Lower bounds for the colored mixed chromatic number of some classes of graphs

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Abstract. A colored mixed graph has vertices linked by both colored arcs and colored edges. The chromatic number of such a graph G is defined as the smallest order of a colored mixed graph H such that there exists a (color preserving) homomorphism from G to H. These notions were introduced by Nešetřil and Raspaud in *Colored homomorphisms of colored mixed graphs*, J. Combin. Theory Ser. B **80** (2000), no. 1, 147–155, where the exact chromatic number of colored mixed trees was given. We prove here that this chromatic number is reached by the much simpler family of colored mixed paths. By means of this result we give lower bounds for the chromatic number of colored mixed partial k-trees, outerplanar and planar graphs.

Keywords: graph colorings, graph homomorphisms, colored mixed graphs

Classification: 05C15

1. Introduction

In this paper we study homomorphisms of (n, m)-*colored mixed graphs*, which are graphs with both edges and arcs colored with n and m colors respectively. This notion was introduced by Nešetřil and Raspaud [4] as a common generalization of the notion of edge-colored graphs, obtained by taking $(n, m) = (0, m)$, and the notion of oriented colorings, which arises when $(n, m) = (1, 0)$, (see e.g. [1] and [7] respectively). Formally, an (n, m)-*colored mixed graph* G consists of a set of vertices $V(G)$ linked by arcs $A(G)$ and edges $E(G)$ (satisfying that the underlying undirected graph is simple) together with partitions $A(G) = A_1(G) \cup \cdots \cup A_n(G)$ and $E(G) = E_1(G) \cup \cdots \cup E_m(G)$ where $A_i(G)$ (resp. $E_i(G)$) consists of the set of arcs (resp. edges) colored by color i. For $n = 0$ there are no arcs, and for $m = 0$ there are no undirected edges. In particular, a $(0, 1)$-colored mixed graph is a simple graph, and a $(1, 0)$-colored mixed graph is an oriented graph.

Let G and H be two (n, m)-colored mixed graphs. A *homomorphism* from G to H is a mapping $h\colon V(G) \to V(H)$ satisfying: $(u, v) \in A_i(G)$ implies $(h(u), h(v)) \in A_i(H)$ for every $i \in \{1, \ldots n\}$, and $uv \in E_i(G)$ implies $h(u)h(v) \in E_i(H)$ for every $i \in \{1, \ldots m\}$. In other words, the homomorphisms of colored mixed graphs map edges into edges and arcs into arcs preserving the colors. The existence of a homomorphism from G to H is denoted by $G \to H$. Given a colored mixed graph G, the smallest number of vertices of a colored mixed graph H such that

$G \to H$, is called the *chromatic number* of G. For a simple graph G, the (n,m)-*mixed chromatic number*, denoted by $\chi_{(n,m)}(G)$, is defined as the maximum of the chromatic numbers taken over all the possible (n,m)-colored mixed graphs having as underlying graph G. Note that $\chi_{(0,1)}(G)$ is the ordinary chromatic number, and $\chi_{(1,0)}(G)$ is the oriented chromatic number.

Given a family $\mathcal{F}$ of simple graphs, we denote by $\chi_{(n,m)}(\mathcal{F})$ the maximum of $\chi_{(n,m)}(G)$ taken over all members in $\mathcal{F}$. The most natural question to consider in this framework is whether or not a given family of graphs has a finite (n,m)-mixed chromatic number. When the answer is affirmative, we are interested in determining or bounding this number. Nešetřil and Raspaud [4] proved that for $\mathcal{P}$, the family of planar graphs, we have:

$$\chi_{(n,m)}(\mathcal{P}) \leq 5(2n+m)^4. \tag{1}$$

This is the best known upper bound even for oriented graphs where the corresponding value is 80. This result is a consequence of a more general result dealing with the *acyclic chromatic number* (smallest number of colors needed in an *acyclic coloring*, which is a proper vertex coloring satisfying that every cycle receives at least three colors). Nešetřil and Raspaud [4] proved that for $\mathcal{A}_k$, the family of graphs with acyclic chromatic number at most k, it holds:

$$\chi_{(n,m)}(\mathcal{A}_k) \leq k(2n+m)^{k-1}. \tag{2}$$

Thus (1) follows from (2) and the well-known result of Borodin that every planar graph has acyclic chromatic number at most five [2]. Upper bounds for the (n,m)-mixed chromatic number of partial k-trees and outerplanar graphs are also given as a consequence of (2). A *k-tree* is a simple graph obtained from the complete graph K_k by repeatedly inserting new vertices linked to all vertices of an existing clique of order k. A *partial k-tree* is a subgraph of some k-tree. It is not difficult to see that every partial k-tree has acyclic chromatic number at most $(k+1)$: starting with a proper k-coloring of the complete graph K_k, every newly inserted vertex has exactly k neighbors and can be thus colored using a $(k+1)$-th color. Moreover, this coloring is clearly acyclic since all the neighbors of a newly inserted vertex have pairwise distinct colors.Therefore by (2) we get the following upper bound for the class $\mathcal{T}^k$ of partial k-trees:

$$\chi_{(n,m)}(\mathcal{T}^k) \leq (k+1)(2n+m)^k. \tag{3}$$

Since the class of outerplanar graphs $\mathcal{O}$ is strictly included in $\mathcal{T}^2$, we also get:

$$\chi_{(n,m)}(\mathcal{O}) \leq 3(2n+m)^2. \tag{4}$$

In this paper we study the tightness of (1), (2), (3) and (4). Note that the family of graphs with acyclic chromatic number at most 2, is in fact the family of forest; in this case ($k = 2$) we know the exact colored mixed chromatic number (see (5) in Section 3), and it is not difficult to see that the upper bound given in (2) is tight just in the cases of simple graphs $(n, m) = (0, 1)$, and 2-edge colored graphs $(n, m) = (0, 2)$. For $k \geq 3$ the prospect is very different. The techniques used to prove Theorem 1 suggested that, maybe in this case, the bound is not tight. However, recently Ochem [5] surprisingly proved that the bound in Theorem 1 is tight for every $k \geq 3$ in the class of oriented graphs $(n, m) = (1, 0)$. In Section 2 we extend Ochem's construction to show that: $\chi_{(n,m)}(\mathcal{A}_k) = k(2n + m)^{k-1}$. Concerning the class of planar graphs, the best known lower bound was: $(2n+m)^3+3 \leq \chi_{(n,m)}(\mathcal{P})$ [4]. In Section 4 we improve this, and also provide lower bounds for the (n, m)-mixed chromatic numbers of partial k-trees and outerplanar graphs. For this purpose we determine the exact (n, m)-mixed chromatic number of the class of paths, which is our main result (Theorem 2 in Section 3).

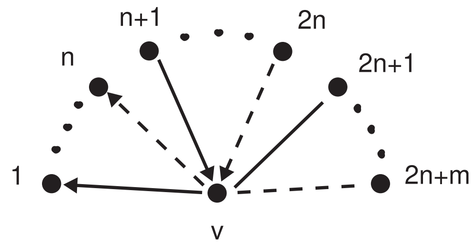

FIGURE 1: The $2n + m$ different types.

Notation: First we give some useful notation to handle (n, m)-colored mixed graphs. Let G be an (n, m)-colored mixed graph. Consider any vertex u of G. Let $N_i^+(u)$ (resp. $N_i^-(u)$) be the set of all vertices in G adjacent from (resp. adjacent to) u by an arc of color i. Similarly, let $N_i^0(u)$ be the set of all vertices in $V(G)$ connected with u by an edge of color i. Note that the maximum number of possible edges and arcs incident to u of particular colors and orientation is $2n+m$. Label these possibilities from 1 to $2n + m$ as it is shown in Figure 1. According to this, we define the *type neighborhood* of a vertex u as:

$$N_i(u) = \begin{cases} N_i^+(u) & \text{for } 1 \leq i \leq n, \\ N_{(i-n)}^-(u) & \text{for } n+1 \leq i \leq 2n, \\ N_{(i-2n)}^0(u) & \text{for } 2n+1 \leq i \leq 2n+m. \end{cases}$$

Now we say that an ordered pair (u, v) of adjacent vertices in G has *type* $i \in \{1, \ldots 2n + m\}$ if $v \in N_i(u)$. In such a case we write $t(u, v) = i$. Given a set of vertices $X \subseteq V(G)$, $N_i(X) = \{v \in V(G) : t(u,v) = i, u \in X\}$. Observe that it may be happen that $X \cap N_i(X) \neq \emptyset$.

2. Graphs with bounded acyclic chromatic number

Recall that $\mathcal{A}_k$ is the family of graphs with acyclic chromatic number at most k.

Theorem 1. *For every $k \geq 3$ and every $m \geq 0$ and $n \geq 0$, $\chi_{(n,m)}(\mathcal{A}_k) = k(2n+m)^{k-1}$.*

PROOF: When $(n,m) = (0,1)$, the statement holds, since $\chi_a(K_k) = \chi(K_k) = k$. For $(n,m) \neq (0,1)$ and $k \geq 3$, we will construct an (n,m)-colored mixed graph with chromatic number at least $k(2n+m)^{k-1}$ such that the acyclic chromatic number of its underlying graph is at most k.

Consider a complete bipartite graph B with independent sets $U = \{u_1, u_2, \ldots u_{(2m+n)^{k-1}}\}$ and $W = \{w_1, w_2, \ldots w_{k-1}\}$. We can color and orient the edges of B in such a way that the sequences of types of vertices in U are pairwise distinct. That is, for every pair of different vertices $u_i, u_j \in U$, we have:

$$(t(u_i, w_1), t(u_i, w_2) \ldots t(u_i, w_{k-1})) \neq (t(u_j, w_1), t(u_j, w_2) \ldots t(u_j, w_{k-1})).$$

This can be done since there are $(2m+n)^{k-1}$ different vectors of length $k-1$ with entries in $\{1, \ldots 2n+m\}$. Now B is such that, if $G \to H$, the vertices of U necessarily get distinct images. Consider now k disjoint copies $B_1, B_2, \ldots B_k$ of B with their respective stable sets labeled $U_1, U_2, \ldots U_k$ and $W_1, W_2, \ldots W_k$. For each pair of subscripts $1 \leq i < j \leq k$ and each pair of vertices $(x,y) \in U_i \times U_j$, we add an extra vertex $z = z_{ij}(x,y)$, connected to x and y in such a way that $t(x,z) \neq t(y,z)$ (recall that $(n,m) \neq (0,1)$). The obtained (n,m)-colored mixed graph is our graph G. By construction, if $G \to H$, the vertices in $U_1 \cup U_2 \cdots \cup U_k$ get pairwise distinct images. Since $|\bigcup_{i=1}^{k} U_i| = k(2m+n)^{k-1}$, we have $\chi_{(n,m)}(G) \geq k(2n+m)^{k-1}$.

Now we color acyclically the underlying undirected graph G_0 of G as follows. Every vertex in U_i gets color i and all vertices in W_i get pairwise distinct colors in $\{1, 2 \ldots k\} \backslash \{i\}$. Thus every cycle in each copy of B gets at least three different colors. It remains to color the extra vertices. For each pair $(x,y) \in U_i \times U_j$ we color the extra vertex $z_{ij}(x,y)$ by any color in $\{1, 2 \ldots k\} \backslash \{i,j\}$, so that every cycle involving extra vertices has at least three colors and the resulting coloring of G_0 is proper. Thus $G_0 \in \mathcal{A}_k$. $\square$

3. The chromatic number of colored mixed paths

Nešetřil and Raspaud [4] provided the exact (n,m)-mixed chromatic number of $\mathcal{F}$, the class of forests:

$$\chi_{(n,m)}(\mathcal{F}) = 2n + m + \epsilon \tag{5}$$

where $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even. The constructed (n, m)-colored mixed trees which attain that chromatic number have maximum degree $2n + m$. This suggests the question of whether this chromatic number can be improved by simpler classes of trees. Here we show that the (n, m)-mixed chromatic number of forests can be attained by paths.

Theorem 2. *Let $\mathcal{L}$ be the class of paths. Then $\chi_{(n,m)}(\mathcal{L}) = 2n + m + \epsilon$, where $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even.*

PROOF: For any fixed (n, m)-colored mixed complete graph H on $(2n+m+\epsilon)-1$ vertices, we will construct an (n, m)-colored mixed path L such that $L \nrightarrow H$. Since the number of (n, m)-colored mixed complete graphs is finite, the concatenation of all such paths cannot be mapped onto any (n, m)-colored mixed complete graph of that size. Thus we get an (n, m)-colored mixed path with chromatic number $2n + m + \epsilon$.

In order to construct the path L such that $L \nrightarrow H$, where H is a fixed complete (n, m)-colored mixed graph on $(2n+m+\epsilon)-1$ vertices, the key idea is to find a sequence of types: $t_1, \dots , t_r$ where $t_i \in \{1, \dots , 2n+m\}$ and subsets $X_0, X_1, \dots , X_r$ of $V(H)$, with the following properties: $X_0 = V(H)$, $X_i = N_{t_i}(X_{i-1})$ and $X_r = \emptyset$. This allows us to define L. Indeed, define $L := v_0, v_1, \dots , v_r$ where $t(v_{i-1}, v_i) = t_i$. Now, for every homomorphism from L to H, the first vertex of L must be mapped onto a vertex of $X_0 = V(H)$, the second vertex onto a vertex of X_1 and so on. Since X_r is the empty set, no such homomorphism can exist. To find the sequence of types and subsets with the properties defined above, we split the proof into two cases according to the value of ϵ. Before that, we prove a simple but useful counting lemma.

Lemma 1. *Let X be a subset of vertices of a complete (n, m)-colored mixed graph H. Then $\sum_{i=1}^{(2n+m)} |N_i(X)| \leq |X|(|V(H)| - 1)$.*

PROOF: Consider the bipartite (n, m)-colored mixed graph B_X defined as follows. The set of vertices is the disjoint union of a copy of X and a copy of $V(H)$. We add every edge or arc $(x, v) \in X \times V(H)$, with the same color and orientation as (x, v) in H (thus the only edges we do not have in B_X are the ones for which x and v correspond to the same vertex in H). Denote by $E_i(B_X)$ the set of arcs or edges from X to $V(H)$ of type i. Observe that the total number of edges in B_X is $|X|(|V(H)|-1)$. Then $\sum_{i=1}^{2n+m} |E_i(B_X)| = |X|(|V(H)|-1)$. The result follows since $|N_i(X)| \leq |E_i(B_X)|$ for every $i \in \{1, \dots , 2n + m\}$. □

Lemma 2. *For any subset of vertices X of a complete (n, m)-colored mixed graph on $2n + m$ vertices, there exists $i \in \{1, \dots , 2n + m\}$ such that $|N_i(X)| < |X|$.*

PROOF: By Lemma 1, we have $\sum_{i=1}^{(2n+m)} |N_i(X)| \leq |X|(2n + m - 1)$, and the result follows. □

Case 1. $\epsilon = 1$ (m odd or $m = 0$)

Let H be a complete (n, m)-colored mixed graph on $2n+m$ vertices. We start with $X_0 = V(H)$. By means of Lemma 2, we are able to find a strictly decreasing sequence of subsets $|X_0| > |X_1| > \ldots > |X_r|$ and a sequence of types $t_1, \ldots t_r$ such that $X_i = N_{t_i}(X_{i-1})$. Since in every step the size of the subset decreases, eventually we get $X_r = \emptyset$.

Case 2. $\epsilon = 2$ ($m > 0$ even)

Let H be a complete (n, m)-colored mixed graph on $2n+m+1$ vertices. In this case we cannot construct a strictly decreasing sequence of subsets as in Case 1. Instead we can guarantee that if we cannot decrease, then all neighborhoods have the same size.

Lemma 3. *For any subset of vertices X of a complete (n, m)-colored mixed graph on $2n + m + 1$ vertices, either there exists $i \in \{1, \ldots, 2n + m\}$ such that $|N_i(X)| < |X|$, or $|N_i(X)| = |X|$ for all $i \in \{1, \ldots, 2n + m\}$.*

Proof: By Lemma 1 we obtain $\sum_{i=1}^{(2n+m)} |N_i(X)| \leq |X|(2n + m)$ and the result follows. □

Now more work is required. Suppose $X \subset V(H)$ is such that $|N_i(X)| = |X|$ for all $i \in \{1, \ldots, 2n + m\}$. In Lemma 5 we show that in at most three steps we can reduce the size of the subset. In order to prove it, we need the next.

Lemma 4. *In any (n, m)-colored mixed complete graph on $2n + m + 1$ vertices with $m > 0$ even, there exists a vertex incident to at least 2 edges of the same type.*

Proof: Any vertex v of H has degree $2n + m$. If v were not incident to an edge of a particular type, then v would be the desired vertex (being $2n + m$ types in total). Assume that every vertex is incident to exactly one edge of every type. Then any color class of edges would induce a perfect matching of H. This is a contradiction since H has an odd number of vertices. □

Lemma 5. *If a subset of vertices X of a complete (n, m)-colored mixed graph on $2n + m + 1$ vertices with $m > 0$ even is such that $|N_i(X)| = |X|$ for all $i \in \{1, \ldots, 2n + m\}$, then there exists $j, k, l \in \{1, \ldots, 2n + m\}$ such that $|N_l(N_k(N_j(X)))| < |X|$.*

Proof: By Lemma 4, there exists a vertex $u \in V(H)$ incident to at least two edges of type $k \in \{1, \ldots, 2n + m\}$. Since H is a complete (n, m)-colored mixed graph, $u \in N_j(X)$ for some $j \in \{1, \ldots, 2n + m\}$. By hypothesis, $|N_j(X)| = |X|$. We may assume that $N_j(X)$ is such that $|N_i(N_j(X))| = |N_j(X)|$ for every $i \in \{1, \ldots, 2n + m\}$, otherwise by Lemma 3 we are done. Then we have $|N_k(N_j(X))| = |N_j(X)| = |X|$. Name $Y := N_k(N_j(X))$. We will use the bipartite graph B_Y, defined as in the proof of Lemma 1. By construction, there are two vertices $v, w \in Y$ with a common k-neighbor (u). Therefore $|N_k(Y)| < |E_k(B_Y)|$. We suppose $|N_k(Y)| = |Y|$ (otherwise by Lemma 3 we are done). Thus we have $|Y| < |E_k(B_Y)|$, and the result follows since $\sum_{i=1}^{2n+m} |E_i(B_Y)| = |Y|(2n+m)$. □

4. Partial k-trees, outerplanar and planar graphs

In this section we give lower bounds for the (n,m)-mixed chromatic number of the families of partial k-trees, outerplanar and planar graphs, which we denote by $\mathcal{T}^k$, $\mathcal{O}$ and $\mathcal{P}$ respectively. The key idea is to generalize a construction proposed by Sopena [6]. For the class of partial k-trees we use (5), and for the classes of outerplanar and planar graphs we use Theorem 2.

Theorem 3. *Let $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even. Then,*

(1) $(2n+m)^k + \epsilon(2n+m)^{k-1} + (2n+m)^{k-2} + \cdots + 1 \le \chi_{(n,m)}(\mathcal{T}^k)$,
(2) $(2n+m)^2 + \epsilon(2n+m) + 1 \le \chi_{(n,m)}(\mathcal{O})$,
(3) $(2n+m)^3 + \epsilon(2n+m)^2 + (2n+m) + 1 \le \chi_{(n,m)}(\mathcal{P})$.

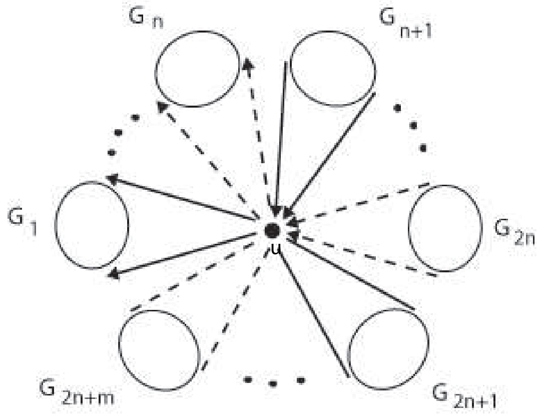

FIGURE 2: Given G we construct G' with higher chromatic number.

We will use the following construction. Let G be an (n,m)-colored mixed graph. Define G' as the (n,m)-colored mixed graph obtained by taking $2n+m$ disjoint copies $G_1, G_2, \ldots, G_{2n+m}$ of G and adding a new vertex u adjacent to all other vertices in such a way that $t(u,v) = i$ for every $v \in G_i$ (see Figure 2). Let G^k be defined inductively by $G^0 := G$ and $G^k := (G^{k-1})'$. By construction, if $G^k \to H$, a vertex in G_i^{k-1} has a different image of a vertex in G_j^{k-1} when $i \neq j$. Moreover the vertex u has a different image from all other vertices. Thus we have:

Remark 1. $\chi_{(n,m)}(G^k) \ge (2n+m)\chi_{(n,m)}(G^{k-1}) + 1$.

Proof of Theorem 3(1): We proceed by induction on k. If $k = 1$ we are done by (5). Suppose now that the result holds up to $(k-1)$ and let $T^{(k-1)}$ be an (n,m)-colored mixed partial $(k-1)$-tree with chromatic number at least: $(2n+m)^{k-1} + \epsilon(2n+m)^{k-2} + (2n+m)^{k-3} + \cdots + 1$. We consider $T^k := (T^{k-1})'$ which is a partial k-tree, and the statement follows by Remark 1. □

Proof of Theorem 3(2): Observe that if G is a path, then G' is an outerplanar graph. Thus, by starting with an (n,m)-colored mixed path with chromatic number at least $2n+m+\epsilon$ (provided by Theorem 2), we get (according to Remark 1) an (n,m)-colored mixed outerplanar graph with the required chromatic number. □

Proof of Theorem 3(3): Observe that if G is an outerplanar graph, then G' is a planar graph. Thus, starting with an (n,m)-colored mixed outerplanar graph with chromatic number at least $(2n+m)^2 + \epsilon(2n+m) + 1$ (provided by Theorem 3(2)), we get (according to Remark 1) an (n,m)-colored mixed planar graph with the required chromatic number. □

5. Conclusions and remarks

In this paper we gave lower bounds for the (n,m)-mixed chromatic number of various classes of graphs. We also computed the exact (n,m)-mixed chromatic number of graphs with bounded acyclic chromatic number and paths. Due to these results, we found interesting the following problems:

1. Regarding the classes of partial k-trees and outerplanar graphs, the lower bounds given in Theorem 3 state that the upper bounds given in (3) and (4) are tight up to a constant multiplicative factor:

$$\chi_{(n,m)}(\mathcal{T}^k) = \Theta((2n+m)^k),$$
$$\chi_{(n,m)}(\mathcal{O}) = \Theta((2n+m)^2).$$

We were not able to do the same for the class of planar graphs, where we have:

$$\Omega((2n+m)^3) \leq \chi_{(n,m)}(\mathcal{P}) \leq \Theta((2n+m)^4).$$

We consider closing the gap to be an interesting problem.

2. Most of the work related to the study of homomorphisms as a generalization of colorings has been done in the context of $(1,0)$-mixed graphs, which are actually oriented graphs [7]. In this case the $(1,0)$-mixed chromatic number (oriented chromatic number) of the family of planar graphs can be significantly lowered when considering planar graphs with large girth, as Borodin, Kostochka, Nešetřil, Raspaud and Sopena showed [3]. We think results of this kind can be extended to the class of (n,m)-colored mixed graphs.

3. We think it is an interesting problem to ask what is the shortest path that attains the (n,m)-mixed chromatic number given in Theorem 2.

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