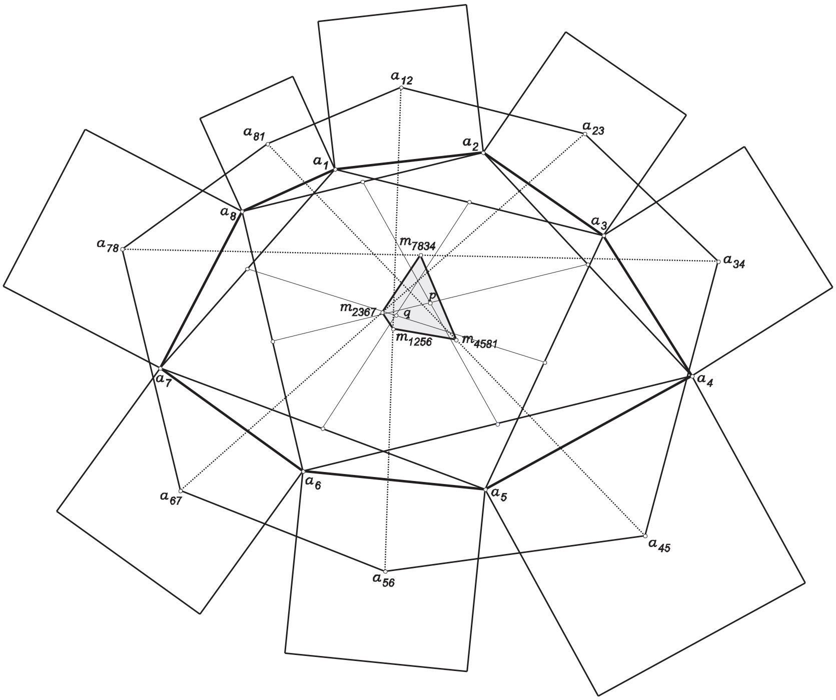

FIGURE 11

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(*Received* November 10, 2007, *revised* March 18, 2008)