Some results on $L\Sigma(\kappa)$-spaces

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Abstract. We present several results related to $L\Sigma(\kappa)$-spaces where κ is a finite cardinal or ω; we consider products and some constructions that lead from spaces of these classes to other spaces of similar classes.

Keywords: upper semicontinuous mappings, products, Lindelöf Σ-spaces

Classification: 54D20, 54C60, 54B10

All spaces in this article are assumed to be Tychonoff (= completely regular Hausdorff). We use terminology and notation as in [Eng2]. For multivalued mappings we do not require that images of points all be nonempty; if $p\colon X \to Y$ is a multivalued mapping and $A \subset X$, then $p(A)$ is defined as $\bigcup\{\, p(x) : x \in A \,\}$. The composition of two multivalued mappings $p\colon X \to Y$ and $q\colon Y \to Z$ is defined by the rule $(q \circ p)(x) = q(p(x))$. A multivalued mapping $p\colon X \to Y$ is *upper semicontinuous* if for every open set V in Y the set $\{\, x \in X : p(x) \subset V \,\}$ is open in X, or, equivalently, if for every point x in X and every neighborhood V of $p(x)$ in Y there is a neighborhood U of x in X such that $p(U) \subset V$.

It is well-known that the composition of compact-valued upper semicontinuous mappings is compact-valued upper semicontinuous. In fact, it is easy to prove that a mapping is compact-valued upper semicontinuous iff it is the composition of a continuous single-valued function, the inverse of a perfect mapping and the inverse of a closed embedding (see, e.g., [KOS]).

The symbol $\mathfrak{c}$ denotes the cardinality of the continuum. If κ is an infinite cardinal, $A(\kappa)$ denotes the one-point compactification of a discrete space of cardinality κ. The symbol I stands for the closed interval $[0,1]$.

Let $\mathcal{K}$ be a cover of a space X. A family $\mathcal{N}$ of subsets of X is called a *network modulo* $\mathcal{K}$ if for every element K of $\mathcal{K}$ and a neighborhood U of K, there is an element N of $\mathcal{N}$ such that $K \subset N \subset U$ [Nag].

The second author acknowledges support from CONACyT research project 61161/2006.

Given a cardinal κ, finite or infinite, a space X is called an *$L\Sigma(<\kappa)$-space* [KOS] if it satisfies one of the following equivalent conditions:

There is a second-countable space M and a compact-valued upper semicontinuous mapping $p\colon M \to X$ such that $p(M) = X$ and $w(p(z)) < \kappa$ for each $z \in M$;

or

There are a compact cover $\mathcal{K}$ of X such that $w(K) < \kappa$ for every $K \in \mathcal{K}$ and a countable network modulo $\mathcal{K}$ in X.

X is an *$L\Sigma(\leq \kappa)$-space* if it is an $L\Sigma(< \kappa^+)$-space. X is an *$L\Sigma(\kappa)$-space* if it is an $L\Sigma(\leq \kappa)$-space and is not an $L\Sigma(< \kappa)$-space; this concept is especially important in the case of finite cardinals κ. Of course, for finite κ, the weights of images of points and of the elements of the compact covers in the above characterizations can be replaced by the cardinalities.

The classes of $L\Sigma(< \kappa)$-spaces are invariant with respect to closed subspaces, continuous images and countable unions. Obviously, all $L\Sigma(\kappa)$-spaces are Lindelöf Σ-spaces in the sense of [Nag]; it is easy to see that $L\Sigma(\leq 1)$-spaces are exactly the spaces of countable network weight. The class of $L\Sigma(2)$-spaces includes the Double Arrow space, one-point compactifications of uncountable discrete spaces of cardinality less or equal to the continuum, and the one-point compactifications of Ψ-like spaces (that is, the spaces of the form $\Psi(\mathcal{A})$, where $\mathcal{A}$ is an almost disjoint family on ω; see Section 2 for a detailed description). Assuming $\mathrm{MA}(\omega_1)$, all scattered compact spaces of height 3 and cardinality ω_1 are in $L\Sigma(\leq 3)$ [KOS].

If $\kappa \geq \mathfrak{c}$, then $L\Sigma(\leq \kappa)$-spaces are exactly Lindelöf Σ-spaces of network weight $\leq \kappa$.

1. Products of $L\Sigma(n)$-spaces

It is easy to see, using the fact that the product of compact-valued upper semicontinuous mappings is upper semicontinuous, that the product of an $L\Sigma(\kappa)$-space with an $L\Sigma(\lambda)$-space is an $L\Sigma(\leq \lambda\cdot\kappa)$-space. However, if λ and κ are finite, it turns out that the product may be of the "type" lower than $\lambda \cdot \kappa$. For example, the one-point compactification $A(\omega_1)$ of the discrete space of cardinality ω_1 is an $L\Sigma(2)$-space; as shown in [KOS], for every $n \in \omega$, $A(\omega_1)^n$ is $L\Sigma(n + 1)$. On the other hand, if $\omega_2 \leq \mathfrak{c}$, then the space $A(\omega_2)$ is also an $L\Sigma(2)$-space, but its square is in $L\Sigma(4)$. Thus, it may be interesting to find the exact $L\Sigma$-classes for products of some $L\Sigma(n)$-spaces. Several problems of this type were posed in [KOS] and [Oku]; here we present solutions to some of these problems.

Theorem 1.1. *Suppose $m, n \in \omega$, X is an $L\Sigma(m)$-space and Y is an $L\Sigma(n)$-space. Then $X \times Y$ is an $L\Sigma(k)$-space, where $n + m - 1 \leq k \leq mn$.*

Proof: Let $p_1\colon M_1 \to X$ and $p_2\colon M_2 \to Y$ be upper semicontinuous mappings from second countable spaces M_1 and M_2 onto X and Y such that p_1 is at most m-valued and p_2 is at most n-valued. Then the mapping $p_1 \times p_2\colon M_1 \times M_2 \to X \times Y$

(defined by the rule $(p_1 \times p_2)(m_1, m_2) = p_1(m_1) \times p_2(m_2)$) is upper semicontinuous and onto $X \times Y$. This proves that $X \times Y$ is an $L\Sigma(\leq mn)$-space; therefore, $X \times Y$ is an $L\Sigma(k)$-space for some $k \leq mn$.

To prove the second part of the inequality, suppose for contradiction that $X \times Y \in L\Sigma(k)$ and $k \leq n + m - 2$. Fix a second countable space M and an at most k-valued upper semicontinuous mapping $p\colon M \to X \times Y$ such that $p(M) = X \times Y$. Let π_X, π_Y be the projections of the product $X \times Y$; put

$$A = \{\, z \in M : |\pi_X(p(z))| \leq m - 1 \,\}.$$

Since the composition $\pi_X \circ p$ is upper semicontinuous and $X \notin L\Sigma(\leq m - 1)$, there is a point $x_0 \in X$ such that $x_0 \notin \pi_X(p(A))$, hence $(\{x_0\} \times Y) \cap p(A) = \emptyset$.

Let $B = M \setminus A$ and $q\colon B \to Y$ be the multivalued mapping defined by the rule:

$$q(z) = \pi_Y\big(p(z) \cap \big(\{x_0\} \times Y\big)\big).$$

Since $\{x_0\} \times Y$ is closed in $X \times Y$, the mapping q is upper semicontinuous, and from $p(M) = X \times Y$ and $(\{x_0\} \times Y) \cap p(A) = \emptyset$ it follows that $q(B) = Y$. For every $z \in B$, $p(z)$ has at most $n + m - 2$ points, and at least $m - 1$ of these points have their projections on X different from x_0. Hence, $q(z)$ contains at most $n - 1$ points. Thus, q is an upper semicontinuous, at most $(n - 1)$-valued mapping from the second countable space B onto the space Y, a contradiction with the assumption that Y is an $L\Sigma(n)$-space. □

Corollary 1.2. *If X is an $L\Sigma(n)$-space for some $n \in \omega$, then X^m is an $L\Sigma(k)$-space for some $k \geq mn - m + 1$.*

In particular,

Corollary 1.3. *If there is an $n \in \omega$ such that X^m is an $L\Sigma(\leq n)$-space for every $m \in \omega$, then X has a countable network.*

It was shown in [KOS] that if X^ω is an $L\Sigma(< \omega)$-space, then there is an $n \in \omega$ such that X^m is an $L\Sigma(\leq n)$-space for every $m \in \omega$; it was also shown that, consistently, this implies that X has a countable network. Corollary 1.3 now allows to prove this in ZFC (thus giving an answer to Question 7.4 in [KOS]):

Corollary 1.4. *If X^ω is an $L\Sigma(< \omega)$-space, then X has a countable network (and hence X^ω is in fact an $L\Sigma(\leq 1)$-space).*

Another interesting corollary of Theorem 1.1 is

Corollary 1.5. *If X is an $L\Sigma(m)$-space for some $m \in \omega$, and Y is an $L\Sigma(n)$-space for some $n \in \omega$, $n \geq 2$, then $X \times Y$ is not homeomorphic to X.*

In particular, if X is an $L\Sigma(m)$-space for some $m \in \omega$, $m \geq 2$, then all finite powers of X are pairwise non-homeomorphic.

Since the classes of $L\Sigma(\leq n)$-spaces are invariant with respect to closed subspaces and continuous images, we may further strengthen Corollary 1.5.:

Corollary 1.6. *If X is an $L\Sigma(m)$-space for some $m \in \omega$, and Y is an $L\Sigma(n)$-space for some $n \in \omega$, $n \geq 2$, then $X \times Y$ is not homeomorphic to a continuous image of any closed subspace of X.*

Corollary 1.7. *If X is an $L\Sigma(\leq n)$-space for some $n \in \omega$, and there are natural k and m such that $k < m$ and X^m is a continuous image of a closed subspace of X^k, then X has a countable network.*

For example,

Corollary 1.8. *Let X be the Double Arrow space. If $m, n \in \omega$ and $n > m$, then X^n cannot be embedded into a continuous image of X^m.*

For some individual spaces, in particular, for products of given spaces, finding the exact $L\Sigma(k)$-class where they belong appears a non-trivial task. For example, it is still not clear whether the square of the Double Arrow space is in $L\Sigma(3)$ or $L\Sigma(4)$ (Problem 1(132) in [Oku]).

The next theorem solves Problem 3(134) in [Oku].

Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω. Recall that the space $\Psi(\mathcal{A})$ is defined as the union $\omega \cup \mathcal{A}$ with the topology in which the points of ω are isolated, and basic neighborhoods of the points $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus F$ where $F \subset A$ is finite. Clearly, $\Psi(\mathcal{A})$ is a Hausdorff zero-dimensional (hence Tychonoff) locally compact space. Let $\alpha\Psi(\mathcal{A})$ be its one-point compactification. Then $\alpha\Psi(\mathcal{A})$ is an $L\Sigma(2)$-space, because it is a countable union of singletons (points of ω) and the subspace homeomorphic to $A(|\mathcal{A}|)$; the latter space is in $L\Sigma(2)$, and the class $L\Sigma(2)$ is invariant with respect to countable unions (see [KOS]). Problem 3(134) in [Oku] was whether the square of a space $\alpha\Psi(\mathcal{A})$ can be an $L\Sigma(3)$-space and whether it can be an $L\Sigma(4)$-space.

Theorem 1.9. *Let $\mathcal{A}$, $\mathcal{B}$ be uncountable almost disjoint families of infinite subsets of ω, and let $X = \alpha\Psi(\mathcal{A}) \times \alpha\Psi(\mathcal{B})$. Then*

X is an $L\Sigma(3)$-space iff both $\mathcal{A}$ and $\mathcal{B}$ have cardinality ω_1;

X is an $L\Sigma(4)$-space iff one of the families $\mathcal{A}$, $\mathcal{B}$ has cardinality greater than ω_1.

Proof: Since both $\alpha\Psi(\mathcal{A})$ and $\alpha\Psi(\mathcal{B})$ are $L\Sigma(2)$-spaces, their product is an $L\Sigma(\leq 4)$-space. By Theorem 1.1, X is not $L\Sigma(2)$, so it is either $L\Sigma(3)$ or $L\Sigma(4)$.

If one of the families $\mathcal{A}$, $\mathcal{B}$ has cardinality greater or equal to ω_2, then the one-point compactification of the corresponding Ψ-space contains a closed copy of $A(\omega_2)$ while the other contains a closed copy of $A(\omega_1)$. Hence, the product X contains a closed subspace homeomorphic to $A(\omega_2) \times A(\omega_1)$, which is not an $L\Sigma(\leq 3)$-space by (the remark after the proof of) Theorem 4.7 in [KOS]. Since the class of $L\Sigma(\leq 3)$-spaces is hereditary with respect to closed subspaces, this proves that X cannot be an $L\Sigma(3)$-space.

On the other hand, if both $\mathcal{A}$ and $\mathcal{B}$ have cardinality ω_1, then each of them is the union of a countable space and the space $A(\omega_1)$. It follows that X is the union of a countable set, countably many copies of $A(\omega_1)$, and a copy of $A(\omega_1) \times A(\omega_1)$. Since each of these spaces is in $L\Sigma(\leq 3)$, the space X is in $L\Sigma(\leq 3)$. $\square$

Corollary 1.10. *If $\mathfrak{c} = \omega_1$, then for any uncountable almost disjoint families $\mathcal{A}$, $\mathcal{B}$ on ω, the product $\alpha\Psi(\mathcal{A}) \times \alpha\Psi(\mathcal{B})$ is an $L\Sigma(3)$-space.*

2. One-point compactifications

In [KOS], the consistently positive answer to Question 7.4 was obtained by showing that a counterexample would have to be a strong S-space and an $L\Sigma(n)$-space for some $n \in \omega$. It appears natural to ask if this kind of spaces can exist. In this section we present a construction that shows, in particular, that the answer is "yes".

Theorem 2.1. *Let X be a locally compact space. Suppose that for some $n, m \in \omega$ there exist an $L\Sigma(\leq n)$-space Y and a continuous mapping $j\colon X \to Y$ such that $j(X) = Y$ and $|j^{-1}(y)| \leq m$ for all $y \in Y$. Then the one-point compactification αX of X is an $L\Sigma(\leq nm + 1)$-space.*

Proof: If X is compact, then the mapping j is perfect, so its inverse is upper semicontinuous and at most m-valued. If $p\colon M \to Y$ is an upper semicontinuous at most n-valued mapping from a second countable space M onto Y, then the composition $j^{-1} \circ p$ is upper semicontinuous, onto X, and at most nm-valued, so $\alpha X = X$ is an $L\Sigma(\leq nm)$-space.

Thus, we may assume that X is not compact. Let ∞ be the point such that $\{\infty\} = \alpha X \setminus X$.

Let $p\colon M \to Y$ be an upper semicontinuous mapping from a second-countable space M onto Y such that $|p(z)| \leq n$ for every $z \in M$. Define a multivalued mapping $q\colon M \to X$ by putting

$$q(z) = j^{-1}(p(z)) \cup \{\infty\}.$$

Obviously, the mapping q is onto αX and is at most $(nm + 1)$-valued, so to complete the proof, it remains to verify that q is upper semicontinuous.

Let z_0 be a point of M and U an open neighborhood of $q(z_0)$ in αX; we need to find a neighborhood V of z_0 in M so that $q(V) \subset U$.

Since $\infty \in U$, the set $K = X \setminus U$ is compact. Put $W = Y \setminus j(K)$. The set W is open in Y and contains $p(z_0)$, so by the upper semicontinuity of p, there is a neighborhood V of z_0 in M such that $p(V) \subset W$. Then $q(V) = \{\infty\} \cup j^{-1}(p(V)) \subset \{\infty\} \cup j^{-1}(W) \subset U$, and the proof is complete. □

Corollary 2.2. *If X is a locally compact space, and X admits a continuous bijection onto a second-countable space, then αX is an $L\Sigma(2)$-space.*

The Kunen Line and the Todorčević line [Todor] are locally compact, admit weaker second-countable topologies, and are strong S-spaces. Since the Todorčević line is constructed assuming $\mathfrak{b} = \omega_1$, we arrive at the following.

Corollary 2.3. *Assume $\mathfrak{b} = \omega_1$. Then there exists a strong S-space which is an $L\Sigma(2)$-space.*

Arguments similar to that of the proof of Theorem 2.1 lead to the following versions:

Theorem 2.4. *Let X be a locally compact space. Suppose there exist an $L\Sigma(<\omega)$-space Y and a continuous finite-to-one mapping $j\colon X \to Y$ such that $j(X) = Y$. Then the one-point compactification αX of X is an $L\Sigma(<\omega)$-space.*

Theorem 2.5. *Let X be a locally compact space. Suppose there exist an $L\Sigma(<\omega)$-space Y and a continuous mapping $j\colon X \to Y$ such that $j(X) = Y$ and $j^{-1}(y)$ is compact and metrizable for every $y \in Y$. Then the one-point compactification αX of X is an $L\Sigma(\leq \omega)$-space.*

Recall that a mapping $j\colon X \to Y$ is called *compact-covering* if for every compact set K in Y there is a compact set F in X such that $j(F) = K$.

Theorem 2.6. *Let X be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space Y and a continuous compact-covering bijection $j\colon X \to Y$. Then the one-point compactification αX of X is an $L\Sigma(\leq \omega)$-space.*

In all three latter theorems the mapping q is defined in the same way as in the proof of Theorem 2.1, and the upper semicontinuity of q is verified by the same argument. In Theorem 2.4, q is trivially finite-valued, and in Theorem 2.5, q has compact metrizable images of points because finite unions of metrizable compacta are metrizable compacta. In Theorem 2.6, the compactness and metrizability of images of points under q are verified as follows: there is a compact subset C of X such that $p(z) \subset j(C)$; since j is a continuous bijection, the restriction of j to C is a homeomorphism. Thus, $q(z)$ is the union of the set $j^{-1}(p(z))$, homeomorphic to $p(z)$, and a singleton, hence compact metrizable.

It is not clear if it is possible to omit the requirement that j be compact-covering in Theorem 2.6. Hence,

Problem 2.7. Let X be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space Y and a continuous bijection $j\colon X \to Y$. Must the one-point compactification αX of X be an $L\Sigma(\leq \omega)$-space?

It is also not clear whether Theorem 2.6 remains true if we require that j be finite-to-one instead of being a bijection. The reason of course is that the preimage of a compact metrizable space under a perfect finite-to-one mapping need not be metrizable, so the argument as above does not work. Hence,

Problem 2.8. Let X be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space Y and a continuous finite-to-one compact-covering mapping $j\colon X \to Y$. Must the one-point compactification αX of X be an $L\Sigma(\leq \omega)$-space?

Problem 2.9. Let X be a locally compact space. Suppose there exist an $L\Sigma(\leq \omega)$-space Y and a continuous finite-to-one mapping $j\colon X \to Y$ such that $j(X) = Y$. Must the one-point compactification αX of X be an $L\Sigma(\leq \omega)$-space?

3. The Alexandroff duplicates

One of intriguing questions in the theory of $L\Sigma(\leq \omega)$-spaces is the following (Question 7.5 in [KOS]; also Problem 13(144) in [Oku]): *Let X be an $L\Sigma(\leq \omega)$-space and let $p\colon X \to Y$ be a finite-valued upper semicontinuous mapping such that $p(X) = Y$. Must Y be an $L\Sigma(\leq \omega)$-space?*

Below we prove that the answer is positive for a particular case of the Alexandroff duplicate of an $L\Sigma(\leq \omega)$-space; this gives a positive answer to Problem 15(146) in [Oku].

Recall that the *Alexandroff duplicate* $AD(X)$ of a space X is $X \times 2$ with the topology defined as follows: the points of $X \times \{1\}$ are isolated, and basic neighborhoods of the points $(x, 0)$ are of the form $(U \times 2) \setminus \{(x, 1)\}$ where U is a neighborhood of x in X (see [Eng1] for a discussion of this construction). It is easy to see that the mapping $\pi\colon AD(X) \to X$ defined by the rule $\pi((x, i)) = x$ is two-to-one and perfect, so its inverse is 2-valued upper semicontinuous.

Theorem 3.1. *If X is an $L\Sigma(\leq \omega)$-space, then so is $AD(X)$.*

Proof: Fix a second-countable space M and an upper semicontinuous compact-valued mapping $p\colon M \to X$ so that $p(M) = X$ and $w(p(z)) \leq \omega$ for every $z \in M$. Since the cardinalities of M and of $p(z)$, $z \in M$, are at most $\mathfrak{c}$, we have $|X| \leq \mathfrak{c}$, and we may fix a one-to-one function (not necessarily continuous) $j\colon X \to I = [0, 1]$. Define a multivalued mapping $q\colon M \times I \to AD(X)$ by the rule:

$$q(z, t) = \big(p(z) \times \{0\}\big) \cup \big((p(z) \cap j^{-1}(t)) \times \{1\}\big).$$

Since for every $(z, t) \in M \times I$ the set $j^{-1}(t)$ contains at most one point, the images of points under q are compact and metrizable. Let us verify that q is upper semicontinuous.

Let $(z_0, t_0) \in M \times I$, and let U be a neighborhood of $q(z_0, t_0)$; we need to find a neighborhood V of (z_0, t_0) so that $q(V) \subset U$. Since $p(z_0)$ is compact, there is a neighborhood W of $p(z_0)$ in X and a finite set $F \subset X$ such that $F \cap j^{-1}(t_0) = \emptyset$ and $U \supset (W \times 2) \setminus (F \times \{1\})$. Indeed, for every point $x \in p(z_0)$ we can fix a standard open neighborhood $(W_x \times 2) \setminus \{(x, 1)\}$ of $(x, 0)$ contained in U; choose a finite subfamily $W_{x_1}, \dots, W_{x_n}$ of the family $\{ W_x : x \in p(z_0) \}$ so that $p(z_0) \subset \bigcup_{i=1}^n W_{x_i}$, and put $W = \bigcup_{i=1}^n W_{x_i}$ and $F = \{x_1, \dots, x_n\} \setminus j^{-1}(t_0)$.

Let $S = j(F)$; then S is finite and $t_0 \notin S$. By the upper semicontinuity of p, there is an open neighborhood G of z_0 in M such that $p(G) \subset W$. Put $V = G \times (I \setminus S)$. Now if $(z, t) \in V$, then $p(z) \subset W$ and $p(z) \cap j^{-1}(t) \subset W \setminus F$, so $q(z, t) \subset (W \times 2) \setminus (F \times \{1\}) \subset U$, and V is as required.

Let us now verify that q is onto $AD(X)$. If $x \in X$, then there is $z_0 \in M$ such that $x \in p(z_0)$. Put $t_0 = j(x)$. Then both $(x, 0)$ and $(x, 1)$ are in $q(z_0, t_0)$.

Thus, there is an upper semicontinuous compact-valued mapping with metrizable images of points from a second-countable space $M \times I$ onto $AD(X)$, and the proof is complete. $\square$

Theorem 3.1 gives the positive answer to Problem 15(146) in [Oku].

A space X is called a $KL\Sigma(\leq \omega)$-*space* if there is a *compact* second-countable space M and a compact-valued upper semicontinuous mapping $p\colon M \to X$ such that $p(M) = X$ and $w(p(z)) \leq \omega$ for all $z \in M$ [KOS]. It is observed in [KOS] that a compact $L\Sigma(\leq \omega)$-space need not be a $KL\Sigma(\leq \omega)$-space. The same argument as in the proof of Theorem 3.1 can be used to prove the following:

Theorem 3.2. *If X is a $KL\Sigma(\leq \omega)$-space, then so is $AD(X)$.*

Of course, the same argument works for the next statement:

Theorem 3.3. *Let κ be an infinite cardinal. If $|X| \leq \mathfrak{c}$ and X is an $L\Sigma(\leq \kappa)$-space ($KL\Sigma(\leq \kappa)$-space), then so is $AD(X)$.*

The condition "$|X| \leq \mathfrak{c}$" in Theorem 3.3 cannot be omitted unless $2^\kappa \leq \mathfrak{c}$. Indeed, if $2^\kappa > \mathfrak{c}$, let $X = 2^\kappa$ (with the product topology). Trivially, $X \in KL\Sigma(\leq \kappa)$. On the other hand, every $L\Sigma(\leq \kappa)$-space is a union of at most $\mathfrak{c}$ subspaces of weight at most κ, so its network weight is at most $\kappa \cdot \mathfrak{c}$. The network weight of $AD(2^\kappa)$ is 2^κ, so it cannot be an $L\Sigma(\leq \kappa)$-space.

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(*Received* February 8, 2008, *revised* July 17, 2008)