Comment.Math.Univ.Carolin. 49,4 (2008)533–539

Pseudocomplemented directoids

Ivan Chajda

Abstract. Directoids as a generalization of semilattices were introduced by J. Ježek and R. Quackenbush in 1990. We modify the concept of a pseudocomplement for commutative directoids and study several basic properties: the Glivenko equivalence, the set of the so-called boolean elements and an axiomatization of these algebras.

Keywords: commutative directoid, λ-lattice, pseudocomplement, boolean elements

Classification: 06A12, 06D15, 06C15, 06A99

The concept of a pseudocomplemented semilattice was introduced by O. Frink [2], see also [1] for reader's convenience. Recall that $\mathcal{S} = (S; \wedge, ^*, 0)$ is a pseudocomplemented semilattice if $(S; \wedge)$ is a meet-semilattice with the least element 0 and for each $a \in S$, a^* is the greatest element of S such that $a \wedge a^* = 0$; a^* is called the pseudocomplement of a.

The concept of a semilattice was generalized by J. Ježek and R. Quackenbush [3] as follows.

Let $(D; \leq)$ be an ordered set which is downward directed, i.e. for any $a, b \in D$ the set $L(a, b) = \{x \in D; x \leq a \text{ and } x \leq b\}$ of all common lower bounds is non-void. For $a, b \in D$ we choose an arbitrary element $c \in L(a, b)$ and define $a \sqcap b = b \sqcap a = c$ if a, b are non-comparable and $a \sqcap b = b \sqcap a = a$ if $a \leq b$. The groupoid $(D; \sqcap)$ is called a *commutative directoid*. It was shown in [3] (see also [1]) that commutative directoids are axiomatized by three simple axioms

(D1) $x \sqcap x = x$;
(D2) $x \sqcap y = y \sqcap x$;
(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$.

Of course, $x \leq y$ if and only if $x \sqcap y = x$.

If an ordered set $(D; \leq)$ is upward directed, i.e. $U(a, b) = \{x \in D; a \leq x \text{ and } b \leq x\} \neq \emptyset$ for each a, b, we can define dually $a \sqcup b = b \sqcup a \in U(a, b)$ arbitrarily if a, b are non-comparable and $a \sqcup b = b \sqcup a = b$ if $a \leq b$. The resulting groupoid $(D; \sqcup)$ is again a commutative directoid, (D1)–(D3) are satisfied when $\sqcap$ is replaced by $\sqcup$. In this case, $x \leq y$ if and only if $x \sqcup y = y$. Let us recall the concept of a λ-lattice

This work is supported by the Research Project MSM 6198952914 by Czech Government.

introduced by V. Snášel [5]. An algebra $\mathcal{L} = (L; \sqcup, \sqcap)$ of type $(2,2)$ is called a *λ-lattice* if $(L; \sqcup)$ and $(L; \sqcap)$ are commutative directoids and the absorption laws

$$x \sqcap (x \sqcup y) = x \quad \text{and} \quad x \sqcup (x \sqcap y) = x$$

are satisfied.

Our aim is to introduce the concept of pseudocomplementation in commutative directoids in the way that ensures basic properties similar to those of pseudocomplemented semilattices (see e.g. [1], [2], [4]). Of course, it can be done in several ways but not each of them gets "nice" results. Hence, one choice is as follows.

Let $\mathcal{A} = (A; \sqcap, 0)$ be a commutative directoid with the least element 0. Let $a \in A$. An element $b \in A$ is called the *pseudocomplement* of a if it is the greatest element with the property $a \wedge b = 0$ (where $a \wedge b$ means $\inf(a, b)$ and the notation $a \wedge b = 0$ means that $a \wedge b$ exists and is equal to 0). The pseudocomplement of a will be denoted by a^*. If there exists the pseudocomplement a^* for each $a \in A$, $\mathcal{A}$ will be called a *pseudocomplemented directoid* and will be denoted by $\mathcal{A} = (A; \sqcap, {}^*, 0)$.

Since $a \sqcap b$ is a common lower bound of a, b, we have $a \sqcap b \leq a \wedge b$ whenever the infimum $a \wedge b$ exists. Hence, we obtain immediately the following.

Lemma 1. *If the pseudocomplement a^* of $a \in A$ exists then $a \sqcap a^* = 0$. If $b \leq a^*$ then $a \sqcap b = 0$.*

Let us note that the converse of the second assertion of Lemma 1 does not hold. If a^* exists and $a \sqcap b = 0$ then b need not be bellow a^*. Moreover, the condition $a \wedge a^* = 0$ in the definition of pseudocomplement cannot be replaced by $a \sqcap a^* = 0$ to obtain this assertion as one can recognize from the following.

Example 1. Let $A = \{0, p, a, b, 1\}$ and the directoid $\mathcal{A} = (A; \sqcap)$ is visualized in Figure 1, where $a \sqcap b = 0$.

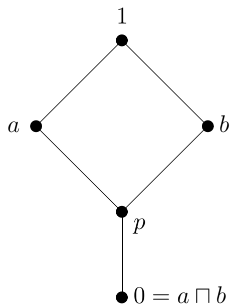

FIGURE 1

Then b is the greatest element with $a \sqcap b = 0$ but $p \leq b$ and $a \sqcap p = p \neq 0$.

Pseudocomplements in directoids satisfy several properties which are well-known in pseudocomplemented semilattices, see e.g. [2].

Lemma 2. *Let* $\mathcal{A} = (A; \sqcap, ^*, 0)$ *be a pseudocomplemented directoid,* $a, b \in A$. *Then*

(1) $a \leq a^{**}$;
(2) $a \leq b$ *implies* $b^* \leq a^*$;
(3) $a \leq b$ *implies* $a^{**} \leq b^{**}$;
(4) $a^{***} = a^*$;
(5) 0^* *is the greatest element of* $\mathcal{A}$; *denote* $0^* = 1$;
(6) $1^* = 0$;
(7) $0^{**} = 0$;
(8) *if* $a = b^*$ *then* $a = a^{**}$;
(9) $a \sqcap (a^* \sqcap b)^* = a$.

Proof: (1) Since $a \wedge a^* = 0$ and a^{**} is the greatest element of $\mathcal{A}$ with $a^{**} \wedge a^* = 0$, we obtain $a \leq a^{**}$.

(2) Since $b \wedge b^* = 0$ and $a \leq b$ then also $a \wedge b^*$ exists and $a \wedge b^* \leq b \wedge b^* = 0$, i.e. $a \wedge b^* = 0$. Hence, $b^* \leq a^*$.

(3) Follows immediately by (2).

(4) By (1) we have $a \leq a^{**}$ and $a^* \leq a^{***}$. By (2) we obtain from $a \leq a^{**}$ also $a^* \geq a^{***}$ thus $a^{***} = a^*$.

(5) Since $0 \wedge x = 0$ for each $x \in A$, we have $x \leq 0^*$ thus 0^* is the greatest element of $\mathcal{A}$, i.e. $0^* = 1$.

(6) Since $1 \wedge x = x$ for each $x \in A$, we have $1^* = 0$.

(7) 0^{**} is the pseudocomplement of $0^* = 1$, i.e. $0^{**} = 1^* = 0$.

(8) If $a = b^*$ then $a^{**} = b^{***} = b^* = a$.

(9) Since $a^* \sqcap b \leq a^*$, we apply (2) and (1) to obtain $(a^* \sqcap b)^* \geq a^{**} \geq a$ whence $a \sqcap (a^* \sqcap b)^* = a$. □

For a pseudocomplemented directoid $\mathcal{A} = (A; \sqcap, ^*, 0)$ we denote by $B(A) = \{x^*; x \in A\}$ the set of *boolean elements*. Due to (8) of Lemma 2, $B(A) = \{x \in A; x = x^{**}\}$. Denote by $D(A) = \{x \in A; x^* = 0\}$, the set of *dense elements* of $\mathcal{A}$. Let us introduce the following concept. Let $\mathcal{L} = (L; \sqcup, \sqcap)$ be a λ-lattice with the least element 0 and the greatest element 1. For $a \in L$, an element $b \in L$ is called the *complement* of a if $a \sqcap b = 0$ and $a \sqcup b = 1$. $\mathcal{L}$ is called *complemented* if every element of $\mathcal{L}$ has a complement. This enables us to reveal the structure of boolean elements in a pseudocomplemented directoid.

Theorem 1. *Let* $\mathcal{A} = (A; \sqcap, ^*, 0)$ *be a pseudocomplemented directoid. Define the binary operations* $\cap$ *and* $\sqcup$ *on* $B(A)$ *as follows*

$$x \sqcup y = (x^* \sqcap y^*)^* \text{ and } x \cap y = (x \sqcap y)^{**}.$$

Then $\mathcal{B}(A) = (B(A); \sqcup, \cap, ^*, 0, 1)$ *is a complemented* λ*-lattice (where* a^* *is a complement of* $a \in B(A)$*).*

PROOF: Assume $a, b \in B(A)$. Due to (8) of Lemma 2, $a^*, a \sqcup b$ and $a \cap b$ belong to $B(A)$. Due to (6) and (7), also $0, 1 \in B(A)$ thus $\mathcal{B}(A) = (B(A); \sqcup, \cap, ^*, 0, 1)$ is an algebra of type $(2, 2, 1, 0, 0)$. By (9) of Lemma 2, $a \sqcup b = (a^* \sqcap b^*)^*$ is a common upper bound of a and b. If $a \le b$ then $b^* \le a^*$ and $(a^* \sqcap b^*)^* = b^{**} = b$ thus $(B(A); \sqcup)$ is a commutative directoid.

Further, $a \cap b = (a \sqcap b)^{**} \le a^{**} = a$, similarly $a \cap b \le b$ thus $a \cap b$ is a common lower bound of a and b. If $a \le b$ then $a \cap b = (a \sqcap b)^{**} = a^{**} = a$ thus $(B(A); \cap)$ is also a commutative directoid. Further, $x \sqcap y \le x$ yields $(x \sqcap y)^* \ge x^*$ thus

$$x \sqcup (x \cap y) = x \sqcup (x \sqcap y)^{**} = (x^* \sqcap (x \sqcap y)^*)^* = x^{**} = x$$

and $x = x^{**} \le (x^* \sqcap y^*)^*$ yields

$$x \cap (x \sqcup y) = x \cap (x^* \sqcap y^*)^* = (x \sqcap (x^* \sqcap y^*)^*)^{**} = x^{**} = x,$$

i.e. $\mathcal{B}(A)$ satisfies the DeMorgan laws and hence it is a λ-lattice. Evidently, 0 is the least and 1 the greatest element of $B(A)$.

Finally, $a \cap a^* = (a \sqcap a^*)^{**} = 0^{**} = 0$ and $a \sqcup a^* = (a^* \sqcap a^{**})^* = (a^* \sqcap a)^* = 0^* = 1$ thus a^* is a complement of a for any $a \in B(A)$. $\square$

Remark. For a pseudocomplemented directoid $\mathcal{A} = (A; \sqcap, ^*, 0)$ and $a, b \in B(A)$, $a \sqcap b$ need not be a boolean element and $\mathcal{B}(A)$ need not be a lattice, see the following example.

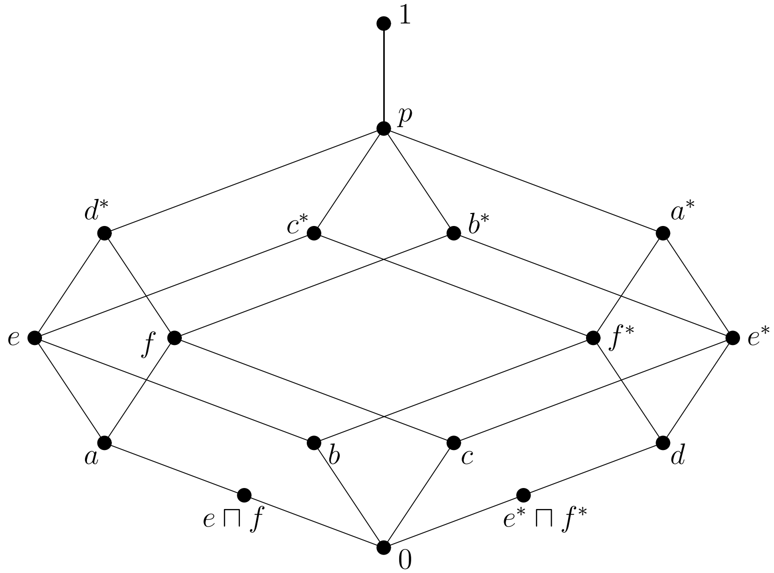

FIGURE 2

Example 2. Consider a pseudocomplemented directoid whose diagram is depicted in Figure 2, where $b^* \sqcap c^* = a$ and $x \sqcap y = x \wedge y$ for $\{x, y\} \neq \{e, f\}$ and $\{x, y\} \neq \{e^*, f^*\}$. Let x^* be the pseudocomplement of x. Then $D(A) = \{p, 1\}, B(A) = \{0, a, b, c, d, e, f, a^*, b^*, c^*, d^*, e^*, f^*, 1\}$. Clearly $e \sqcap f \notin B(A)$ and $e^* \sqcap f^* \notin B(A)$ but $e \cap f = (e \sqcap f)^{**} = a^{**} = a \in B(A)$ and $e^* \cap f^* = d^{**} = d \in B(A)$. The complemented λ-lattice $\mathcal{B}(A) = (B(A); \sqcup, \cap, ^*, 0, 1)$ is visualized in Figure 3. It is evident that $\mathcal{B}(A)$ is not a lattice since $b \vee c$ and $b^* \wedge c^*$ do not exist.

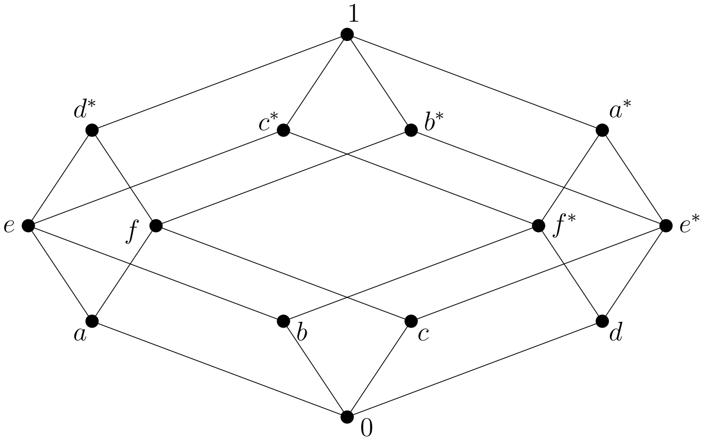

Figure 3

The converse of Theorem 1 is evident. If $\mathcal{B} = (B; \sqcup, \sqcap, ^*, 0, 1)$ is a complemented λ-lattice where a^* is a pseudocomplement of $a \in B$ then B is the set of all boolean elements of the pseudocomplemented directoid $\mathcal{D} = (B; \sqcap, ^*, 0)$.

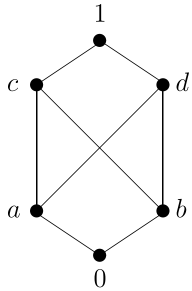

Figure 4

On the other hand, every commutative directoid $\mathcal{A} = (A; \sqcap)$ with the least element p and the greatest element 1 can be considered as the set of dense elements of a certain pseudocomplemented directoid. Namely, we can add an element 0 and define $x \sqcap 0 = 0$ for any $x \in A$ and $x^* = 0$ for $x \in A$, $0^* = 1$. Then $\mathcal{A}_0 = (A \cup \{0\}, \sqcap, ^*, 0)$ is clearly a pseudocomplemented directoid with $D(A_0) = A$. However, $D(A)$ need not have a least element in general. Consider e.g. the directoid in Figure 4 where $c \sqcap d = a$ and $x \sqcap y = x \wedge y$ otherwise. Then clearly $a^* = b$, $b^* = a$ and $c^* = d^* = 1^* = 0$, $0^* = 1$. Thus $B(A) = \{0, a, b, 1\}$ (which is even a lattice with respect to $\sqcup$ and $\cap$) and $D(A) = \{c, d, 1\}$, i.e. it has not a least element.

Lemma 3. *Let $\mathcal{A} = (A; \sqcap, ^*, 0)$ be a pseudocomplemented directoid. Define a binary relation Φ on A as follows*

$$\langle x, y\rangle \in \Phi \quad \textit{if and only if} \quad x^* = y^*.$$

Then Φ is an equivalence on A and

(a) *for each $x \in A$, the class $[x]_\Phi$ has the greatest element which is x^{**};*
(b) *for each $x \in A$, the class $[x]_\Phi$ contains a unique element of $B(A)$ which is x^{**}.*

Proof: It is straightforward that Φ is an equivalence on A. Assume $a \in [x]_\Phi$. Then $a^* = x^*$, i.e. $x^{**} = a^{**} \geq a$. Moreover, $x^{***} = x^*$ thus $x^{**} \in [x]_\Phi$, i.e. x^{**} is the greatest element of $[x]_\Phi$. Further, $x^{**} \in B(A)$. Assume $b \in B(A) \cap [x]_\Phi$. Then $b = b^{**} = x^{**}$ thus x^{**} is the unique element of $B(A)$ included in $[x]_\Phi$. $\square$

Due to the equivalent formulation as for semilattices in [2], call Φ the *Glivenko equivalence* of $\mathcal{A} = (A; \sqcap, ^*, 0)$.

The following result is an easy consequence.

Corollary. *Let $\mathcal{A} = (A; \sqcap, ^*, 0)$ be a pseudocomplemented directoid and Φ the Glivenko equivalence on $\mathcal{A}$. Define the following operations on the quotient set A/Φ:*

$[x]_\Phi \sqcap [y]_\Phi = [x \sqcap y]_\Phi$;
$[x]_\Phi \sqcup [y]_\Phi = [(x^* \sqcap y^*)^*]_\Phi$;
${[x]_\Phi}^* = [x^*]_\Phi$.

Then $[1]_\Phi = D(A)$, $[0]_\Phi = \{0\}$ and $\mathcal{A}/\Phi = (A/\Phi; \sqcup, \sqcap, ^*, D(A), \{0\})$ is a complemented λ-lattice isomorphic to $\mathcal{B}(A)$.

The afore mentioned isomorphism $h : A/\Phi \to B(A)$ is given by $h([x]_\Phi) = x^{**}$.

In the definition of a pseudocomplement in directoids, we had to use infimum, which is not an operation of directoids. Hence, we are interested if pseudocomplemented directoids can be axiomatized in the language containing only its operations.

Theorem 2. *An algebra $\mathcal{A} = (A; \sqcap, ^*, 0)$ of type $(2, 1, 0)$ is a pseudocomplemented directoid if and only if it satisfies the axioms* (D1), (D2), (D3) *and*

(P1) $x \sqcap 0 = 0$;

(P2) $x \sqcap 0^* = x$;
(P3) $x \sqcap (x^* \sqcap y) = 0$;
(P4) $x \sqcap (y \sqcap z) = 0$ *for each* $z \in A \quad \Rightarrow \quad y \sqcap x^* = y$.

Proof: Every pseudocomplemented directoid surely satisfies (D1)–(D3), (P1) and (P2). Since $x^* \sqcap y \leq x^*$, Lemma 1 yields $x \sqcap (x^* \sqcap y) = 0$ which is (P3). Since x^* is the pseudocomplement of x, $x \wedge x^*$ exists and is equal to 0 and x^* is the greatest element of this property. Assume $x \sqcap (y \sqcap z) = 0$ for each $z \in A$ and let $x \wedge y$ do not exist. Then there is $a \neq 0$ such that $a \leq x$, $a \leq y$. Hence

$$x \sqcap (y \sqcap a) = x \sqcap a = a \neq 0,$$

a contradiction. Thus $x \wedge y$ exists and is equal to 0. Hence $y \leq x^*$, i.e. $y \sqcap x^* = y$ proving (P4).

Conversely, assume (D1)–(D3), (P1) and (P2). Then $\mathcal{A}$ is a commutative directoid with the least element 0 and the greatest element 0^*. Denote $1 = 0^*$. Taking $y = 1$ in (P3) we obtain $x \sqcap x^* = 0$. Assume $y \leq x^*$. Then $x^* \sqcap y = y$ and (P3) yields $x \sqcap y = x \sqcap (x^* \sqcap y) = 0$. Hence, $x \wedge x^*$ exists and is equal to 0.

Assume now that $x \wedge y = 0$. Since $x \sqcap y \leq x \wedge y$ and $y \sqcap z \leq y$ for any $z \in A$, we obtain $x \sqcap (y \sqcap z) \leq x \wedge (y \sqcap z) \leq x \wedge y = 0$ thus $x \sqcap (y \sqcap z) = 0$. By (P4) we conclude $y \leq x^*$, i.e. x^* is really the greatest element of A with $x \wedge x^* = 0$. □

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Department of Algebra and Geometry, Palacký University Olomouc, Tomkova 40, 779 00 Olomouc, Czech Republic

E-mail: chajda@inf.upol.cz

(*Received* January 21, 2008, *revised* April 23, 2008)