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The Lindelöf property and pseudo-$\aleph_1$-compactness in spaces and topological groups

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Abstract. We introduce and study, following Z. Frolík, the class $\mathcal{B}(\mathcal{P})$ of regular P-spaces X such that the product $X \times Y$ is pseudo-$\aleph_1$-compact, for every regular pseudo-$\aleph_1$-compact P-space Y. We show that every pseudo-$\aleph_1$-compact space which is locally $\mathcal{B}(\mathcal{P})$ is in $\mathcal{B}(\mathcal{P})$ and that every regular Lindelöf P-space belongs to $\mathcal{B}(\mathcal{P})$. It is also proved that all pseudo-$\aleph_1$-compact P-groups are in $\mathcal{B}(\mathcal{P})$.

The problem of characterization of subgroups of $\mathbb{R}$-factorizable (equivalently, pseudo-$\aleph_1$-compact) P-groups is considered as well. We give some necessary conditions on a topological P-group to be a subgroup of an $\mathbb{R}$-factorizable P-group and deduce that there exists an ω-narrow P-group that cannot be embedded as a subgroup into any $\mathbb{R}$-factorizable P-group.

The class of σ-products of second-countable topological groups is especially interesting. We prove that *all subgroups* of the groups in this class are perfectly κ-normal, $\mathbb{R}$-factorizable, and have countable cellularity. If, in addition, H is a closed subgroup of a σ-product of second-countable groups, then H is an Efimov space and satisfies $\mathrm{cel}_\omega(H) \le \omega$.

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1. Introduction

Lindelöfness and pseudo-$\aleph_1$-compactness (this and other notions will be defined later in this section) are intimately related properties. For example, according to Glicksberg's theorem in [9], every continuous real-valued function f defined on a product space $X = \prod_{i\in I} X_i$ admits a continuous factorization through the projection p_J of X to a subproduct $X_J = \prod_{i\in J} X_i$, for some countable set $J \subseteq I$, if and only if X is pseudo-$\aleph_1$-compact. The same conclusion remains valid for continuous real-valued functions defined on an arbitrary Lindelöf subspace Y of the product space X (see [16]). The stronger Lindelöf property of Y appears here to compensate the fact that the restrictions to Y of projections p_J need not be open, even when considered as mappings onto their images.

The Lindelöf property seems to be considerably stronger than pseudo-$\aleph_1$-compactness. However, the two properties coincide, for example, in the class of paracompact spaces. The list of interrelations between these properties can be as long as one wishes, but we would like to focus attention on two groups of problems related to the preservation of pseudo-$\aleph_1$-compactness under taking products, and factorization of continuous real-valued functions on topological groups. The former group of problems has its origin in Frolík's articles [7], [8] where the class $\mathcal{B}$ of spaces was defined, while the latter takes us directly to the class of $\mathbb{R}$-factorizable topological groups studied in [11], [17], [18], [19]. A special emphasis is given here to *P-spaces* in which every G_δ-set is open. It is worth mentioning that a *P-group* G, *i.e.*, a topological group which is a P-space, is $\mathbb{R}$-factorizable if and only if the space G is pseudo-$\aleph_1$-compact (see [19, Theorem 4.16]).

The article is organized as follows. In Section 2 we study the class $\mathfrak{B}(\mathcal{P})$ of regular P-spaces X such that the product $X \times Y$ is pseudo-$\aleph_1$-compact, for every regular pseudo-$\aleph_1$-compact P-space Y. We show that every pseudo-$\aleph_1$-compact space which is locally $\mathfrak{B}(\mathcal{P})$ is in $\mathfrak{B}(\mathcal{P})$ and that every regular Lindelöf P-space is in $\mathfrak{B}(\mathcal{P})$. For topological groups, the latter result admits a more general form, namely, every pseudo-$\aleph_1$-compact P-group is in $\mathfrak{B}(\mathcal{P})$ (see Theorem 2.8).

In Section 3 we consider the problem of characterization of subgroups of $\mathbb{R}$-factorizable (equivalently, pseudo-$\aleph_1$-compact) P-groups. It is known (see [17, Section 5]) that every subgroup of an $\mathbb{R}$-factorizable group is ω-narrow and that every ω-narrow topological group can be embedded as a closed subgroup into an $\mathbb{R}$-factorizable group. It seems natural to conjecture that the subgroups of $\mathbb{R}$-factorizable P-groups are exactly the ω-narrow P-groups. It was shown in [11] that if H is a topological subgroup of an $\mathbb{R}$-factorizable P-group, then H can be embedded as a *closed* subgroup into another $\mathbb{R}$-factorizable P-group. In a sense, this fact might be considered as an implicit confirmation of the validity of the conjecture. We give in Theorem 3.1 new necessary conditions on a topological P-group to be a subgroup of an $\mathbb{R}$-factorizable P-group and deduce in Corollary 3.3 that there exists an ω-narrow P-group that cannot be embedded as a subgroup into any $\mathbb{R}$-factorizable P-group.

Our aim in Section 4 is to study σ-products of second-countable topological groups and establish several common properties of *all* subgroups of these σ-products. It is proved in Theorem 4.6 that all subgroups in question are perfectly κ-normal and $\mathbb{R}$-factorizable. If, in addition, H is a closed subgroup of a σ-product of second-countable groups, then H is an Efimov space (*i.e.*, the closure of the union of an arbitrary family of G_δ-sets in H is a zero-set) and satisfies $\mathrm{cel}_\omega(H) \leq \omega$. In particular, the cellularity of H is countable, and the same conclusion clearly remains valid if H is not assumed to be closed. The main technical tool in the proof of these results is the existence of a good lattice of continuous retractions of the group H onto its subgroups of weight less than or equal to 2^ω (see Theorem 4.3).

1.1 Notation and terminology. A space X is called *pseudo-$\aleph_1$-compact* if every locally finite family of non-empty open sets in X is countable. It is easy to see that every Lindelöf space as well as every space of countable cellularity is pseudo-$\aleph_1$-compact. It is also clear that a continuous image of a pseudo-$\aleph_1$-compact space is pseudo-$\aleph_1$-compact.

A *P-space* is a space in which every G_δ-set is open. A *P-group* is a topological group with the same property. Evidently, every regular P-space has a base of clopen sets. Hence, every P-group is zero-dimensional.

We use the concepts of the σ-product and Σ-product in Sections 2 and 4. Suppose that $\{X_i : i \in I\}$ is a family of spaces and $\Pi = \prod_{i\in I} X_i$ is the topological product of this family. Given a point $p \in \Pi$, we put

$$\mathrm{diff}(x, p) = \{i \in I : \pi_i(x) \neq \pi_i(p)\}$$

for every $x \in \Pi$, where $\pi_i\colon \Pi \to X_i$ is the projection of the product space Π onto the factor X_i. Making use of the function diff, we define

$$\sigma\Pi(p) = \{x \in \Pi : |\,\mathrm{diff}(x, p)| < \omega\}$$

and

$$\Sigma\Pi(p) = \{x \in \Pi : |\,\mathrm{diff}(x, p)| \leq \omega\}.$$

It is clear that both $\sigma\Pi(p)$ and $\Sigma\Pi(p)$, called the *σ-product* and *Σ-product* of the family $\{X_i : i \in I\}$, respectively, are dense subspaces of Π, for any choice of $p \in \Pi$. The point p is called the *center* of the spaces $\sigma\Pi(p)$ and $\Sigma\Pi(p)$.

Occasionally, the product space $\Pi = \prod_{i\in I} X_i$ is given the *box* or *ω-box* topology. The standard base of the box topology on Π is formed by the sets of the form $U = \prod_{i\in I} U_i$, where each U_i is open in X_i. Basic open sets in the ω-box topology on Π have the similar form $U = \prod_{i\in I} U_i$ with each U_i open in X_i, but there must be at most countably many indices $i \in I$ with $U_i \neq X_i$. We then put

$$\mathrm{coord}(U) = \{i \in I : U_i \neq X_i\}.$$

It is clear that the box topology and ω-box topology on Π coincide when the index set I is countable.

Suppose that $f\colon X \to Y$ and $g\colon X \to Z$ are continuous mappings. We write $f \prec g$ if there exists a continuous mapping $\varphi\colon f(X) \to Z$ such that $g = \varphi \circ f$. Let $\mathcal{F}$ be a family of continuous mappings of X elsewhere. Given a subfamily γ of $\mathcal{F}$, we denote by $\Delta\gamma$ the diagonal product of the mappings from γ considered as a mapping of X onto its image. It is clear that $\Delta\gamma$ is continuous for every $\gamma \subseteq \mathcal{F}$. If τ is an infinite cardinal, we say that $\mathcal{F}$ is *τ-directed* if for every $\gamma \subseteq \mathcal{F}$ with $|\gamma| \leq \tau$, there exists $f \in \mathcal{F}$ such that $f \prec \Delta\gamma$. The family $\mathcal{F}$ is *τ-complete* if it is τ-directed and for every subfamily $\{f_\alpha : \alpha < \tau\} \subseteq \mathcal{F}$ satisfying $f_\beta \prec f_\alpha$ whenever $\alpha < \beta < \tau$, the mapping $\Delta_{\alpha<\tau} f_\alpha$ belongs to $\mathcal{F}$. If $\mathcal{F}$ is τ-complete and generates the topology of X, we say that $\mathcal{F}$ is a *τ-lattice.*

We use $w(X)$, $c(X)$, and $\psi(X)$ to denote the weight, cellularity, and pseudocharacter of a space X, respectively. The power of the continuum is $\mathfrak{c} = 2^\omega$. If

$\kappa \geq \omega$ is a cardinal, we put $2^{<\kappa} = \sum_{\lambda<\kappa} 2^\lambda$. If λ is a cardinal and X is a set, then $[X]^\lambda$ denotes the family of all subsets Y of X satisfying $|Y| \leq \lambda$.

All topological groups are assumed to be Hausdorff. A topological group G is *ω-narrow* if for every neighbourhood U of the neutral element in G, there exists a countable set $C \subseteq G$ such that $CU = G$. The class of ω-narrow groups is closed under taking arbitrary products, subgroups, and continuous homomorphic images. By Guran's theorem in [10], a group G is ω-narrow if and only if G is topologically isomorphic to a subgroup of a product of second-countable groups.

A topological group G is called *$\mathbb{R}$-factorizable* if for every continuous real-valued function f on G, there exists a continuous homomorphism p of G onto a second-countable group H such that $p \prec f$. It is well known that every $\mathbb{R}$-factorizable group is ω-narrow, but the converse is false (see Proposition 5.3 and Example 5.14 in [17]). In fact, it is shown in [5] that every uncountable ω-narrow Abelian group G is an image under a continuous one-to-one homomorphism of an ω-narrow group $G^\square$ that fails to be $\mathbb{R}$-factorizable.

2. The class $\mathfrak{B}(\mathcal{P})$

In what follows we denote by $\mathfrak{B}(\mathcal{P})$ the class of all regular P-spaces X such that for every regular pseudo-$\aleph_1$-compact P-space Y, the product $X \times Y$ is also pseudo-$\aleph_1$-compact. Naturally, all spaces in this section are assumed to be regular.

The next proposition is an easy consequence of the definition of the class $\mathfrak{B}(\mathcal{P})$:

Proposition 2.1. *Let Y be a regular P-space.*

(a) *If Y is continuous image of a space $X \in \mathfrak{B}(\mathcal{P})$, then $Y \in \mathfrak{B}(\mathcal{P})$.*
(b) *$X \times Y \in \mathfrak{B}(\mathcal{P})$ if and only if $X, Y \in \mathfrak{B}(\mathcal{P})$.*

Here is an auxiliary result we need for the proof of Proposition 2.3 below. It follows from [15, Lemma 1].

Lemma 2.2. *If a regular P-space X contains a locally countable family $\mathcal{A}$ of open sets with $|\mathcal{A}| = \tau > \omega$, then X also contains a discrete family of open set of the same cardinality τ.*

Proposition 2.3. *Let X be a pseudo-$\aleph_1$-compact P-space such that each point of X has a neighborhood that belongs to $\mathfrak{B}(\mathcal{P})$. Then $X \in \mathfrak{B}(\mathcal{P})$.*

Proof: Let Y be a regular pseudo-$\aleph_1$-compact P-space, and suppose that γ is an uncountable locally finite family of open sets in $X \times Y$. Let us show that γ is countable.

For each $x \in X$, take a neighbourhood U_x of x in X with $U_x \in \mathfrak{B}(\mathcal{P})$. Since the product space $U_x \times Y$ is pseudo-$\aleph_1$-compact, the family $\gamma_x = \{O \in \gamma : O \cap (U_x \times Y) \neq \emptyset\}$ is countable. This means that the family $\mathcal{F} = \{\pi_X(O) : O \in \gamma\}$ of open subsets of X is locally countable, where $\pi_X \colon X \times Y \to X$ is the projection. To finish the proof, it suffices to apply Lemma 2.2. $\square$

We say that a space X has property $(\mathcal{F})$ if for every uncountable pairwise disjoint family $\mathcal{U}$ of non-empty open sets in X, there exists a subfamily $\{U_\alpha : \alpha < \omega_1\}$ of distinct elements of $\mathcal{U}$ such that

$$\bigcap_{F\in\mathcal{F}} \overline{\bigcup_{\alpha\in F} U_\alpha} \neq \emptyset,$$

for every countably complete filter $\mathcal{F}$ of subsets of ω_1.

Theorem 2.4. *Suppose that a regular P-space X has property $(\mathcal{F})$. Then $X \in \mathfrak{B}(\mathcal{P})$.*

PROOF: Let Y be a pseudo-$\aleph_1$-compact P-space. We shall prove that every uncountable family $\mathcal{W}$ of non-empty open sets in $X \times Y$ has an accumulation point. Without loss of generality we can assume that $\mathcal{W}$ consists of rectangular sets, *i.e.*, $\mathcal{W} = \{U_\alpha \times V_\alpha : \alpha < \omega_1\}$. Let us consider two cases:

1) There exist a point $x \in X$ and a subset $A \subseteq \omega_1$ such that $|A| = \aleph_1$ and every neighbourhood of x intersects every element of $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, except perhaps a countable number of them. By Lemma 2.2, the family $\{V_\alpha : \alpha \in A\}$ cannot be locally countable. Let $y \in Y$ be a complete accumulation point of $\{V_\alpha : \alpha \in A\}$. It is clear that (x, y) is an accumulation point of $\{U_\alpha \times V_\alpha : \alpha \in A\}$ and, therefore, of $\mathcal{W}$.

2) For every uncountable set $A \subseteq \omega_1$ and every point $x \in X$, there exists a neighbourhood U of x such that $\{\alpha \in A : U_\alpha \cap U = \emptyset\}$ is uncountable. Since X is a regular pseudo-$\aleph_1$-compact P-space, we can construct by recursion a set $A \subseteq \omega_1$ with $|A| = \aleph_1$ and non-empty open sets $W_\alpha \subseteq U_\alpha$, for $\alpha \in A$, such that the elements of the family $\{W_\alpha : \alpha \in A\}$ are pairwise disjoint. Again, we can assume that $A = \omega_1$. By Lemma 2.2, the family $\{V_\alpha : \alpha < \omega_1\}$ has a complete accumulation point $y \in Y$. Denote by $\mathcal{N}(y)$ the neighbourhood base of Y at y. For every $V \in \mathcal{N}(y)$, let $\varphi(V) = \{\alpha \in \omega_1 : V \cap V_\alpha \neq \emptyset\}$. It is easy to see that $\{\varphi(V) : V \in \mathcal{N}(y)\}$ is a base of a countably complete filter $\mathcal{F}$ in ω_1. Since $X \in \mathfrak{B}(\mathcal{P})$, we can find a point $x \in \bigcap_{F\in\mathcal{F}} \overline{\bigcup_{\alpha\in F} W_\alpha}$. One readily checks that (x, y) is an accumulation point of the family $\{W_\alpha \times V_\alpha : \alpha < \omega_1\}$ and, hence, of $\mathcal{W}$. $\square$

Since every Lindelöf space has property $(\mathcal{F})$, the theorem above implies the following:

Corollary 2.5. *Every Lindelöf P-space is in $\mathfrak{B}(\mathcal{P})$.*

Let $\Pi = \prod_{i\in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of spaces, and suppose that $a \in \Pi$. By a theorem of W. Comfort in [3], the σ-product $\sigma\Pi(a) \subseteq \Pi$ endowed with the ω-box topology inherited from the product space Π is Lindelöf provided that each X_i is countable. Here we extend this result to σ-products of Lindelöf P-spaces. Our argument, however, is close to that given in [3].

Proposition 2.6. *An arbitrary σ-product of Lindelöf P-spaces endowed with the ω-box topology is a Lindelöf P-space.*

Proof: Let $\{X_i : i \in I\}$ be a family of Lindelöf P-spaces. Pick a point $a \in \Pi = \prod_{i\in I} X_i$ and consider the σ-product $\sigma\Pi(a)$ of this family with center at a. Since each X_i is a P-space, the product Π with the ω-box topology is also a P-space, and so is the subspace $\sigma\Pi(a)$ of Π. Suppose that γ is a cover of $\sigma\Pi(a)$ by basic ω-boxes in X. It suffices to show that γ contains a countable subcover of $\sigma\Pi(a)$.

Given a set $J \subseteq I$, we put

$$\sigma_J(a) = \{x \in \sigma\Pi(a) : \mathrm{diff}(x, a) \subseteq J\}.$$

If $J \subseteq I$ is countable, then $\sigma_J(a)$ is the union of the countable family $\{\sigma_F(a) : F \subseteq J,\ |F| < \omega\}$, where every summand $\sigma_F(a) \cong \prod_{i\in F} X_i$ is Lindelöf by Noble's theorem in [14]. Therefore, $\sigma_J(a)$ is also Lindelöf.

Let J_0 be a countable non-void subset of I. Since $\sigma_{J_0}(a)$ is Lindelöf, there exists a countable subfamily γ_0 of γ such that $\sigma_{J_0}(a) \subseteq \bigcup\gamma_0$. Suppose that for some $n \in \omega$, we have defined increasing sequences

$$J_0 \subseteq \ldots \subseteq J_n \subseteq I \ \text{ and } \ \gamma_0 \subseteq \ldots \subseteq \gamma_n \subseteq \gamma,$$

where $|J_n| \le \omega$ and $|\gamma_n| \le \omega$. Then the set $J_{n+1} = J_n \cup \bigcup\{\mathrm{coord}(W) : W \in \gamma_n\}$ is countable, so $\sigma_{J_{n+1}}(a)$ is Lindelöf and we can find a countable subfamily γ_{n+1} of γ such that $\gamma_n \subseteq \gamma_{n+1}$ and $\sigma_{J_{n+1}}(a) \subseteq \bigcup\gamma_{n+1}$.

Consider the set $J^* = \bigcup_{n\in\omega} J_n \subseteq I$ and the family $\gamma^* = \bigcup_{n\in\omega}\gamma_n \subseteq \gamma$. Clearly, γ^* is countable, and we claim that γ^* covers $\sigma\Pi(a)$. Indeed, let $x \in \sigma\Pi(a)$ be arbitrary. Since the set $\mathrm{diff}(x, a)$ is finite, there exists $n \in \omega$ such that $\mathrm{diff}(x, a) \cap J^* \subseteq J_n$. Denote by y the point of $\sigma_{J^*}(a)$ such that $y_i = x_i$ for each $i \in J^*$. Then $y \in \sigma_{J_n}(a)$. Since γ_n covers $\sigma_{J_n}(a)$, we can find $W \in \gamma_n$ such that $y \in W$. Then $\mathrm{coord}(W) \subseteq J_{n+1} \subseteq J^*$, whence it follows that $x \in W$. This proves that γ^* covers $\sigma\Pi(a)$ and, hence, $\sigma\Pi(a)$ is Lindelöf. □

Combining Proposition 2.6 and Corollary 2.5, we obtain the following:

Corollary 2.7. *Let $\Pi = \prod_{i\in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of regular Lindelöf P-spaces and $S = (\sigma\Pi(a))_\omega$ the corresponding σ-product that carries the ω-box topology, where $a \in \Pi$. Then $S \in \mathfrak{B}(\mathcal{P})$.*

It is known that an arbitrary product of pseudo-$\aleph_1$-compact P-groups is pseudo-$\aleph_1$-compact (it suffices to combine Theorems 4.16 and 5.5 of [19]). The theorem below complements this fact.

Theorem 2.8. *Every pseudo-$\aleph_1$-compact P-group belongs to $\mathfrak{B}(\mathcal{P})$.*

Proof: Let G be a pseudo-$\aleph_1$-compact P-group and $\{U_\alpha : \alpha < \omega_1\}$ a disjoint family of non-empty open sets in G. We can suppose that the family consists of clopen sets. Moreover, combining item (b) of Lemma 2.1 and Lemma 3.29 of [19], we may assume that each U_α has the form $p_\alpha^{-1}p_\alpha(U_\alpha)$, where $p_\alpha\colon G \to H_\alpha$ is a continuous homomorphism and H_α is a discrete countable group. Consider the diagonal product of these homomorphisms, say $f\colon G \to \prod_{\alpha<\omega_1} H_\alpha = \Pi$. It is easy to see that $f^{-1}f(U_\alpha) = U_\alpha$ for every $\alpha < \omega_1$. Observe that the product group Π and its subgroup $f(G)$ have weight $\leq \aleph_1$. Denote by H the topological group obtained when we endow $f(G)$ with the quotient topology with respect to f. Then H is a pseudo-$\aleph_1$-compact P-group and, since the topology of H is finer than the topology of $f(G)$, we have that $\psi(H) \leq \aleph_1$. Hence, combining Corollaries 3.32 and 4.11 of [19], we conclude that H is Lindelöf. The mapping f is a quotient homomorphism and, consequently, is open.

Suppose that $\mathcal{F}$ is a countably complete filter of subsets of ω_1. Since H is a Lindelöf space, we have $\bigcap_{F\in\mathcal{F}} \overline{\bigcup_{\alpha\in F} f(U_\alpha)} \neq \emptyset$. Making use of the fact that f is open, we deduce that

$$\begin{aligned}\emptyset \neq f^{-1}\Big(\bigcap_{F\in\mathcal{F}} \overline{\bigcup_{\alpha\in F} f(U_\alpha)}\Big) &= \bigcap_{F\in\mathcal{F}} f^{-1}\Big(\overline{\bigcup_{\alpha\in F} f(U_\alpha)}\Big)\\ &= \bigcap_{F\in\mathcal{F}} \overline{f^{-1}\Big(\bigcup_{\alpha\in F} f(U_\alpha)\Big)}\\ &= \bigcap_{F\in\mathcal{F}} \overline{\bigcup_{\alpha\in F} f^{-1}f(U_\alpha)}\\ &= \bigcap_{F\in\mathcal{F}} \overline{\bigcup_{\alpha\in F} U_\alpha}.\end{aligned}$$

It follows that $G \in \mathfrak{B}(\mathcal{P})$. □

We close this section with an example of a pseudo-$\aleph_1$-compact P-space which is not in $\mathfrak{B}(\mathcal{P})$. The existence of such a space shows that Theorem 2.8 is not valid outside the class of topological groups. The construction that follows was outlined by Alan Dow and it is placed here with his kind permission.

For brevity, we slightly change the usual terminology and call a family γ of subsets of an infinite cardinal κ *almost disjoint* if $|X \cap Y| < \kappa$, for all distinct $X, Y \in \gamma$. The next lemma is well known (see [4] or [12]), but we supply the reader with a short proof.

Lemma 2.9. *If $2^{<\kappa} = \kappa$, there exists an almost disjoint family of 2^κ subsets of κ.*

Proof: Let S be the set of all 0-1 sequences of length less than κ, *i.e.*, $S = \bigcup_{\alpha<\kappa}\{0,1\}^\alpha$. Clearly $|S| = \kappa$. For every $f\colon \kappa \to \{0,1\}$, let A_f be the set $A_f = \{s \in S : s \subset f\} = \{f \restriction \alpha : \alpha \in \kappa\}$. It is easy to see that $|A_f \cap A_g| < \kappa$ if $f \neq g$; thus, $\{A_f : f \in \{0,1\}^\kappa\}$ is an almost disjoint family of 2^κ subsets of S. Since $|S| = \kappa$, we are done. □

From now on, we shall assume that $2^{\aleph_1} = \aleph_2$. Hence, there exists a maximal almost disjoint family Γ of subsets of ω_1 with $|\Gamma| = \aleph_2$, *i.e.*, $\Gamma = \{A_\alpha : \alpha \in \omega_2\}$. Let us use the function $\sigma = \{(0,\alpha)\}$ as index instead of α. For each σ, there exists a maximal almost disjoint family of subsets of A_σ of cardinality $\aleph_2$. The indices now will be extensions of σ to $\{0,1\}$ with values in ω_2. Continuing this way, we obtain a family $\{A_\sigma : \sigma \in \omega_2^{<\omega}\} \subseteq [\omega_1]^{\omega_1}$ such that for every $\sigma\colon \{0,1,\ldots,n\} \to \omega_2$ and all ordinals $\alpha, \beta \in \omega_2$ with $\alpha \neq \beta$, we have $A_\sigma \supseteq A_{\sigma\wedge\alpha} \cup A_{\sigma\wedge\beta}$ and $|A_{\sigma\wedge\alpha} \cap A_{\sigma\wedge\beta}| < \omega_1$. Here $\sigma \wedge \alpha$ is the function with domain $\{0,1,\ldots,n+1\}$ that extends σ and takes the value α in $n+1$. We may assume that

$$A_\sigma = \bigcup_{\alpha\in\omega_2} A_{\sigma\wedge\alpha}$$

and that for each $A \subseteq \omega_1$ with $|A| = \omega_1$, there exists $\sigma \in \omega_2^{<\omega}$ such that $A_\sigma \subseteq A$ (see [4, Theorem 12.11]).

Take $X = \omega_1 \cup \omega_2^{<\omega}$. Each point $x \in \omega_1$ is declared to be isolated in X, and if $x \in \omega_2^{<\omega_1}$, basic neighbourhoods of x in X are the sets of the form

$$\{x\} \cup A_x \setminus \bigcup\{A_{x\wedge\alpha} : \alpha \in S\}, \tag{1}$$

where $S \in [\omega_2]^\omega$. It is easy to see that X is Hausdorff and the family of basic neighborhoods of every point $x \in \omega_2^{<\omega_1}$ is closed under countable intersections. Therefore, X is a P-space. Moreover, the sets in (1) are clopen, so X is regular and, hence, Tychonoff. Now, let

$$T_e = \{\sigma \in [\omega_2]^{<\omega} : |\operatorname{dom}\sigma| \text{ is even}\} \text{ and } \ X_e = \omega_1 \cup T_e.$$

We consider X_e as a subspace of X. Since, for every $A \in [\omega_1]^{\omega_1}$, there exists $\sigma \in T_e$ such that $|A_\sigma \setminus A| \leq \omega$, the space X_e is pseudo-ω_1-compact. Similarly, with $T_o = \{\sigma \in [\omega_2]^{<\omega} : |\operatorname{dom}\sigma| \text{ is odd}\}$, the subspace $X_o = \omega_1 \cup T_o$ of X is Hausdorff and zero-dimensional. Again, X_o is a pseudo-ω_1-compact P-space. Finally, $X_e \times X_o$ is not pseudo-ω_1-compact since the diagonal $\{(\alpha,\alpha) : \alpha \in \omega_1\}$ consists of isolated points in $X \times X$, is contained in $X_e \times X_o$, and has no accumulation points in $X_e \times X_o$.

3. Subgroups of $\mathbb{R}$-factorizable P-groups

It was shown in [11] that every subgroup of an $\mathbb{R}$-factorizable P-group can be embedded as a *closed* subgroup into another $\mathbb{R}$-factorizable P-group, and that closed subgroups of $\mathbb{R}$-factorizable P-groups may fail to be $\mathbb{R}$-factorizable. It is also clear that an arbitrary subgroup of an $\mathbb{R}$-factorizable P-group is an ω-narrow P-group. These facts motivated the authors to ask whether every ω-narrow P-group is a subgroup of an $\mathbb{R}$-factorizable P-group (see [11, Problem 4.1]). In

Corollary 3.3 below we answer this question in the negative. To some extent, this seems to be unexpected, since every ω-narrow group is a *closed subgroup* of an $\mathbb{R}$-factorizable group [17, Theorem 5.15].

We start with a theorem describing an interesting property of subgroups of $\mathbb{R}$-factorizable P-groups.

Theorem 3.1. *Let H be a subgroup of an $\mathbb{R}$-factorizable P-group. Then H is ω-narrow and every continuous homomorphic image G of H satisfying $w(G) \le \aleph_1$ is a subgroup of a Lindelöf topological group.*

Proof: Suppose that H is a subgroup of an $\mathbb{R}$-factorizable P-group H_0. By [17, Proposition 5.3], every $\mathbb{R}$-factorizable group is ω-narrow, while every subgroup of an ω-narrow group is ω-narrow. Hence, the group H is ω-narrow as well.

Let $h\colon H \to G$ be a continuous homomorphism onto a group G with $w(G) \le \aleph_1$. We will prove that the Raĭkov completion ϱG of G is Lindelöf. Indeed, let $\{U_\alpha : \alpha < \omega_1\}$ be a local base at the identity of G. For each $\alpha < \omega_1$, let $\pi_\alpha\colon H_0 \to L_\alpha$ be a continuous homomorphism to a second-countable group L_α such that $\pi_\alpha^{-1}(V_\alpha) \cap H \subseteq h^{-1}(U_\alpha)$, for some neighbourhood V_α of the identity in L_α. Consider the diagonal product $\pi = \Delta_{\alpha<\omega_1}\pi_\alpha\colon H_0 \to \prod_{\alpha<\omega_1} L_\alpha$. Then our definition of π implies that $H \cap \ker\pi \subseteq \ker h$. It is also clear that $L_0 = \pi(H_0)$ is a subgroup of $\prod_{\alpha<\omega_1} L_\alpha$, so that $w(L_0) \le \aleph_1$. Denote by L the underlying group L_0 endowed with the quotient topology with respect to the homomorphism $\pi\colon H_0 \to L$. According to [19, Lemma 2.1 c)], L is a P-group. Since the identity isomorphism of L onto L_0 is continuous, we have that $\psi(L) \le \psi(L_0) \le w(L_0) \le \aleph_1$. Also, notice that L is a continuous homomorphic image of an $\mathbb{R}$-factorizable P-group H_0. Hence, by [19, Theorem 4.16], L is Lindelöf. Since L is a P-group, it follows from [19, Proposition 2.3] that the group L is Raĭkov complete. Let $K = \pi(H)$. Now, from the inclusion $H \cap \ker\pi \subseteq \ker h$ it follows that there exists a homomorphism $g\colon K \to G$ such that $h = g \circ p$, where $p = \pi \restriction H$. Our choice of the families $\{U_\alpha : \alpha < \omega_1\}$, $\{V_\alpha : \alpha < \omega_1\}$ and the definition of π imply that the homomorphism g is continuous at the identity of K, *i.e.*, g is continuous. We describe the situation in the following diagram:

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, "h"'] \arrow[dr, "p"] & H_0 \arrow[dr, "\pi"] & \\ G & K \arrow[l, "g"'] \arrow[r, hook] & L \end{tikzcd}$$

Denote by K^* the closure of K in L. Since the group L is Lindelöf and Raĭkov complete, so is K^*. In particular, $K^* \cong \varrho K$. Therefore, the homomorphism g admits an extension to a continuous homomorphism $g^*\colon K^* \to \varrho G$. Finally, since $G = h(H) = g(K)$, we see that G is a dense subgroup of the Lindelöf group $g^*(K^*)$ which in its turn is a subgroup of ϱG. $\square$

Corollary 3.2. *Let H be a Raĭkov complete P-group with $w(H) = \aleph_1$. If H is not Lindelöf, then it cannot be embedded as a subgroup into an $\mathbb{R}$-factorizable P-group.*

PROOF: Suppose to the contrary that H is a subgroup of an $\mathbb{R}$-factorizable P-group, but H is not Lindelöf. Clearly, the group H is ω-narrow. Since H is a P-group and $w(H) = \aleph_1$, Theorem 3.1 implies that H is a subgroup of a Lindelöf group, say, H_0. Since H is Raĭkov complete, it must be closed in H_0. Hence, H is Lindelöf. This contradiction completes the proof. ☐

In [18], an example of a Raĭkov complete, ω-narrow, non-Lindelöf P-group H with $w(H) = \aleph_1$ was constructed. Therefore, according to Corollary 3.2, we have:

Corollary 3.3. *There exists an ω-narrow P-group that cannot be embedded as a subgroup into any $\mathbb{R}$-factorizable P-group.*

4. σ-products of topological groups and their subgroups

The cellularity is not monotonous when passing to a subspace — $c(Y)$ can be arbitrarily bigger than $c(X)$, for a subspace Y of a space X. Neither is the cellularity monotonous in topological groups, but the inequality $c(Y) \leq 2^{c(X)}$ holds in this case, for any subgroup Y of a topological group X [17, Theorem 4.28]. We show in Theorem 4.6 that the situation is completely different in the case of σ-products of second-countable groups — their subgroups have countable cellularity, are perfectly κ-normal and $\mathbb{R}$-factorizable.

First we need several auxiliary results regarding continuous retractions defined on Lindelöf subspaces of Σ-products of "small" spaces.

Lemma 4.1. *Let $\Pi = \prod_{i\in I} X_i$ be the product of a family of spaces satisfying $|X_i| \leq \mathfrak{c}$ for each $i \in I$, $p \in \Pi$, and $\Sigma\Pi(p)$ the corresponding Σ-product of this family. Then every Lindelöf subspace H of $\Sigma\Pi(p)$ has an $\aleph_1$-complete lattice of continuous retractions that can be identified with some family of projections of H to subproducts $\Pi_J = \prod_{i\in J} X_i$ with $|J| \leq \mathfrak{c}$.*

PROOF: Let $[I]^{\leq\mathfrak{c}}$ be the family of all subsets J of the index set I satisfying $|J| \leq \mathfrak{c}$. For every $J \in [I]^{\leq\mathfrak{c}}$, we put

$$H(J) = \{x \in H : \mathrm{diff}(x,p) \subseteq J\}.$$

Then we define a family $\mathcal{L} \subseteq [I]^{\leq\mathfrak{c}}$ by

$$\mathcal{L} = \{J \in [I]^{\leq\mathfrak{c}} : p_J(H) = p_J(H(J))\},$$

where $p_J\colon \Pi \to \Pi_J = \prod_{i\in J} X_i$ is the natural projection. It is clear that the restriction of p_J to $H(J)$ is a topological embedding of $H(J)$ to Π_J, for each

$J \in [I]^{\le\mathfrak{c}}$. Identifying $H(J)$ with $p_J(H(J))$, we conclude that the restriction of p_J to H is a continuous retraction of H onto $H(J)$, for each $J \in \mathcal{L}$.

To finish the proof of the theorem, it suffices to establish the following two properties of the family $\mathcal{L}$:

Claim 1. For every $A \in [I]^{\le\mathfrak{c}}$, there exists $J \in \mathcal{L}$ such that $A \subseteq J$.

Claim 2. If $\{J_\alpha : \alpha \in \omega_1\}$ is an increasing subfamily of $\mathcal{L}$, then the set $J = \bigcup_{\alpha\in\omega_1} J_\alpha$ also belongs to $\mathcal{L}$.

It will follow from Claim 1 that the family $\{p_J \restriction H : J \in \mathcal{L}\}$ is $\aleph_1$-directed and generates the topology of H, while Claim 2 will imply that this family of retractions is $\aleph_1$-complete.

Let us start with Claim 1. Since $H \subseteq \Sigma\Pi(p)$ and $|X_i| \le \mathfrak{c}$, for each $i \in I$, it follows that $|p_B(H)| \le \mathfrak{c}$, for each $B \subseteq I$ satisfying $|B| \le \mathfrak{c}$. Take an arbitrary set $A \subseteq I$ with $|A| \le \mathfrak{c}$ and put $A_0 = A$. Suppose that for some $\alpha < \omega_1$, we have defined a sequence $\{A_\nu : \nu < \alpha\}$ of subsets of I and a sequence $\{H_\nu : \nu < \alpha\}$ of subsets of H satisfying the following conditions for all $\nu, \mu < \alpha$:

(i) $A_\nu \subseteq A_\mu$ and $H_\nu \subseteq H_\mu$ if $\nu < \mu$;
(ii) $|A_\nu| \le \mathfrak{c}$ and $|H_\nu| \le \mathfrak{c}$;
(iii) $p_{A_\nu}(H_\nu) = p_{A_\nu}(H)$;
(iv) $\mathrm{diff}(x,p) \subseteq A_{\nu+1}$, for each $x \in H_\nu$.

If α is limit, we put $A_\alpha = \bigcup_{\nu<\alpha} A_\nu$. By (ii), we have that $|A_\alpha| \le \mathfrak{c}$. Since $|p_{A_\alpha}(H)| \le \mathfrak{c}$, there exists a subset H_α of H such that $\bigcup_{\nu<\alpha} H_\nu \subseteq H_\alpha$, $|H_\alpha| \le \mathfrak{c}$, and $p_{A_\alpha}(H_\alpha) = p_{A_\alpha}(H)$. It is easy to see that the sequences $\{A_\nu : \nu \le \alpha\}$ and $\{H_\nu : \nu \le \alpha\}$ satisfy (i)–(iv) at this step.

If $\alpha = \beta + 1$, we put $A_\alpha = A_\beta \cup \bigcup\{\mathrm{diff}(x,p) : x \in H_\beta\}$. By (ii), the set A_α satisfies $|A_\alpha| \le \mathfrak{c}$. Therefore, there exists a subset H_α of H such that $H_\beta \subseteq H_\alpha$, $|H_\alpha| \le \mathfrak{c}$, and $p_{A_\alpha}(H_\alpha) = p_{A_\alpha}(H)$. Again, the sequences $\{A_\nu : \nu \le \alpha\}$ and $\{H_\nu : \nu \le \alpha\}$ satisfy (i)–(iv). This finishes our construction.

We claim that the set $J = \bigcup_{\alpha<\omega_1} A_\alpha$ belongs to $\mathcal{L}$. It is clear from (ii) that $|J| \le \mathfrak{c}$, so it suffices to verify the equality $p_J(H) = p_J(H(J))$. This follows from (iii), (iv), and the fact that the space H is Lindelöf. Indeed, take an arbitrary point $y \in p_J(H)$ and choose $x \in H$ with $p_J(x) = y$. According to (iii) and (iv), the intersection $p_{A_\alpha}^{-1}(p_{A_\alpha}(x)) \cap H(J)$ is non-empty. Since the space $H(J)$ is Lindelöf (as a closed subset of H), we infer that $\bigcap_{\alpha<\omega_1} p_{A_\alpha}^{-1}(p_{A_\alpha}(x)) \cap H(J) \ne \emptyset$ or, equivalently, $p_J^{-1}(y) \cap H(J) \ne \emptyset$. It follows that $p_J(H) = p_J(H(J))$. Claim 1 is proved.

The proof of Claim 2 is almost the same as the final part of the proof of Claim 1. Indeed, let $\{J_\alpha : \alpha < \omega_1\} \subseteq \mathcal{L}$ be an increasing sequence of length $\aleph_1$ of subsets of I. This means, in particular, that the equality $p_{J_\alpha}(H) = p_{J_\alpha}(H(J_\alpha))$ holds for each $\alpha < \omega_1$. Put $J = \bigcup_{\alpha<\omega_1} J_\alpha$ and take an arbitrary point $y \in p_J(H)$. Again, choose $x \in H$ satisfying $p_J(x) = y$. If $\alpha < \omega_1$, the obvious inclusion

$H(J_\alpha) \subseteq H(J)$ and the equality $p_{J_\alpha}(H) = p_{J_\alpha}(H(J_\alpha))$ together imply that the intersection $p_{J_\alpha}^{-1}(p_{J_\alpha}(x)) \cap H(J)$ is non-empty. Since $H(J)$ is Lindelöf, it follows that $H(J) \cap \bigcap_{\alpha<\omega_1} p_{J_\alpha}^{-1}(p_{J_\alpha}(x)) \neq \emptyset$. We conclude that $H(J) \cap p_J^{-1}(y) \neq \emptyset$ and, hence, $y \in p_J(H(J))$. This proves Claim 2 and the lemma. $\square$

Similarly, one can prove the following result (we omit the corresponding argument):

Lemma 4.2. *Let $\Pi = \prod_{i\in I} X_i$ be the product of a family of spaces satisfying $|X_i| \leq \aleph_1$ for each $i \in I$, and $\sigma\Pi(p)$ the corresponding σ-product of this family, for some $p \in \Pi$. Then every Lindelöf subspace H of $\sigma\Pi(p)$ has an $\aleph_1$-complete lattice of continuous retractions that can be identified with some family of projections of H to subproducts $\Pi_J = \prod_{i\in J} X_i$ with $|J| \leq \aleph_1$.*

In the case when the factors of the Σ-product $\Sigma\Pi(p)$ in Lemma 4.1 are second-countable groups and H is a closed subgroup of the corresponding σ-product $\sigma\Pi(p)$, we can find an $\aleph_1$-complete lattice consisting of continuous homomorphic retractions of H onto its subgroups of weight less than or equal to $\mathfrak{c}$. These retractions will automatically be open.

Theorem 4.3. *Let $\Pi = \prod_{i\in I} G_i$ be the product of a family of second-countable topological groups G_i, and $\sigma\Pi$ the corresponding σ-product of the same family of groups. Then every closed subgroup H of $\sigma\Pi$ is Lindelöf and has an $\aleph_1$-complete lattice of continuous open homomorphisms onto topological groups of weight $\leq \mathfrak{c}$. In fact, this lattice consists of continuous homomorphic retractions of H that can be identified with a family of projections of H to subproducts $\Pi_J = \prod_{i\in J} G_i$, where $|J| \leq \mathfrak{c}$.*

PROOF: Since the subproduct Π_J is second-countable (hence, is Lindelöf) for every finite set $J \subseteq I$, the subspace $\sigma\Pi$ of Π is Lindelöf, by a result from [13]. Therefore, the closed subgroup H of $\sigma\Pi$ is also Lindelöf. By assumptions of the theorem, each group G_i has a countable base and, hence, satisfies $|G_i| \leq \mathfrak{c}$. So, we can apply Lemma 4.1 to find an $\aleph_1$-complete lattice $\mathcal{P}$ of projections of H to subproducts Π_J with $|J| \leq \mathfrak{c}$ such that each of these projections can be naturally identified with a retraction of H onto its subgroup. Since each continuous retraction is a quotient mapping, we conclude that the family $\mathcal{P}$ consists of open homomorphisms onto groups of weight $\leq \mathfrak{c}$. $\square$

For a space Y, we write $\mathrm{cel}_\omega(Y) \leq \omega$ if for every family γ of G_δ-sets in Y, there exists a countable subfamily λ of γ such that $\bigcup\lambda$ is dense in $\bigcup\gamma$. The space Y satisfying $\mathrm{cel}_\omega(Y) \leq \omega$ is called *ω-cellular*.

Lemma 4.4. *Let Y be a subspace of a product space $X = \prod_{i\in I} X_i$. Suppose that $\mathrm{cel}_\omega(p_J(Y)) \leq \omega$, for every $J \subseteq I$ with $|J| \leq \aleph_1$. Then $\mathrm{cel}_\omega(Y) \leq \omega$.*

PROOF: For every non-empty set $B \subseteq I$, let p_B be the natural projection of the product space X to $X_B = \prod_{i\in B} X_i$. A non-empty subset P of the space X is called a *canonical* G_δ*-set* if it has the form $P = \prod_{i\in I} P_i$, where each P_i is a G_δ-set in X_i, and $P_i \neq X_i$ for at most countably many indices $i \in I$. The countable set

$$B(P) = \{i \in I : P_i \neq G_i\}$$

is called the *base* of P. It is clear that if $B \subseteq I$ is the base of a canonical G_δ-set P in X, then $P = p_B^{-1} p_B(P)$ and $p_B(P)$ is a G_δ-set in X_B. In fact, $p_J(P)$ is a G_δ-set in X_J, for each $J \subseteq I$.

Suppose that γ is a family of G_δ-sets in Y such that $\bigcup \lambda$ is not dense in $\bigcup \gamma$, for any countable family $\lambda \subseteq \gamma$. By recursion on $\alpha < \omega_1$, we can define families $\{P_\alpha : \alpha < \omega_1\} \subseteq \gamma$ and $\{Q_\alpha : \alpha < \omega_1\}$ satisfying the following conditions for all $\alpha, \beta < \omega_1$:

(i) Q_α is a non-empty G_δ-set in Y and $Q_\alpha \subseteq P_\alpha$;
(ii) $Q_\alpha \cap \overline{\bigcup_{\nu<\alpha} P_\nu} = \emptyset$.

For every $\alpha < \omega_1$, pick a point $x_\alpha \in Q_\alpha$ and take a canonical open set U_α in X such that $x_\alpha \in U_\alpha$ and $U_\alpha \cap P_\nu = \emptyset$, for each $\nu < \alpha$. Then choose a canonical G_δ-set F_α in X such that $x_\alpha \in F_\alpha$ and $F_\alpha \cap Y \subseteq Q_\alpha$. Clearly, there exists a countable set $J_\alpha \subseteq I$ containing the base of F_α such that $U_\alpha = p_{J_\alpha}^{-1} p_{J_\alpha}(U_\alpha)$ and $F_\alpha = p_{J_\alpha}^{-1} p_{J_\alpha}(F_\alpha)$.

Let $J = \bigcup_{\alpha<\omega_1} J_\alpha$. Then $|J| \leq \aleph_1$, and the definition of J implies that $U_\alpha = p_J^{-1} p_J(U_\alpha)$ and $F_\alpha = p_J^{-1} p_J(F_\alpha)$ for each $\alpha < \omega_1$. Evidently, $p_J(F_\alpha)$ is a G_δ-set in X_J and $R_\alpha = p_J(F_\alpha) \cap p_J(Y)$ is a non-empty G_δ-set in $p_J(Y)$, for each $\alpha < \omega_1$. It is also clear that $V_\alpha = p_J(U_\alpha) \cap p_J(Y)$ is a non-empty open set in $p_J(Y)$ since $p_J(x_\alpha) \in V_\alpha$, and we claim that $V_\alpha \cap R_\nu = \emptyset$ for each $\nu < \alpha$.

Indeed, if $\nu < \alpha$, it follows from the equalities $U_\alpha = p_J^{-1} p_J(U_\alpha)$, $F_\nu = p_J^{-1} p_J(F_\nu)$, and $U_\alpha \cap (F_\nu \cap Y) = \emptyset$ that

$$\emptyset = p_J(U_\alpha \cap F_\nu \cap Y) = p_J(U_\alpha) \cap p_J(F_\nu \cap Y) = p_J(U_\alpha) \cap p_J(F_\nu) \cap p_J(Y).$$

This implies immediately that $V_\alpha \cap R_\nu = \emptyset$. In addition, we have that $p_J(x_\alpha) \in V_\alpha \cap R_\alpha \neq \emptyset$, for each $\alpha < \omega_1$. Therefore, the family $\theta = \{R_\alpha : \alpha < \omega_1\}$ of G_δ-sets in $p_J(Y)$ does not contain a countable subfamily whose union is dense in the union of θ. This finishes the proof. □

We say that a Tychonoff space X is an *Efimov space* if for every family γ of G_δ-sets in X, the closure of $\bigcup \gamma$ is a zero-set in X. The class of Efimov spaces is quite wide; it includes arbitrary products of regular second-countable spaces [6], [21] and of Lindelöf Σ-groups [17].

Theorem 4.5. *Let H be a closed subgroup of a σ-product of a family of second-countable topological groups. Then $\mathrm{cel}_\omega(H) \leq \omega$, and H is an Efimov space.*

Proof: Suppose that H is a closed subgroup of $\sigma\Pi$, where $\Pi = \prod_{i\in I} G_i$ is the product of a family of second-countable groups.

First, we show that $\mathrm{cel}_\omega(H) \le \omega$. By Lemma 4.4, it suffices to verify that $\mathrm{cel}_\omega(p_B(H)) \le \omega$ for each $B \subseteq I$ with $|B| \le \aleph_1$. Given such a set B, we apply Theorem 4.3 to find $J \subseteq I$ such that $B \subseteq J$, $|J| \le \mathfrak{c}$, and the restriction of p_J to H is a retraction of H. In particular, $p_J(H)$ is a closed subgroup of H and, hence, of both $\sigma\Pi$ and $\sigma\Pi_J$ (we identify each face Π_J with the corresponding subgroup of Π). It is clear that $p_B(H) \subseteq \sigma\Pi_B$, for each $B \subseteq I$. According to [1, Theorem 1], $\sigma\Pi_J$ is a Lindelöf Σ-space for each $J \subseteq I$ with $|J| \le \mathfrak{c}$. Therefore, so is the closed subgroup $p_J(H)$ of $\sigma\Pi_J$. According to [20, Theorem 2] (see also [17, Theorem 4.14]), the Lindelöf Σ-group $p_J(H)$ satisfies $\mathrm{cel}_\omega(p_J(H)) \le \omega$. Since $p_B(H)$ is a continuous image of $p_J(H)$, we have that $\mathrm{cel}_\omega(p_B(H)) \le \omega$. This implies the inequality $\mathrm{cel}_\omega(H) \le \omega$.

It remains to show that H is an Efimov space. Let γ be a family of G_δ-sets in H. We can assume without loss of generality that every element $P \in \gamma$ has the form $P = p_B^{-1}(E) \cap H$, where B is a countable subset of I, $p_B\colon \Pi \to \Pi_B = \prod_{i\in B} G_i$ is the projection, and E is a G_δ-set in $p_B(H)$. Since $\mathrm{cel}_\omega(H) \le \omega$, we can find a countable subfamily λ of γ and a countable set $B \subseteq I$ such that $\bigcup\lambda$ is dense in $\bigcup\gamma$ and each $P \in \lambda$ has the form $P = p_B^{-1}(E_P) \cap H$, for some G_δ-set E_P in $p_B(H)$. Again, we use Theorem 4.3 to find a set $J \subseteq I$ such that $B \subseteq J$, $|J| \le \mathfrak{c}$, and the restriction $\pi = p_J \restriction H$ is a retraction of H. Since every retraction is quotient, the homomorphism π is open. It follows from $B \subseteq J$ that for every element $P \in \lambda$, there exists a G_δ-set F_P in $p_J(H)$ such that $P = p_J^{-1}(F_P) \cap H$. Arguing as above, we conclude that $p_J(H)$ is a closed subgroup of $\sigma\Pi_J$ and, therefore, is a Lindelöf Σ-group. It is known, however, that every Lindelöf Σ-group is an Efimov space [20]. Let $D = \bigcup\{F_P : P \in \lambda\}$. Since $\bigcup\lambda$ is dense in $\bigcup\gamma$ and the homomorphism π is open, we have:

$$\overline{\bigcup\gamma} = \overline{\bigcup\lambda} = \overline{\pi^{-1}(D)} = \pi^{-1}(\overline{D}),$$

where the closure of D is taken in $p_J(H)$. Since D is the union of a family of G_δ-sets in $p_J(H)$ and the group $p_J(H)$ is an Efimov space, it follows that $\overline{D}$ is a zero-set in $p_J(H)$. Therefore, $\overline{\bigcup\gamma}$ is a zero-set in H. The theorem is proved. $\Box$

We recall that a Tychonoff space X is called *perfectly κ-normal* (or an *Oz*-space, see [Bla]) if the closure of every open set in X is a zero-set. Clearly, every Efimov space is perfectly κ-normal, but not vice versa. We also need one more concept. A subspace Y of a Tychonoff space X is said to be *z-embedded* in X if for every zero-set Z in Y, there exists a zero-set Z' in X such that $Z' \cap Y = Z$. The following result is an easy corollary of Theorem 4.5.

Theorem 4.6. *Let K be an arbitrary subgroup of the σ-product of a family of second-countable topological groups. Then K has countable cellularity, is perfectly κ-normal and $\mathbb{R}$-factorizable.*

Proof: Let K be a subgroup of $\sigma\Pi$, where Π is the product of a family of second-countable topological groups. Denote by H the closure of K in $\sigma\Pi$. It follows from Theorem 4.5 that $c(H) \le \mathrm{cel}_\omega(H) \le \omega$. Since K is dense in H, we conclude that $c(K) = c(H) \le \omega$.

Again, by Theorem 4.5, H is an Efimov space and, therefore, is perfectly κ-normal. It follows from [2] that every dense subspace of H is perfectly κ-normal and z-embedded in H, which is the case of K. According to [17, Theorem 5.17], every z-embedded subgroup of an $\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable as well. Since Lindelöf topological groups are $\mathbb{R}$-factorizable (see [17, Theorem 5.5]) and the group $H = \overline{K}$ is Lindelöf by Theorem 4.3, we conclude that the group K is also $\mathbb{R}$-factorizable. □

Corollary 4.7. *Let $\Pi = \prod_{i\in I} G_i$ be the product of a family of second-countable topological groups. Then every subgroup of $\sigma\Pi$ is z-embedded in Π.*

Proof: It follows from [17, Theorem 5.16] that if a subgroup K of an arbitrary topological group G is $\mathbb{R}$-factorizable, then K is z-embedded in G. It remains to refer to Theorem 4.6. □

5. Open problems

Problem 5.1. *Denote by $\mathfrak{B}(\aleph_1)$ the class of (regular, Tychonoff) spaces X with the property that the product $X \times Y$ is pseudo-$\aleph_1$-compact, for every pseudo-$\aleph_1$-compact (regular, Tychonoff) space Y. Find an internal characterization of the spaces in the class $\mathfrak{B}(\aleph_1)$.*

Our last problem is motivated by Corollary 2.7 and the results of Section 4.

Problem 5.2. *Let X and Y be pseudo-$\aleph_1$-compact subspaces of the σ-product $\sigma\Pi(p)$ of a family $\{X_i : i \in I\}$ of regular Lindelöf P-spaces satisfying $|X_i| \le \aleph_1$, for each $i \in I$, where $\Pi = \prod_{i\in I} X_i$ carries the ω-box topology. Is the product space $X \times Y$ pseudo-$\aleph_1$-compact?*

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