On weak solutions of steady Navier-Stokes equations for monatomic gas

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Abstract. We use L^∞ estimates for the inverse Laplacian of the pressure introduced by Plotnikov, Sokolowski and Frehse, Goj, Steinhauer together with the nonlinear potential theory due to Adams, Hedberg, to get a priori estimates and to prove existence of weak solutions to steady isentropic Navier-Stokes equations with the adiabatic constant $\gamma > \frac{1}{3}(1+\sqrt{13}) \approx 1.53$ for the flows powered by volume non-potential forces and with $\gamma > \frac{1}{8}(3+\sqrt{41}) \approx 1.175$ for the flows powered by potential forces and arbitrary non-volume forces. According to our knowledge, it is the first result that treats in three dimensions existence of weak solutions in the physically relevant case $\gamma \leq \frac{5}{3}$ with arbitrary large external data. The solutions are constructed in a rectangular domain with periodic boundary conditions.

Keywords: steady compressible Navier-Stokes equations, periodic domain, isentropic flow, existence of the weak solution, potential theory

Classification: 35Q, 76N

1. Introduction

Evolution of a viscous compressible fluid is described by the density $\varrho(t,\boldsymbol{x})$, the velocity field $\boldsymbol{u}(t,\boldsymbol{x})$, and the temperature $\vartheta(t,\boldsymbol{x})$, which are functions of the time t and the spatial coordinates $\boldsymbol{x}$. These quantities have to satisfy the fundamental conservation laws, namely the conservation of

(1.1) mass: $$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0,$$

(1.2) linear momentum: $$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \mathbb{S} + \varrho \boldsymbol{f} + \boldsymbol{g},$$

(1.3) energy: $$\partial_t(\varrho e) + \operatorname{div}(\varrho e \boldsymbol{u}) + \operatorname{div} \boldsymbol{q} = \mathbb{S} : \nabla \boldsymbol{u} - p \operatorname{div} \boldsymbol{u}.$$

In (1.1)–(1.3), $\boldsymbol{q}$ denotes the heat flux and $\mathbb{S}$ the viscous stress tensor, p represents the pressure and e the internal energy. The dependence of these quantities on the state variables ϱ, ϑ, $\boldsymbol{u}$, and their derivatives characterises the physical nature of the gas and will be discussed later. Finally $\varrho \boldsymbol{f}$ and $\boldsymbol{g}$ denote external volume and non volume forces.

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In the case of the small velocity gradient and/or small viscosities, the dissipation (i.e. transformation of the kinetic energy into heat) may be neglected. Similarly, in the case of small heat conductivity of the gas and/or small temperature gradients, the heat flux $\boldsymbol{q}$ may be neglected, as well. A flow that fulfills both these physical assumptions is called *adiabatic*. If one rewrites equation (1.3) in terms of the specific entropy s (defined by the Gibbs law, namely $\vartheta \, \mathrm{d}s = \mathrm{d}e - p\varrho^{-2} \, \mathrm{d}\varrho$), it appears that in the adiabatic case, the specific entropy is constant along trajectories of fluid particles. This implies that the pressure has a particular form

$$p(\varrho) = a\varrho^{\gamma}, \quad a > 0, \quad \gamma > 1, \tag{1.4}$$

where $\gamma \geq 1$ is the so called *adiabatic constant* and $a > 0$ is a constant along any trajectory. In the sequel, we will assume that the flow is *isentropic*, which means that a is constant across all trajectories. As the pressure is a function of the sole density, equations (1.1)–(1.2) become an independent system, while, once $(\varrho, \boldsymbol{u})$ is known, (1.3) is an independent equation to determine the temperature field.

To complete system (1.1)–(1.2) it remains to specify $\mathbb{S}$. We consider *Newtonian* fluid, which is characterized by the viscous stress tensor

$$\mathbb{S} := \mu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) + \lambda \operatorname{div} \boldsymbol{u} \, \mathbb{I}, \tag{1.5}$$

where μ and λ are constant viscosity coefficients which have to satisfy thermodynamic constraints

$$\mu > 0, \; 2\mu + 3\lambda > 0. \tag{1.6}$$

In this paper we deal with the existence of steady (i.e. time independent) solutions $(\varrho, \boldsymbol{u})$ to the system of equations for the isentropic flow of the Newtonian fluid which reads

$$\operatorname{div}(\varrho \boldsymbol{u}) = 0 \tag{1.7}$$

$$\operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} - (\mu + \lambda) \nabla \operatorname{div} \boldsymbol{u} + \nabla p(\varrho) = \varrho \boldsymbol{f} + \boldsymbol{g}, \tag{1.8}$$

with $p(\varrho) = \varrho^{\gamma}$, where we have taken $a = 1$ without loss of generality.

It is shown in statistical physics that the adiabatic constant γ in (1.4) depends on the number M of the degrees of freedom of the molecules of the gas. One has $\gamma = \frac{5}{3} \approx 1.66$ for the mono-atomic gas, $\gamma = \frac{7}{5} = 1.4$ for the air and in general $\gamma = \frac{M+2}{M}$. Parameters similar to γ appear in the complete theory of the viscous compressible fluids described by the full Navier-Stokes-Fourier system (1.1)-(1.3), and from the mathematical point of view, they determine the quality of density estimates. That is why the simplified isentropic model for compressible fluids is important, in spite of its slightly contradictory physical background.

The first existence result for the system (1.7)–(1.8) is due to the pioneering work of Lions [10]. This work assumes $\gamma > \frac{5}{3}$. Later, Novotný, Novo [12] have adapted a method of Feireisl [5] to prove existence in the case of the potential $\boldsymbol{f}$ (and arbitrary $\boldsymbol{g}$) with $\gamma > \frac{3}{2}$, see also [15]. Recently, Frehse, Goj, Steinhauer [8] and Plotnikov, Sokolowski [16] have independently obtained new L^∞ estimates for the quantity $\Delta^{-1}p$ and have proposed several methods to improve estimates of the density. Both works however assume a priori bound for L^1 norm of $\varrho \boldsymbol{u}^2$ which is not available for the general system (1.7)–(1.8). When this paper was completed, we have learned about the paper by Plotnikov, Sokolowski [17], where the authors consider the same problem with the Dirichlet boundary conditions, and where however the physical condition $\int_\Omega \varrho = m$ of the conservation of total mass may be violated. They use similar bootstrapping argument and obtain existence with the coefficient $\gamma > \frac{4}{3}$.

The main goals of this paper are:

(1) to put the Frehse, Goj, Steinhauer [8] and the Plotnikov, Sokolowski [16] estimates into the context of the modern potential theory (see Adams, Hedberg [1]);
(2) to show how the L^∞ estimate of $\Delta^{-1}p$ can be combined with the standard energy and density bounds even without the a priori L^1 bound for $\varrho \boldsymbol{u}^2$;
(3) to use these observations to prove existence of solutions for small values of γ, namely $\gamma > \frac{1}{3}(1+\sqrt{13}) \approx 1.53$ in the case of three dimensional flows and arbitrary $\boldsymbol{f}$, and $\gamma > \frac{1}{8}(3+\sqrt{41}) \approx 1.175$, if $\boldsymbol{f}$ is potential.

The condition for the general $\boldsymbol{f}$ allows to treat at least the monatomic gas. As the estimate of $\Delta^{-1}p$ is essentially of the local character we limit ourselves to the periodic boundary conditions and periodic domain. In order to guarantee existence of space periodic solutions, we assume $\boldsymbol{f}$ and $\boldsymbol{g}$ with certain symmetries.

The paper is organised as follows. In the next section we formulate Theorem 1, the main result of the paper. The rest of the paper is devoted to its proof. In Section 3 we derive L^∞ estimates for $\Delta^{-1}p$. Then in Section 4 we use the nonlinear potential theory due to Adams, Hedberg [1] to find a convenient L^1 bound for the quantity $p\boldsymbol{u}^2$. In Section 5 we use this estimate together with standard energy and density bounds to estimate the density in the space $L^{\gamma q}$, $q = \frac{3\gamma}{\gamma+2}$. This piece of information, combined with the recently discovered compactness properties of the so called effective viscous flux and with the notion of the renormalized solutions to the continuity equation (cf. P.L. Lions' results [10] and [6], [15]), makes possible to prove compactness of the set of weak solutions as well as to construct weak solutions via a several level approximation scheme, in the same manner as in [12]. The approximation process leading to the existence theorem is described and investigated in Section 6. The limit passage from one level to another is standard, see e.g. [15]. Nevertheless, the necessary modifications in the construction of approximations to accommodate the periodic boundary conditions, as well as the last (and the most delicate) limit process are performed in all details in Sections 6.1 and 6.2.

2. Formulation of the problem and main results

We consider equations (1.7)–(1.8) on a periodic cell

$$\Omega = \Big([-\pi, +\pi]\big|_{\{-\pi,\pi\}}\Big)^3 \tag{2.1}$$

with the periodic boundary conditions and $\boldsymbol{f}$, $\boldsymbol{g}$ with symmetry

$$\begin{aligned} &\boldsymbol{f}_i(\boldsymbol{x}) = -\boldsymbol{f}_i(\boldsymbol{Y}_i(\boldsymbol{x})), \qquad \boldsymbol{f}_i(\boldsymbol{x}) = \boldsymbol{f}_i(\boldsymbol{Y}_j(\boldsymbol{x})) \qquad \text{and} \\ &\boldsymbol{g}_i(\boldsymbol{x}) = -\boldsymbol{g}_i(\boldsymbol{Y}_i(\boldsymbol{x})), \qquad \boldsymbol{g}_i(\boldsymbol{x}) = \boldsymbol{g}_i(\boldsymbol{Y}_j(\boldsymbol{x})) \qquad \text{for} \;\; i \neq j, \;\; i,j \in \{1,2,3\}, \end{aligned} \tag{2.2}$$

where

$$\boldsymbol{Y}_i(\dots, x_i, \dots) = (\dots, -x_i, \dots).$$

This implies the same symmetry of $\boldsymbol{u}$, and ϱ with the symmetry

$$\varrho(\boldsymbol{x}) = \varrho(\boldsymbol{Y}_i(\boldsymbol{x})) \quad \text{for} \;\; i = 1,2,3. \tag{2.3}$$

Consequently the investigated problem can be viewed also as the problem on the cube $(0,\pi)^3$ with slip boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\,\mathbb{S}\,\boldsymbol{\tau} = 0 \quad \text{both on} \;\; \partial\,(0,\pi)^3,$$

see Ebin [4].

Let G stand for a domain in $\mathbb{R}^3$ or for $\mathbb{R}^3$ or for the periodic cell Ω. Throughout the whole paper we shall write $L^p(G)$ for the Lebesgue spaces, $W^{k,p}(G)$, $k \in \mathbb{N}$, for the Sobolev spaces, $C^k(G)$ resp. $C^k(\overline{G})$ for the k-times continuously differentiable functions on G resp. $\overline{G}$, $C_0(G)$ for the continuous functions with compact support in G, and $\mathcal{D}(G)$ for $C^\infty(G) \cap C_0(G)$. The spaces of vector valued functions have the vector space as the next argument (e.g. $L^p(G;\mathbb{R}^3)$ resp. $W^{k,p}(G;\mathbb{R}^3)$ are Lebesgue resp. Sobolev spaces of $\mathbb{R}^3$-valued functions). If there is no danger of confusion, we write simply $L^p(G;\mathbb{R}^3) = L^p(G)$ and $W^{k,p}(G;\mathbb{R}^3) = W^{k,p}(G)$. The corresponding norms are $\|\cdot\|_{L^p(G)}$, $\|\cdot\|_{W^{k,p}(G;\mathbb{R}^3)}$, and so on. If G is Ω, we write simply $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{p,\Omega} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)} = \|\cdot\|_{k,p,\Omega} = \|\cdot\|_{k,p}$. By prime we denote dual spaces (e.g. $\mathcal{D}'(G)$ is the space of distributions — dual to $\mathcal{D}(G)$; $(L^q(\Omega))' = L^{q'}(\Omega)$, where q' is the dual index to q, i.e. $\frac{1}{q'} + \frac{1}{q} = 1$; $(W^{1,q}(\mathbb{R}^3))' = W^{-1,q'}(\mathbb{R}^3)$, etc.) Furthermore we introduce spaces of symmetric functions: for example, $W^{1,2}_{\mathrm{sym}}(\Omega;\mathbb{R}^3)$ stands for the (vector valued) functions from $W^{1,2}(\Omega;\mathbb{R}^3)$, that enjoy symmetric property (2.2) and $L^p_{\mathrm{sym}}(\Omega)$ denotes (scalar) functions from $L^p(\Omega)$ that satisfy symmetry (2.3). A set as an index of a measure (or a function) means the measure restricted to the set, e.g. p_Ω is the measure $p_\Omega(M) = p(\Omega \cap M) = \int_{\Omega\cap M} p$.

Suppose for a moment that $(\varrho, \boldsymbol{u})$ is a classical solution to (1.7)–(1.8) and let $b \in C^1(0,\infty)$. Multiplying continuity equation (1.7) by $b'(\varrho)$, we obtain the renormalized continuity equation

$$\operatorname{div}(b(\varrho)\boldsymbol{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big) \operatorname{div} \boldsymbol{u} = 0. \tag{2.4}$$

To keep this equation valid even for a weak solution $\varrho \in L^\gamma(\Omega)$ and $\boldsymbol{u} \in W^{1,2}(\Omega;\mathbb{R}^3)$ (see Definition 1 later on) we require that (2.4) is satisfied in the sense of distributions $\mathcal{D}'(\Omega)$ for any

$$\begin{cases} b \in C\big([0,\infty)\big) \cap C^1\big((0,\infty)\big) \\ \sup\limits_{t\in(0,1)} \big|t^\alpha b'(t)\big| < \infty, \qquad \text{for some } \ \alpha \in [0,1), \\ \sup\limits_{t\in(1,\infty)} \big|t^{-\alpha} b'(t)\big| < \infty, \qquad \text{for some } \ \alpha \le \dfrac{\gamma}{2} - 1. \end{cases} \tag{2.5}$$

Similarly, we take a scalar product of momentum equation (1.8) with $\boldsymbol{u}$ and we integrate over Ω. Using continuity equation (1.7) and taking advantage of the periodicity of solutions, after several integrations by parts, we obtain the energy equality

$$\int_\Omega \mu|\nabla \boldsymbol{u}|^2 + (\mu+\lambda)|\operatorname{div}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho \boldsymbol{f}\cdot\boldsymbol{u} + \boldsymbol{g}\cdot\boldsymbol{u} \,\mathrm{d}\boldsymbol{x}. \tag{2.6}$$

Of course, due to the presence of the weakly lower semi-continuous functionals

$$\nabla\boldsymbol{u} \to \int_\Omega |\nabla\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x}, \quad \nabla\boldsymbol{u} \to \int_\Omega |\operatorname{div}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x},$$

on $L^2(\Omega;\mathbb{R}^3)$, for weak solutions, we can expect only the energy inequality

$$\int_\Omega \mu|\nabla \boldsymbol{u}|^2 + (\mu+\lambda)|\operatorname{div}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \le \int_\Omega \varrho \boldsymbol{f}\cdot\boldsymbol{u} + \boldsymbol{g}\cdot\boldsymbol{u} \,\mathrm{d}\boldsymbol{x}. \tag{2.7}$$

Last but not least, integrating momentum equation (1.8) over the periodic cell Ω, in accordance with the periodicity of solutions, we obtain the compatibility relation

$$\int_\Omega \varrho\boldsymbol{f} + \boldsymbol{g} \,\mathrm{d}\boldsymbol{x} = 0. \tag{2.8}$$

This condition is automatically satisfied by any solution induced by $\boldsymbol{f}$ and $\boldsymbol{g}$ with symmetry (2.2). Finally, we denote by $m > 0$ the total mass of the gas in the volume Ω.

Following the terminology of [15] we define a *renormalized bounded energy weak solution* of the periodic problem (1.7)–(1.8) on the domain Ω as follows:

Definition 1. Let the viscosity coefficients μ, λ satisfy (1.6). Suppose that $\gamma > 1$ and $m > 0$ are given constants and assume that both $\boldsymbol{f}, \boldsymbol{g} \in L^\infty(\Omega)$ satisfy (2.2). We say that a couple $(\varrho, \boldsymbol{u})$ is a renormalised bounded energy weak solution of the periodic problem (1.7)–(1.8) on the periodic cell Ω if

$$\varrho \in L^\gamma_{\mathrm{sym}}(\Omega),\ \ \boldsymbol{u} \in W^{1,2}_{\mathrm{sym}}(\Omega; \mathbb{R}^3), \tag{2.9}$$

$$\int_\Omega \varrho \,\mathrm{d}\boldsymbol{x} = m, \tag{2.10}$$

the renormalised continuity equation (2.4) is valid for any b satisfying (2.5), the momentum equation (1.8) holds in $\mathcal{D}'(\Omega)$, and (2.7) is satisfied.

Remark 1. In view of (2.9) the simple density argument can be used to see that (1.8) holds even in $(W^{1,q}(\Omega; \mathbb{R}^3))'$ for any $q \geq \max(2, \frac{3\gamma}{2\gamma-3})$.

Now we are ready to state the main result.

Theorem 1. *Let Ω, m, μ, λ, $\boldsymbol{f}$, $\boldsymbol{g}$ satisfy the hypotheses of Definition 1. Let*

$$\gamma > \gamma_{\mathrm{gen.}} := \frac{1}{3}(1 + \sqrt{13}) \approx 1.53 \tag{2.11}$$

or let $\boldsymbol{f}$ be potential and

$$\gamma > \gamma_{\mathrm{pot.}} := \frac{1}{8}(3 + \sqrt{41}) \approx 1.175. \tag{2.12}$$

Then there exists a renormalised bounded energy weak solution $(\varrho, \boldsymbol{u})$ of the periodic problem (1.7)–(1.8) *which satisfies*

$$\varrho \in L^{\gamma q}(\Omega), \qquad q = \frac{3\gamma}{2+\gamma}\,. \tag{2.13}$$

Weak solutions are constructed via several approximation levels described in Section 6. The last approximation leading to the final system (1.7)–(1.8) consists in investigating the same system where $p(\varrho) = \varrho^\gamma$ is replaced by the modified pressure $p_\delta(\varrho) = \varrho^\gamma + \delta\varrho^\beta$, where $\beta > 6$ (for technical reasons) and δ is a positive parameter. Existence of weak solutions to these equations is well known, cf. [10] or [15], modulo some changes in proofs in order to accommodate the periodic boundary conditions as explained in Section 6.

3. A potential estimate

Let $(\varrho_\delta, \boldsymbol{u}_\delta)$ be a sequence of renormalized bounded energy weak solutions to the problem (1.7)–(1.8), where, as well as in sequel, p stands for p_δ. Our aim is to derive for ϱ_δ sufficiently strong estimates independent of $\delta > 0$ in terms of the external data $\|\boldsymbol{f}\|_\infty$, $\|\boldsymbol{g}\|_\infty$ (and, of course, of the coefficients μ, λ).

Choose $\boldsymbol{y} \in \Omega$. Since the periodic problem is invariant with respect to the translation of the periodic cell, we can assume $y = 0$. As in [8] and [16], the main estimate of this section can be obtained testing formally the momentum equation (1.8) by $\varphi(x) = (x-y)|x-y|^{-1}$. Since this is not an admissible test function in the sense of Remark 1, we shall truncate it as follows:

$$\varphi = (x-y)\eta(|x-y|),$$
$$\eta(t) = \begin{cases} \frac{1}{r} - \frac{1}{R} & \text{on } [0,r) \\ \frac{1}{t} - \frac{1}{R} & \text{on } [r,R) \\ 0 & \text{on } [R,\infty) \end{cases}$$

where $0 < r < \frac{\pi}{2} < R < \pi$. Denoting $\mathbb{P} = \varrho\boldsymbol{u}\otimes\boldsymbol{u} + p\,\mathbb{I}$ and $\boldsymbol{n} = \frac{(x-y)}{|x-y|}$, a short calculation yields

$$\begin{aligned}
&\frac{1}{r}\int_{B_r} \mathrm{Tr}(\mathbb{P}-\mathbb{S}) + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} \\
&\quad - \frac{1}{R}\int_{B_R} \mathrm{Tr}(\mathbb{P}-\mathbb{S}) + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{B_R\setminus B_r} \frac{\mathrm{Tr}(\mathbb{P}-\mathbb{S}) - (\mathbb{P}-\mathbb{S}):\boldsymbol{n}\otimes\boldsymbol{n}}{|\boldsymbol{x}-\boldsymbol{y}|} + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} = 0,
\end{aligned} \tag{3.1}$$

where $B_s = \{\boldsymbol{x} : |\boldsymbol{x}-\boldsymbol{y}| < s\}$. Since $\varrho \in L^\beta(\Omega)$ for a fixed δ, we realize that the term $Q := \big(\mathrm{Tr}(\mathbb{P}-\mathbb{S}) + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\big)$ belongs to $L^1(\Omega)$. Thus the Lebesgue point property implies

$$\frac{1}{r}\int_{B_r} Q\,\mathrm{d}\boldsymbol{x} = \frac{4\pi\,r^2}{|B_r|}\int_{B_r} Q\,\mathrm{d}\boldsymbol{x} \to 0 \quad \text{as} \quad r \to 0.$$

Rearranging the remaining terms in (3.1) and estimating the resulting right-hand side, we obtain

$$\begin{aligned}
\sup_{r>0}\int_{B_R\setminus B_r} \frac{\mathrm{Tr}\,\mathbb{P} - \mathbb{P}:\boldsymbol{n}\otimes\boldsymbol{n}}{|\boldsymbol{y}-\boldsymbol{x}|}\,\mathrm{d}\boldsymbol{x} &\le \frac{1}{R}\int_{B_R} \mathrm{Tr}(\mathbb{P}-\mathbb{S}) + (\varrho\boldsymbol{f}+\boldsymbol{g})\cdot(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{B_R} \frac{2|\mathbb{S}|}{|\boldsymbol{y}-\boldsymbol{x}|} + |\varrho\boldsymbol{f}+\boldsymbol{g}|\,\mathrm{d}\boldsymbol{x} \le C(1 + \|\mathbb{P}\|_{1,\Omega} + \|\mathbb{S}\|_{2,\Omega} + \|\varrho\|_{1,\Omega}).
\end{aligned}$$

Here and in the sequel, C is a generic positive constant independent of δ. Next, we observe that

$$\mathrm{Tr}\,\mathbb{P} - \mathbb{P}:\boldsymbol{n}\otimes\boldsymbol{n} = \varrho\boldsymbol{u}^2 + 3p - (\varrho(\boldsymbol{u}\cdot\boldsymbol{n})^2 + p) \ge 2p \ge 0.$$

Thus, recalling the structure of $\mathbb{S}$, see (1.5), we get

$$\int_{B_R} \frac{2p}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x} \le C\big(1+\|\varrho\boldsymbol{u}^2\|_{1,\Omega}+\|p\|_{1,\Omega}+\|\boldsymbol{u}\|_{1,2,\Omega}\big). \tag{3.2}$$

Finally, denoting the periodic extension of p from $L^1(\Omega)$ to $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ again by p and extending the integral at the left-hand side of (3.2) to the whole $\mathbb{R}^3$, we arrive at

$$\begin{aligned}(\Delta^{-1}p_\Omega)[\boldsymbol{y}] := \int_{\mathbb{R}}^3 \frac{p_\Omega(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x} &\le \int_{B_R}\frac{p}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x}+\frac{1}{R}\int_\Omega p\,\mathrm{d}\boldsymbol{x}\\ &\le C(1+\|\varrho\boldsymbol{u}^2\|_{1,\Omega}+\|p\|_{1,\Omega}+\|\boldsymbol{u}\|_{1,2,\Omega}).\end{aligned} \tag{3.3}$$

4. An application of the potential theory

In this part we will apply the general potential theory developed by Adams, Hedberg [1] to obtain a convenient estimate for $p\boldsymbol{u}^2$. A similar estimate has been proved in [16], in a direct way. A slightly weaker one, for the quantity $p|\boldsymbol{u}|$, was derived in [8] via the theory of Morrey spaces. The main advantage of our approach is that accurate expressions for the best constants (see (4.9)) of estimates are obtained, which will be crucial for the bootstrapping argument in Section 5.

We shall say that a function g on $\mathbb{R}^N$ is a *radially decreasing convolution kernel* if $g(\boldsymbol{x}) = g_0(|\boldsymbol{x}|)$, for some non-negative, lower semi-continuous, non-increasing function g_0 on $\mathbb{R}^+$ for which $\int_0^1 g_0(t)t^{N-1}\,\mathrm{d}t < \infty$. The key ingredient of our proof is the following theorem.

Theorem 2 ([1, Theorem 7.2.1]). *Let g be a radially decreasing convolution kernel, and let $\mu \in \mathcal{M}^+(\mathbb{R}^N)$ be a positive Radon measure. Then for $1 < p \le q < \infty$ the following properties of μ are equivalent:*

(a) *there is a constant A_1 such that*

$$\Big(\int_{\mathbb{R}^N} |g\star f|^q\,\mathrm{d}\mu\Big)^{1/q} \le A_1\|f\|_p \tag{4.1}$$

for all $f \in L^p(\mathbb{R}^N)$;

(b) *there is a constant A_2 such that*

$$\|g\star\mu_K\|_{p'} \le A_2\,\mu(K)^{1/q'} \tag{4.2}$$

for all compact sets $K \subset \mathbb{R}^N$.

Moreover, the least possible values of A_1 and A_2 are the same. As a matter of fact one can take $A_1 = A_2$.

The following preliminary material is taken again from [1, Chapter 1]. We shall be concerned with the *Bessel kernels* G_α, which are defined for any real (or even complex) index α via the Fourier transform by the formula

$$G_\alpha := \mathcal{F}^{-1}\big((1+|\boldsymbol{\xi}|^2)^{-\frac{\alpha}{2}}\big). \tag{4.3}$$

The Bessel kernel G_α is radially decreasing convolution kernel, in particular it is real and positive. It has exponential decay at infinity and the following asymptotics at zero

$$G_\alpha(\boldsymbol{x}) \le C(\alpha, N)|\boldsymbol{x}|^{\alpha-N} \quad \text{as } |\boldsymbol{x}| \to 0, \quad \text{for } 0 < \alpha < N. \tag{4.4}$$

Due to the definition (4.3) it is easy to see that the kernels G_α form a group, namely

$$G_\alpha \star G_\beta = G_{\alpha+\beta}. \tag{4.5}$$

For the kernel G_α one can define the *Bessel potential space*

$$L^{\alpha,p}(\mathbb{R}^N) := \{\varphi = G_\alpha \star f \mid f \in L^p(\mathbb{R}^N)\},$$

with the norm $\|G_\alpha \star f\|_{L^{\alpha,p}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)}$. The fundamental theorem of A.P. Calderon [2] identifies these spaces with the Sobolev spaces.

Theorem 3 ([1, Theorem 1.2.3])**.** *For $\alpha \in \mathbb{N}$, $W^{\alpha,p}(\mathbb{R}^N) = L^{\alpha,p}(\mathbb{R}^N)$, $1 < p < \infty$, with equivalence of norms. In particular, for all $\varphi \in W^{\alpha,p}(\mathbb{R}^N)$ there exists a unique $f \in L^p(\mathbb{R}^N)$ such that $\varphi = G_\alpha \star f$, and there is a constant A such that*

$$A^{-1}\|\varphi\|_{L^{\alpha,p}(\mathbb{R}^N)} \le \|\varphi\|_{W^{\alpha,p}(\mathbb{R}^N)} \le A\|\varphi\|_{L^{\alpha,p}(\mathbb{R}^N)}.$$

Due to Theorem 3, for any $u^i \in W^{1,2}(\Omega)$ there exists a unique $f \in L^2(\Omega)$ such that $E(u^i) = G_1 \star f$, where $E : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^3)$ is a continuous extension operator. Now, we are in the position to use Theorem 2 with $N = 3$, $p = q = 2$, $\mu = p_\Omega \, \mathrm{d}\boldsymbol{x}$, $g = G_1$ and f. First we apply Fubini's theorem to check the condition (b) of Theorem 2

$$\|G_1 \star p_{\Omega\cap K}\|_2^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_1(\boldsymbol{y}-\boldsymbol{x})\, p_{\Omega\cap K}(\boldsymbol{y})\; G_1(\boldsymbol{z}-\boldsymbol{x})\, p_{\Omega\cap K}(\boldsymbol{z}) \;\mathrm{d}\boldsymbol{y}\;\mathrm{d}\boldsymbol{z}\;\mathrm{d}\boldsymbol{x} \tag{4.6}$$

$$= \int_{\mathbb{R}^3} \big((G_1 \star G_1) \star p_{\Omega\cap K}\big)(\boldsymbol{z})\, p_{\Omega\cap K}(\boldsymbol{z}) \;\mathrm{d}\boldsymbol{z} \tag{4.7}$$

$$\le \|G_2 \star p_\Omega\|_\infty \, p_\Omega(K) \le C\|\Delta^{-1}p_\Omega\|_\infty \, p_\Omega(K) \le A_2^2\, p_\Omega(K), \tag{4.8}$$

where on the last line we have used (4.5), (4.4), (3.3), and we have put

$$A_2^2 = C(1 + \|\varrho \boldsymbol{u}^2\|_1 + \|p\|_1 + \|\boldsymbol{u}\|_{1,2}). \tag{4.9}$$

Finally, using the statement (a) of Theorem 2 and Theorem 3 we conclude that

$$\begin{aligned} \|p\boldsymbol{u}^2\|_{L^1(\Omega)} &= \sum_{i=1}^{3} \int_{\mathbb{R}}^{3} E(u^i)^2\, p_\Omega \,\mathrm{d}\boldsymbol{x} \\ &\le \sum_{i=1}^{3} A_1^2 \|E(u^i)\|^2_{L^{1,2}(\mathbb{R}^3)} \le C\, A_2^2 \|\boldsymbol{u}\|^2_{W^{1,2}(\Omega)} \,. \end{aligned} \tag{4.10}$$

5. Bootstrapping argument

There are two standard estimates for the renormalized bounded energy weak solutions we have not yet exploited. First, if we use the energy inequality (2.7), Korn's inequality, the Young inequality, and the Sobolev imbeddings we arrive at the estimate

$$\|\boldsymbol{u}\|_{1,2} \le C(\Omega) \|\boldsymbol{f}\|_\infty \|\varrho\|_{\frac{6}{5}}. \tag{5.1}$$

Second, we introduce the so called *Bogovskii operator*, which is a particular solving operator

$$\mathcal{B}:\ \varphi \in L^q(\Omega) \to \boldsymbol{v} \in W^{1,q}(\Omega;\mathbb{R}^3), \quad 1 < q < \infty \tag{5.2}$$

of the problem

$$\left\{ \begin{array}{ll} \operatorname{div} \boldsymbol{v} = \varphi - \int_\Omega \varphi \,\mathrm{d}\boldsymbol{x} & \text{in } (-\pi,\pi)^3, \\ \boldsymbol{v} = 0 & \text{on } \partial(-\pi,\pi)^3. \end{array} \right\} \tag{5.3}$$

The operator $\mathcal{B}$ is continuous, namely $\|v\|_{1,q} \le C\|\varphi\|_q$. For details see [15, Section 3] and references quoted there. In view of Remark 1 we can test (1.8) by the function $\mathcal{B}[\varphi]$, where $\varphi \in L^{q'}(\Omega)$, $1 < q \le 2$, to get

$$\begin{aligned} \int_\Omega p \operatorname{div}(\mathcal{B}[\varphi]) \,\mathrm{d}\boldsymbol{x} &= \int_\Omega (\mathbb{S} - \varrho \boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\mathcal{B}[\varphi] - (\varrho\boldsymbol{f} + \boldsymbol{g})\cdot\mathcal{B}[\varphi] \,\mathrm{d}\boldsymbol{x} \\ &\le C\big(\|\boldsymbol{u}\|_{1,2} + \|\varrho\boldsymbol{u}^2\|_q + \|\varrho\|_{\frac{6}{5}}\|\boldsymbol{f}\|_\infty + \|\boldsymbol{g}\|_\infty\big)\|\varphi\|_{q'} \,. \end{aligned} \tag{5.4}$$

For $\gamma q > \frac{6}{5}$, the Young inequality together with (5.1) yields

$$\|p\|_q = \sup_{\varphi \in L^{q'}(\Omega)} \|\varphi\|_{q'}^{-1} \int_\Omega p \left(\operatorname{div}\mathcal{B}[\varphi] + \int_\Omega \varphi \,\mathrm{d}\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x} \le C\big(1 + \|\varrho\boldsymbol{u}^2\|_q\big). \tag{5.5}$$

Next, we split the right-hand side,

$$\|\varrho \boldsymbol{u}^2\|_q^q = \int_\Omega (\varrho^\gamma \boldsymbol{u}^2)^b \boldsymbol{u}^c \, \mathrm{d}\boldsymbol{x}, \quad q = \gamma b,\ 2q = 2b + c,$$

and apply the Hölder inequality to get

$$\|\varrho \boldsymbol{u}^2\|_q^q \le \|\varrho^\gamma \boldsymbol{u}^2\|_1^b \|\boldsymbol{u}\|_6^c, \tag{5.6}$$

provided

$$b + \frac{c}{6} \le 1 \text{ or equivalently } q \le \frac{3\gamma}{\gamma + 2}\,. \tag{5.7}$$

With help of estimates (5.1), (5.5) we can rewrite (4.10) as

$$\|p\boldsymbol{u}^2\|_1 \le C(1 + \|\varrho \boldsymbol{u}^2\|_{1+\varepsilon}) \|\varrho\|_{\frac{6}{5}}^2. \tag{5.8}$$

Further application of the Hölder inequality together with the imbedding $L^6(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and with (5.1) yields

$$\|p\boldsymbol{u}^2\|_1 \le C(1 + \|\varrho\|_{\frac{3}{2}+\varepsilon} \|\varrho\|_{\frac{6}{5}}^2) \|\varrho\|_{\frac{6}{5}}^2. \tag{5.9}$$

In (5.8), (5.9), ε can be chosen arbitrary from the interval $(0, \varepsilon_0)$ where ε_0 is sufficiently small and C depends on ε_0 but is independent of ε. Taking into account (5.5), (5.6), and (5.9) we arrive at

$$\|p\|_q^q = \|\varrho^\gamma + \delta\varrho^\beta\|_q^q \le C(1 + \|\varrho\|_{\frac{3}{2}+\varepsilon}^b \|\varrho\|_{\frac{6}{5}}^{2b+2q}). \tag{5.10}$$

In the next step we shall interpolate the norms at the right-hand side of (5.10) between $L^1(\Omega)$ and $L^{\gamma q}(\Omega)$ as follows

$$\|\varrho\|_r \le \|\varrho\|_1^x \|\varrho\|_{\gamma q}^y = C\|\varrho\|_{\gamma q}^y, \ y = \frac{\gamma q}{(\gamma q - 1)} \frac{(r-1)}{r}\,. \tag{5.11}$$

Applying (5.11) to (5.10) with r successively equal to $\frac{3}{2} + \varepsilon$ and $\frac{6}{5}$, under the necessary conditions $\gamma q > \frac{3}{2}$ and $\gamma q \ge \frac{6}{5}$ respectively, and noticing that $\|\varrho\|_1 = m$, we get

$$\|\varrho^\gamma + \delta\varrho^\beta\|_q^q \le C(1 + \|\varrho\|_{\gamma q}^{z + \hat{C}\varepsilon}), \quad z = \frac{\gamma q}{\gamma q - 1}\left(\frac{b}{3} + \frac{2b + 2q}{6}\right), \ \hat{C} \le \frac{\gamma q}{\gamma q - 1}\frac{2}{3} b. \tag{5.12}$$

This formula yields

$$\|\varrho^\gamma + \delta\varrho^\beta\|_q \le C(\Omega, m, \boldsymbol{f}, \boldsymbol{g})$$

provided

$$\gamma q > z = \frac{\gamma q}{\gamma q - 1}\,\frac{\gamma + 2}{3\gamma}\,q. \tag{5.13}$$

The expression $\frac{\gamma q}{\gamma q-1}$ is a decreasing function of q, consequently (5.13) can be understood as an inequality to determine the lower bound for q. Thus, in accordance with (5.7), $q = \frac{3\gamma}{\gamma+2}$ represents the optimal choice of q. Then (5.13) reduces to $\gamma q > 2$ or equivalently $3\gamma^2 - 2\gamma - 4 > 0$. The latter inequality leads directly to the condition $\gamma > \gamma_{\text{gen.}}$ (2.11).

If the volume force $\boldsymbol{f}$ is potential, the term $\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}$ on the right-hand side of (2.7) is zero thanks to (1.7). Thus we obtain, instead of (5.1), a priori bound for $\|\boldsymbol{u}\|_{1,2}$. Consequently (5.9) takes the form

$$\|p\boldsymbol{u}^2\|_1 \le C(1 + \|\varrho\boldsymbol{u}^2\|_{1+\varepsilon}) \tag{5.14}$$

and interpolation (5.6) yields

$$\|\varrho\boldsymbol{u}^2\|_q^q \le C\|\varrho^\gamma\boldsymbol{u}^2\|_1^b\|\boldsymbol{u}\|_6^c \le C(1 + \|\varrho\boldsymbol{u}^2\|_{1+\varepsilon}^b). \tag{5.15}$$

As $b < q$, we get estimate for $\|\varrho\boldsymbol{u}^2\|_q^q$. Using (5.5), we arrive at

$$\|\varrho^\gamma + \delta\varrho^\beta\|_q^q \le C(1 + \|\varrho\boldsymbol{u}^2\|_q^q) \le C(\Omega, m, \boldsymbol{f}, \boldsymbol{g}) \tag{5.16}$$

for every $1 < q \le \frac{3\gamma}{\gamma+2}$ and for all $\gamma > 1$.

Summarizing all estimates, we have

$$\begin{array}{rlrl} \delta^{1/\beta}\varrho_\delta & \text{bounded in } L^{\beta q}(\Omega), & \varrho_\delta & \text{bounded in } L^{\gamma q}(\Omega), \\ \varrho_\delta\boldsymbol{u}_\delta^2 & \text{bounded in } L^q(\Omega), & \boldsymbol{u}_\delta & \text{bounded in } W^{1,2}(\Omega;\mathbb{R}^3) \end{array} \tag{5.17}$$

uniformly with respect to δ, provided $\gamma > \gamma_{\text{gen.}}$, or provided $\gamma > 1$ and $\boldsymbol{f}$ is potential. To prove strong convergence of the density, we shall also need the estimate

$$\|\varrho_\delta\boldsymbol{u}_\delta\|_r \le \|\varrho_\delta\|_{\gamma q}^{\frac{1}{2}}\|\varrho_\delta\boldsymbol{u}_\delta^2\|_q^{\frac{1}{2}} \le C \quad \text{with some } r > \frac{6}{5}\,. \tag{5.18}$$

This is true provided $\frac{5}{6} > \frac{1}{2q}(1 + \frac{1}{\gamma})$ which is equivalent to condition (2.12).

6. Existence of a solution

The first part of this section is devoted to the construction of the bounded energy weak solutions to problem (1.7)–(1.8) by using several level approximation scheme. We also explain (referring to the second part) how to pass to the limit between the levels. In the second part we combine the estimates of Section 5 with the compactness properties of the effective viscous flux and with the convenient estimates of oscillations to the density sequence to carry out the last limit process $\delta \to 0+$.

6.1 Approximations.

In this section we explain how to construct the renormalised bounded energy weak solutions to problem (1.7)–(1.8) on the periodic cell (2.1). We adopt the same chain of approximations as described in Chapter 4 of [15], where a similar problem is treated for larger values of the adiabatic constant and the homogeneous Dirichlet boundary conditions for the velocity. The problem of density estimates for the small adiabatic constants was already treated in Section 5. Due to this fact, we shall concentrate in this part essentially to the changes which are necessary to be operated in order to accommodate the periodic boundary conditions and the symmetries (2.2), (2.3).

To this end, we consider an approximating problem with positive parameters α, ε, and δ:

$$\alpha(\varrho - h) + \operatorname{div}(\varrho \boldsymbol{u}) - \varepsilon \Delta \varrho = 0, \tag{6.1}$$

$$\alpha(h + \varrho)\boldsymbol{u} + \frac{1}{2}\big(\operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \varrho \boldsymbol{u} \nabla \boldsymbol{u}\big) + \nabla(\varrho^{\gamma} + \delta \varrho^{\beta}) - \operatorname{div} \mathbb{S} = \varrho \boldsymbol{f} + \boldsymbol{g}, \tag{6.2}$$

on the periodic cell Ω. Here h is a smooth periodic function with the symmetry (2.3) satisfying $\int_{\Omega} h = m$. Further, ρ and $\boldsymbol{u}$ are unknowns that have to obey symmetries (2.2) and (2.3), respectively. Notice that in this case $\boldsymbol{u} \cdot n$ and $\partial_n \varrho$ necessarily vanish on $\partial(-\pi, \pi)^3$. In order to solve this system we employ the Leray-Schauder fixed point theorem.

Theorem 4 (see [15, Section 1.4.11.8]). *Let X be a Banach space and $D \subset X$ bounded open set. Let $H : \overline{D} \times [0,1] \to X$ be a* homotopy of compact transformations, *which means that H is a compact mapping for every $t \in [0,1]$ and that it is uniformly continuous in t on any bounded set $B \subset \overline{D}$. Let*

$$\omega - H(\omega, t) \neq 0, \quad \forall\, t \in [0,1], \quad \forall\, \omega \in \partial D. \tag{6.3}$$

If there exists $\omega_0 \in D$ such that $H(\omega_0, 0) = \omega_0$, then, for any $t \in [0,1]$, there exists $\omega_t \in D$, satisfying $H(\omega_t, t) = u_t$ as well.

We take $\boldsymbol{v} \in W^{1,\infty}_{\mathrm{sym}}(\Omega; \mathbb{R}^3)$ such that $\|\boldsymbol{v}\|_{1,\infty} \leq K$ for some $K > 0$. Using the standard theory of elliptic operators, see e.g. Nečas [11], we can construct solving operators

$$\Pi_t : \xi \in W^{1,p}_{\mathrm{sym}}(\Omega) \cap \{\textstyle\int_\Omega \xi = m\} \to \varrho_t \in \big(W^{2,p}_{\mathrm{sym}}(\Omega) \cap \{\textstyle\int_\Omega \varrho = m\}\big)$$

to the problems

$$(6.4) \qquad -\varepsilon\Delta\varrho_t = -t\big(\alpha(\xi - h) + \operatorname{div}(\xi \boldsymbol{v})\big) \ \text{ in } \Omega, \qquad \int_\Omega \varrho_t \,\mathrm{d}\boldsymbol{x} = m, \quad t \in [0,1],$$

which, for any $1 < p < \infty$, form a homotopy of compact transformations by virtue of the compact imbedding $W^{2,p}_{\mathrm{sym}}(\Omega) \hookrightarrow\hookrightarrow W^{1,p}_{\mathrm{sym}}(\Omega)$. Testing

$$(6.5) \qquad \alpha(\varrho - h) + \operatorname{div}(\varrho\boldsymbol{v}) - \varepsilon\Delta\varrho = 0$$

(compare with (6.1)) by ϱ and using conveniently a bootstrapping argument we realize that any fixed point $\varrho_t \in W^{1,p}_{\mathrm{sym}}(\Omega) \cap \{\int_\Omega \varrho = m\}$ of Π_t satisfies

$$(6.6) \qquad \big\|\varrho_t\big\|_{1,p} \le C_S(K, p, \varepsilon, \alpha, h),$$

where C_S is a positive constant independent of t. As a consequence the domain

$$D = \{\xi \in W^{1,p}_{\mathrm{sym}}(\Omega) \mid \big\|\xi\big\|_{1,p} \le 2C_S,\ \int_\Omega \varrho = m\}$$

verifies (6.3) with the homotopy $H(\,\cdot\,,t) = \Pi_t(\,\cdot\,)$. We can therefore employ Theorem 4, taking $X = W^{1,p}_{\mathrm{sym}}(\Omega) \cap \{\int_\Omega \varrho = m\}$, to construct the operator S

$$(6.7) \qquad S : \boldsymbol{v} \in W^{1,\infty}_{\mathrm{sym}}(\Omega;\mathbb{R}^3) \to (\varrho = \Pi_1(\varrho)) \in W^{1,p}_{\mathrm{sym}}(\Omega)$$

such that $\varrho = S(\boldsymbol{v})$ solves equation (6.1).

Similarly we define operators $T_t : \boldsymbol{v} \to \boldsymbol{u}_t$, $t \in [0,1]$ as the solving operators to the problems

$$(6.8) \qquad -\mu\Delta\boldsymbol{u} - (\mu + \lambda)\nabla \operatorname{div} \boldsymbol{u} = -t\boldsymbol{F}(S(\boldsymbol{v}), \boldsymbol{v}),$$

on the periodic cell Ω, where

$$(6.9) \quad \boldsymbol{F}(\varrho, \boldsymbol{v}) := \alpha(h + \varrho)\boldsymbol{v} + \frac{1}{2}\operatorname{div}(\varrho\boldsymbol{v} \otimes \boldsymbol{v}) + \frac{1}{2}\varrho\boldsymbol{v}\nabla\boldsymbol{v} + \nabla(\varrho^\gamma + \delta\varrho^\beta) - \varrho\boldsymbol{f} - \boldsymbol{g}.$$

The necessary condition to guarantee the existence of solutions to this system is $\int_\Omega \boldsymbol{F} = 0$. This condition is always satisfied provided $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{v}$ and ϱ, h posses symmetries (2.2) and (2.3), respectively.

Referring to the standard results of the regularity to the elliptic systems, see again [11], we conclude that

$$T_t : \boldsymbol{v} \in W^{1,\infty}_{\mathrm{sym}}(\Omega; \mathbb{R}^3) \to \boldsymbol{u}_t \in W^{2,p}_{\mathrm{sym}}(\Omega; \mathbb{R}^3) \hookrightarrow\hookrightarrow W^{1,\infty}_{\mathrm{sym}}(\Omega; \mathbb{R}^3)$$

for any $p > 3$.

We test (6.2) by $\boldsymbol{u}$, where (6.2) can be viewed as the Lamé type system (6.8) with $\boldsymbol{v} = \boldsymbol{u}$. After a long but standard calculation, employing among others (6.1), we get

$$\tag{6.10} \begin{aligned} \int_\Omega \mu|\nabla \boldsymbol{u}|^2 + (\mu+\lambda)|\operatorname{div}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} + \varepsilon\delta\big\|\nabla(\varrho^{\beta/2})\big\|^2_{0,2} \\ \le \int_\Omega (\varrho \boldsymbol{f} + \boldsymbol{g})\cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \alpha C(h), \end{aligned}$$

where $C(h)$ is a positive constant dependent on h. Taking advantage of the symmetries of $\boldsymbol{u}$ and of the fact that $\int_\Omega(\varrho - h) = 0$, one can use the Sobolev and Poincaré type inequalities as well as a bootstrapping via $\boldsymbol{F}(S(\boldsymbol{u}), \boldsymbol{u})$ and the elliptic regularity of (6.8) to conclude that

$$\tag{6.11} \big\|\boldsymbol{u}\big\|_{2,6} + \big\|\varrho\big\|_{0,3\beta} \le C_T(\alpha, \delta, \varepsilon, \boldsymbol{f}, \boldsymbol{g}, h).$$

Now we shall take $K = 2C_T$ in the definition of C_S (see (6.6)) in order to have the operator S well defined.

The domain $D = \{\boldsymbol{v} \in W^{1,\infty}_{\mathrm{sym}}(\Omega, \mathbb{R}^3) \,|\, \big\|\boldsymbol{v}\big\|_{1,\infty} \le 2C_T\}$, verifies (6.3) with $H(\cdot, t) = T_t$. Once again, we can use Theorem 4 with $X = W^{1,\infty}_{\mathrm{sym}}(\Omega)$, to guarantee existence of a fixed point $\boldsymbol{u}_\varepsilon = T_1(\boldsymbol{u}_\varepsilon)$ and then we set $\varrho_\varepsilon = S(\boldsymbol{u}_\varepsilon)$. Evidently, the couple $(\varrho_\varepsilon, \boldsymbol{u}_\varepsilon)$ solves (6.1)–(6.2).

To pass to the limit $\varepsilon \to 0+$, we have on our disposal estimate (6.10) and another estimate

$$\big\|\varrho\big\|_{0,2\beta} \le C(\delta, \boldsymbol{f}, \boldsymbol{g}, h).$$

It can be obtained by testing the momentum equation (6.2) by the Bogovskii operator $\mathcal{B}[\varphi]$, see (5.2), (5.3), using the known bound (6.10), and applying conveniently the Sobolev imbeddings and the Hölder inequality in a way similar to (5.4). Both estimates provide uniform bounds for $\big\|\boldsymbol{u}_\varepsilon\big\|_{1,2}$ and $\big\|\varrho_\varepsilon\big\|_{0,2\beta}$ independent of ε.

These estimates are sufficient to pass to the limit in the continuity equation (6.1), the energy inequality (6.10), and in all terms of the momentum equation (6.2) except the pressure term $p_\delta(\varrho_\varepsilon)$.

To pass to the limit in $p_\delta(\varrho_\varepsilon)$, one needs to show that the weak limits $\boldsymbol{u}$ and ϱ of the sequences $\boldsymbol{u}_\varepsilon$ and ϱ_ε satisfy also the renormalized continuity equation similar to (2.4), namely

$$\tag{6.12} \begin{aligned} \alpha\varrho b'(\varrho) + \operatorname{div}(b(\varrho)\boldsymbol{u}) + (\varrho b'(\varrho) - b(\varrho))\operatorname{div}\boldsymbol{u} \\ = \alpha h b'(\varrho) + \varepsilon\operatorname{div}(b'(\varrho)\nabla\varrho) - \varepsilon b''(\varrho)|\nabla\varrho|^2 \end{aligned}$$

with a convenient function $b \in C^2(0,\infty)$. This equation can be obtained via multiplying equation (6.5) by $b'(\varrho)$. Further, one needs to prove that the quantity

$$P_\delta(\varrho) = p_\delta(\varrho) - (2\mu+\lambda)\operatorname{div}\boldsymbol{u}, \tag{6.13}$$

called *effective viscous pressure*, satisfies the identity

$$\overline{P_\delta(\varrho)b(\varrho)} - \overline{P_\delta(\varrho)}\;\overline{b(\varrho)} = (2\mu+\lambda)\Big(\overline{b(\varrho)\operatorname{div}\boldsymbol{u}} - \overline{b(\varrho)}\;\operatorname{div}\boldsymbol{u}\Big) \tag{6.14}$$

with another convenient function b. Here and in what follows the overlined quantities denote corresponding weak limits in $\mathcal{D}'(\Omega)$.

The same holds for the passage $\alpha \to 0+$, but now, (6.12) is replaced by the renormalized continuity equation (2.4).

Importance of the effective viscous pressure (6.13) and some of their properties was recovered in various contexts by several authors: Lions [10], Serre [18], Hoff [9], Novotný, Padula [14] and [13]. Finally it was successfully used in existence theory by Lions [10]. Its rigorous mathematical realization is deeply related to the quality of density estimates and therefore to the value of γ (resp. β, in the case of limits $\varepsilon \to 0+$ and $\alpha \to 0+$). In fact, the difficulty of the underlying mathematical analysis increases with decreasing values of adiabatic constant. Intimately related to the DiPerna-Lions transport theory and to the Friedrich's lemma about commutators [3], the Lions method is applicable provided ϱ is square integrable. Thus, for general $\boldsymbol{f}$, it could be used without additional restriction as the condition $\gamma > \gamma_{\text{gen.}}$ is equivalent to $\gamma q > 2$ (cf. discussion after (5.13)). To treat also the case of potential $\boldsymbol{f}$ we shall rather apply another method proposed by Feireisl [5] (see also [7]) which is better adapted to investigate small adiabatic constants. We shall describe all details of this approach in the next section.

To conclude, both previous limit procedures, namely $\varepsilon \to 0+$ and $\alpha \to 0+$ have common features with the limit passage $\delta \to 0+$. The latter (most difficult) limit contains all of essential mathematical aspects of limits $\varepsilon \to 0+$, $\alpha \to 0+$. Consequently, the reader can, by himself, adapt the arguments of Section 6.2 to these situations.

6.2 Vanishing artificial pressure.

Let $\varrho_\delta \in L^\beta_{\text{sym}}(\Omega)$, $\boldsymbol{u}_\delta \in W^{1,2}_{\text{sym}}(\Omega;\mathbb{R}^3)$ be sequences of bounded energy renormalized weak solutions to the problem

$$\operatorname{div}\big(b(\varrho_\delta)\boldsymbol{u}_\delta\big) + \big(\varrho_\delta b'(\varrho_\delta - b(\varrho_\delta)\big)\operatorname{div}\boldsymbol{u}_\delta = 0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{6.15}$$

$$\begin{aligned}&\operatorname{div}(\varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta) - \mu\Delta\boldsymbol{u}_\delta - (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u}_\delta\\ &\quad+\nabla(\varrho_\delta^\gamma + \delta\varrho_\delta^\beta) = \varrho_\delta\boldsymbol{f} + \boldsymbol{g} \qquad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^3),\end{aligned} \tag{6.16}$$

$$\begin{aligned}&\int_\Omega \mu|\nabla\boldsymbol{u}_\delta|^2 + (\mu+\lambda)|\operatorname{div}\boldsymbol{u}_\delta|^2\,\mathrm{d}\boldsymbol{x}\\ &\le \int_\Omega(\varrho_\delta\boldsymbol{f}+\boldsymbol{g})\cdot\boldsymbol{u}_\delta\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{6.17}$$

where b is the same as in (2.4). By virtue of the estimates (5.17), (5.18), and the compact imbedding $W^{1,2}(\Omega;\mathbb{R}^3) \hookrightarrow\hookrightarrow L^p(\Omega;\mathbb{R}^3)$, $1 \le p < 6$ we obtain the following limits

$$
\left\{\begin{array}{ll}
\delta\varrho^{\beta} \to 0 & \text{in } \mathcal{D}'(\Omega), \\
\varrho_\delta \rightharpoonup \varrho & \text{weakly in } L^{q\gamma}(\Omega), \\
\boldsymbol{u}_\delta \rightharpoonup \boldsymbol{u} & \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^3), \\
\boldsymbol{u}_\delta \to \boldsymbol{u} & \text{in } L^p(\Omega;\mathbb{R}^3),\ 1 \le p < 6,
\end{array}\right. \tag{6.18}
$$

$$
\left\{\begin{array}{ll}
\varrho_\delta \boldsymbol{u}_\delta \rightharpoonup \varrho\boldsymbol{u} & \text{weakly in } L^r(\Omega),\ \text{for some } r > 6/5, \\
\varrho_\delta \boldsymbol{u}_\delta \otimes \boldsymbol{u}_\delta \rightharpoonup \varrho\boldsymbol{u}\otimes\boldsymbol{u} & \text{weakly in } L^q(\Omega),
\end{array}\right. \tag{6.19}
$$

at least for a chosen subsequence.

Using these facts and the weak lower semi-continuity of the left hand side of (6.17) we can pass to the limit in (6.15)–(6.17) and we get

$$\operatorname{div}(\varrho\boldsymbol{u}) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{6.20}$$

$$\operatorname{div}\big(\overline{b(\varrho)}\boldsymbol{u}\big) + \overline{\big(\varrho b'(\varrho - b(\varrho)\big)\operatorname{div}\boldsymbol{u}} = 0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{6.21}$$

$$\begin{aligned}\operatorname{div}(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) - \mu\Delta\boldsymbol{u}& \\ - (\mu+\lambda)\nabla\operatorname{div}\boldsymbol{u} + \nabla\overline{\varrho^\gamma} &= \varrho\boldsymbol{f} + \boldsymbol{g} \qquad \text{in } \mathcal{D}'(\Omega;\mathbb{R}^3),\end{aligned} \tag{6.22}$$

$$\int_\Omega \mu|\nabla\boldsymbol{u}|^2 + (\mu+\lambda)|\operatorname{div}\boldsymbol{u}|^2 \,\mathrm{d}\boldsymbol{x} \le \int_\Omega (\varrho\boldsymbol{f} + \boldsymbol{g})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \tag{6.23}$$

The proof will be complete provided we show the strong convergence of ϱ_δ in $L^1(\Omega)$. This will be done in several steps following [15]. In the first step we shall prove identity (6.14) with $b = T_k$, $k > 0$, where

$$
\begin{gathered}
T_k(z) = kT\Big(\frac{z}{k}\Big);\ T \in C^\infty(\mathbb{R}^+),\ \text{concave;} \\
T(z) = z \ \text{ for } \ z \le 1;\ T(z) = 2 \ \text{ for } \ z \ge 3.
\end{gathered} \tag{6.24}
$$

In the second step, we deduce from (6.14) an estimate measuring oscillations of the sequence of densities ρ_δ (see formula (6.34)). This information is used in the third step to prove that the couple $(\varrho, \boldsymbol{u})$ satisfies the renormalized continuity equation (see Lemma 7). The last fourth step consists in comparing the weak limit of the renormalized continuity equation for $(\varrho_\delta, \boldsymbol{u}_\delta)$ with the renormalized continuity equation for the weak limit $(\varrho, \boldsymbol{u})$.

Step 1: *Compactness properties of the effective viscous pressure* (6.13). To begin with, we shall briefly recall the definition of the *Riesz operator*

$$\mathcal{R}_{i,j}[v] := \mathcal{F}^{-1}\big(- \boldsymbol{\xi}_i\boldsymbol{\xi}_j\,|\boldsymbol{\xi}|^{-2}\,\mathcal{F}(v)\big) = \nabla_i\nabla_j\Delta^{-1}v, \tag{6.25}$$

where

$$\Delta^{-1}v[\boldsymbol{x}] = \mathcal{F}^{-1}\big(-|\boldsymbol{\xi}|^{-2}\mathcal{F}(v)\big) = \int_{\mathbb{R}}^{3} v(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^{-1}\,\mathrm{d}\boldsymbol{x}. \tag{6.26}$$

It is a continuous operator on $L^p(\mathbb{R}^3)$, $1<p<\infty$ and there holds

$$\mathcal{R}_{i,j} = \mathcal{R}_{j,i}, \quad \int_{\mathbb{R}^3} \mathcal{R}_{i,j}[v]w d\boldsymbol{x} = \int_{\mathbb{R}}^{3} v\mathcal{R}_{i,j}[w]d\boldsymbol{x}.$$

Next we recall the celebrated Div-Curl lemma due to Tartar [19].

Lemma 5. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. Let*

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \ \text{ weakly in } \ L^p(\Omega;\mathbb{R}^N), \quad \boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \ \text{ weakly in } \ L^q(\Omega;\mathbb{R}^N),$$

where $\frac{1}{p}+\frac{1}{q}\le 1$ and let

$$\operatorname{div}\boldsymbol{v}_n \ \text{ and } \ \operatorname{curl}\boldsymbol{w}_n \ \text{ be precompact in } \ W^{-1,s}(\mathbb{R}^3).$$

where $s>1$. Then

$$\boldsymbol{v}_n\cdot\boldsymbol{w}_n \rightharpoonup \boldsymbol{v}\cdot\boldsymbol{w} \quad \text{in } \ \mathcal{D}'(\mathbb{R}^3).$$

An useful and interesting corollary of Lemma 5 is the following commutator lemma (see [6, Corollary 6.1] or [15, Lemma 4.25]).

Lemma 6. *Let $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1$ and*

$$\begin{aligned} f_n \to f \quad &\text{weakly in } \ L^p(\mathbb{R}^3),\\ g_n \to g \quad &\text{weakly in } \ L^q(\mathbb{R}^3). \end{aligned}$$

Then

$$f_n\mathcal{R}_{i,j}(g_n) - g_n\mathcal{R}_{i,j}(f_n) \to f\mathcal{R}_{i,j}(g) - g\mathcal{R}_{i,j}(f) \quad \text{weakly in } \ L^r(\mathbb{R}^3). \tag{6.27}$$

Testing (6.16) by $\eta\boldsymbol{\varphi}_\delta = \eta\nabla\Delta^{-1}(\xi T_k(\varrho_\delta))$ with η, $\xi\in\mathcal{D}(\Omega)$ we obtain

$$\begin{aligned} &\int_\Omega \eta\xi\big(\varrho_\delta^\gamma - (2\mu+\lambda)\operatorname{div}\boldsymbol{u}_\delta\big)T_k(\varrho_\delta)\,\mathrm{d}\boldsymbol{x}\\ &= \mathrm{GoodTerms}_\delta + \int_\Omega \eta\,\underbrace{\mathcal{R}_{i,j}(\xi T_k(\varrho_\delta)\boldsymbol{u}_\delta^j)\varrho_\delta\boldsymbol{u}_\delta^i}_{\mathrm{DivCurl}_\delta}\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_\Omega \boldsymbol{u}_\delta^j\cdot\underbrace{[\xi T_k(\varrho_\delta)\mathcal{R}_{i,j}(\eta\varrho_\delta\boldsymbol{u}_\delta^i) - \eta\varrho_\delta\boldsymbol{u}_\delta^i\mathcal{R}_{i,j}(\xi T_k(\varrho_\delta))]}_{\mathrm{Commutator}_\delta}\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{6.28}$$

$$\begin{aligned}\mathrm{GoodTerms}_\delta = \int_\Omega &\big((\mu+\lambda)\operatorname{div}\boldsymbol{u}_\delta - \varrho_\delta^\gamma\big)\nabla\eta\cdot\boldsymbol{\varphi}_\delta - \delta\varrho_\delta^\beta\operatorname{div}(\eta\boldsymbol{\varphi}_\delta)\\ &+ \big(\mu\nabla\boldsymbol{u}_\delta - \varrho_\delta\boldsymbol{u}_\delta\otimes\boldsymbol{u}_\delta\big)\nabla\eta\otimes\boldsymbol{\varphi}_\delta - \mu\nabla\eta\otimes\boldsymbol{u}_\delta : \nabla\boldsymbol{\varphi}_\delta\\ &+ \mu\boldsymbol{u}_\delta\cdot\nabla\eta(\xi T_k(\varrho_\delta)) - (f\varrho_\delta + g)\eta\boldsymbol{\varphi}_\delta \,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{6.29}$$

Similarly we can test (6.22) by $\eta\boldsymbol{\varphi} = \eta\nabla\Delta^{-1}(\xi\overline{T_k(\varrho)})$ to get

$$\begin{aligned}&\int_\Omega \eta\xi\big(\overline{\varrho^\gamma} - (2\mu+\lambda)\operatorname{div}\boldsymbol{u}\big)\overline{T_k(\varrho)}\,\mathrm{d}\boldsymbol{x}\\ &= \mathrm{GoodTerms} + \int_\Omega \eta\,\underbrace{\mathcal{R}_{i,j}(\xi\overline{T_k(\varrho)}\boldsymbol{u}^j)\varrho\boldsymbol{u}^i}_{\mathrm{DivCurl}}\,\mathrm{d}\boldsymbol{x}\\ &\quad+ \int_\Omega \boldsymbol{u}^j\cdot\underbrace{[\xi\overline{T_k(\varrho)}\mathcal{R}_{i,j}(\eta\varrho\boldsymbol{u}^i) - \eta\varrho\boldsymbol{u}^i\mathcal{R}_{i,j}(\xi\overline{T_k(\varrho)})]}_{\mathrm{Commutator}}\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{6.30}$$

$$\begin{aligned}\mathrm{GoodTerms} = \int_\Omega &\big((\mu+\lambda)\operatorname{div}\boldsymbol{u} - \overline{\varrho^\gamma}\big)\nabla\eta\cdot\boldsymbol{\varphi} + \big(\mu\nabla\boldsymbol{u} - \varrho\boldsymbol{u}\otimes\boldsymbol{u}\big)\nabla\eta\otimes\boldsymbol{\varphi}\\ &- \mu\nabla\eta\otimes\boldsymbol{u} : \nabla\boldsymbol{\varphi} + \mu\boldsymbol{u}\cdot\nabla\eta(\xi\overline{T_k(\varrho)}) - (f\varrho + g)\eta\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{6.31}$$

Next we shall pass to the limit in (6.28) as $\delta\to 0+$. Realizing that $\boldsymbol{\varphi}_\delta\to\boldsymbol{\varphi}$ in any $L^p(\Omega;\mathbb{R}^3)$, $p>1$ and taking into account limits (6.18), (6.19) it is straightforward to show that (GoodTerms$_\delta$) $\to$ (GoodTerms). Furthermore, applying Lemma 5 and Lemma 6 we easily verify that (DivCurl$_\delta$) $\rightharpoonup$ (DivCurl) weakly in $\mathcal{D}'(\Omega)$ and (Commutator$_\delta$) $\rightharpoonup$ (Commutator) weakly in $L^r(\Omega)$, respectively. This is the only place where we need quite restrictive estimate (5.18).

Finally, subtracting (6.30) and the limit of (6.28) as $\delta\to 0+$, we obtain the famous identity for the effective viscous pressure, cf. (6.13), namely

$$\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)} = -(2\mu+\lambda)\big(\overline{T_k(\varrho)\operatorname{div}\boldsymbol{u}} - \overline{T_k(\varrho)}\operatorname{div}\boldsymbol{u}\big)\quad\text{a.e. in }\Omega. \tag{6.32}$$

Step 2: *Defect measure of oscillations.* Using in successive steps the elementary algebraic inequality $(a-b)^\gamma \le a^\gamma - b^\gamma$, $a\ge b\ge 0$, weak lower semi-continuity of convex functionals $\varrho\to\int_\Omega\varrho^\gamma$, $\varrho\to-\int_\Omega T_k(\varrho)$, and (6.32) we succeed to control oscillations of the density sequence ϱ_δ in the following way

$$\begin{aligned}&\limsup_{\delta\to 0+}\int_\Omega |T_k(\varrho) - T_k(\varrho_\delta)|^{\gamma+1}\,\mathrm{d}\boldsymbol{x}\\ &\le \limsup_{\delta\to 0}\int_\Omega(\varrho^\gamma - \varrho_\delta^\gamma)\big(T_k(\varrho) - T_k(\varrho_\delta)\big)\,\mathrm{d}\boldsymbol{x}\\ &\le \int_\Omega \overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)}\,\mathrm{d}\boldsymbol{x} \le C\big\|\operatorname{div}\boldsymbol{u}_\delta\big\|_2 \limsup_{\delta\to 0+}\big\|T_k(\varrho) - T_k(\varrho_\delta)\big\|_2.\end{aligned} \tag{6.33}$$

Hence, thanks to (5.17),

$$\sup_{k>0} \limsup_{\delta\to 0+} \left\| T_k(\varrho) - T_k(\varrho_\delta) \right\|_{\gamma+1} \le C. \tag{6.34}$$

Step 3: *Renormalized continuity equation.* The control of the density oscillations allows us to keep the renormalized continuity equation (2.4) valid for the limits ϱ, $\boldsymbol{u}$ even if the density is not known to be square integrable. More precisely we claim (see e.g. [15, Lemma 4.50]):

Lemma 7. *Let b belong to* (2.5), $\boldsymbol{u}_\delta \rightharpoonup \boldsymbol{u}$ *weakly in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *and* $\varrho_\delta \rightharpoonup \varrho$ *weakly in* $L^s(\Omega)$, $s > 1$ *and suppose that* (6.15), (6.21) *and* (6.34) *hold. Then* $(\varrho, \boldsymbol{u})$ *satisfies renormalized continuity equation* (2.4) *in* $\mathcal{D}'(\Omega)$.

If $s \ge 2$, Lemma 7 is a particular case of the DiPerna-Lions transport theory, which is, in this case, a direct consequence of (6.20) and the Fridrichs' lemma about commutators [3].

If $s \in (1,2)$ one may adapt to the steady situation the "nonsteady" approach of Feireisl [5] (see also [7]). Since $\overline{T_k(\varrho)}$ belongs, in particular, to $L^2(\Omega)$, one can apply the Di-Perna, Lions transport theory to (6.21) with $b = T_k$ to conclude that

$$\begin{aligned} &\operatorname{div}\left(b\big(\overline{T_k(\varrho)}\big)\boldsymbol{u}\right) + \Big(\overline{T_k(\varrho)}b'\big(\overline{T_k(\varrho)}\big) - b\big(\overline{T_k(\varrho)}\big)\Big)\operatorname{div}\boldsymbol{u} \\ &= b'\big(\overline{T_k(\varrho)}\big)\overline{\big(\varrho T_k(\varrho) - T_k(\varrho)\big)\operatorname{div}\boldsymbol{u}}, \end{aligned} \tag{6.35}$$

e.g. for any $b \in C^1([0,\infty)) \cap C_0([0,\infty))$. As the consequence of the weak lower semi-continuity of norms we get

$$\left\|\overline{T_k(\varrho)} - \varrho\right\|_1 \le C k^{1-p}, \quad \left\|T_k(\varrho) - \varrho\right\| \le C k^{1-p}, \qquad \text{for } \ 1 \le p < \gamma q. \tag{6.36}$$

Using this fact and (6.34) one verifies that

$$b'(\overline{T_k(\varrho)})\overline{(\varrho T_k(\varrho) - T_k(\varrho))\operatorname{div}\boldsymbol{u}} \to 0 \quad \text{in} \ \ L^1(\Omega).$$

Consequently (6.35) yields (2.4) for a compactly supported b. The passage to general b given by (2.5) can be performed via the Lebesgue dominated convergence theorem.

Step 4: *Strong convergence of* ρ_δ. Finally we use (2.4) to prove the strong convergence of ϱ_δ in $L^1(\Omega)$. We introduce functions $L_k(z) \approx z\log(z)$ by the equation $tL_k'(t) - L_k(t) = T_k(t)$. Using L_k as b in (2.4) and (6.20) leads to $\int_\Omega T_k(\varrho)\operatorname{div}\boldsymbol{u} = 0$ and $\int_\Omega \overline{T_k \operatorname{div}\boldsymbol{u}} = 0$, respectively. With this information at hand, the revisited proof of formula (6.33) yields

$$\begin{aligned} &\limsup_{\delta\to 0+} \left\| T_k(\varrho) - T_k(\varrho_\delta) \right\|_{\gamma+1}^{\gamma+1} \le C \int_\Omega \operatorname{div}\boldsymbol{u}(T_k(\varrho) - \overline{T_k(\varrho)})\,\mathrm{d}\boldsymbol{x} \\ &\le C \left\| T_k(\varrho) - \overline{T_k(\varrho)} \right\|_1^{\frac{\gamma-1}{2\gamma}} \limsup_{\delta\to 0+} \left\| T_k(\varrho) - T_k(\varrho_\delta) \right\|_{\gamma+1}^{\frac{\gamma+2}{2\gamma}}. \end{aligned} \tag{6.37}$$

Recalling (6.36), the right-hand side of (6.37) tends to zero with k. Now, we write

$$\limsup_{\delta\to 0+}\big\|\varrho_\delta-\varrho\big\|_1\le\big\|\varrho_\delta-T_k(\varrho_\delta)\big\|_1+\limsup_{\delta\to 0+}\big\|T_k(\varrho_\delta)-T_k(\varrho)\big\|_1+\big\|T_k(\varrho)-\varrho\big\|_1.$$

By virtue of (6.36) and (6.37), the right hand side of the above formula tends to zero. Consequently, the sequence ρ_δ converges strongly in $L^s(\Omega)$, for all $1\le s<\gamma q$ and $\overline{\rho^\gamma}$ in equation (6.22) is equal to ρ^γ. This completes the proof of Theorem 1.

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