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Products and projective limits of function spaces

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Abstract. We introduce a notion of a product and projective limit of function spaces. We show that the Choquet boundary of the product space is the product of Choquet boundaries. Next we show that the product of simplicial spaces is simplicial. We also show that the maximal measures on the product space are exactly those with maximal projections. We show similar characterizations of the Choquet boundary and the space of maximal measures for the projective limit of function spaces under some additional assumptions and we prove that the projective limit of simplicial spaces is simplicial.

Keywords: Choquet theory, function space, product, projective limit, simplicial space

Classification: Primary 46A55; Secondary 26B25, 46A32

1. Introduction

Let $\{X_i\}_{i\in I}$ be a family of Choquet simplexes. We can construct a compact convex set X as the state space of the space of all continuous multiaffine functions on $\prod_{i\in I} X_i$. It has been shown in [6] and [16] that X itself is a simplex with extreme points being the evaluation functionals at the points $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ with $x_i \in \operatorname{ext} X_i$ for every $i \in I$. Generalizations to products of arbitrary compact convex sets followed (see [11], [18]). Characterization of maximal measures on the product of two compact convex sets, as the measures whose every 'projection' is a maximal measure, appeared later in [3] and [2].

In Section 3 we transfer these results to the context of function spaces. We first introduce a notion of a product of function spaces with several special products. We compare these products and prove appropriate associative laws. Then we show that the Choquet boundary of a product space is the product of Choquet boundaries. We prove that the product is simplicial if and only if every of the original spaces is simplicial. Finally we show that maximal measures on the product of arbitrary many spaces are exactly those with maximal projections.

In Section 4 we transfer known results from [6] and [13] on projective limits of compact convex sets to function spaces. We use Grossman's definition of the projective limit of function spaces from [10] and prove that the projective limit of simplicial spaces is simplicial. We also derive similar characterizations of the Choquet boundary and maximal measures as in the case of product of function spaces.

2. Preliminaries

Let K be a compact Hausdorff space. We denote by $\mathcal{C}(K)$ the space of all continuous functions on K, by $\mathcal{M}^+(K)$ the set of all positive Radon measures on K and by $\mathcal{M}^1(K)$ the set of all probability Radon measures on K. Let ε_x stand for the Dirac measure at $x \in K$. We say that a linear subspace $\mathcal{H}$ of $\mathcal{C}(K)$ is a *function space*, if it contains 1_K (the function identically 1 on K) and separates the points of K. Let $\mathcal{M}_x(\mathcal{H})$ be the set of all *$\mathcal{H}$-representing measures* for $x \in K$, i.e.,

$$\mathcal{M}_x(\mathcal{H}) := \{\mu \in \mathcal{M}^1(K) : h(x) = \int_K h\, d\mu \ \text{ for every } \ h \in \mathcal{H}\}.$$

The set $\operatorname{Ch}_{\mathcal{H}} K := \{x \in K : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$ is called the *Choquet boundary* of $\mathcal{H}$. It is a G_δ-set if K is metrizable (see [1, Corollary I.5.17]). We denote by $\nabla_{\mathcal{H}} K$ the *Šilov boundary* of $\mathcal{H}$ (see [1, p. 50] for definition) and we remark that $\nabla_{\mathcal{H}} K$ is equal to the closure of $\operatorname{Ch}_{\mathcal{H}} K$ (see [1, Theorem I.5.15] for the proof). A non-empty closed set $E \subset K$ is called *$\mathcal{H}$-extremal*, if $\operatorname{spt} \mu \subset E$ for every $x \in E$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Finally, for every $x \in K$ we denote $F_x(\mathcal{H}) := \bigcup\{\operatorname{spt} \mu : \mu \in \mathcal{M}_x(\mathcal{H})\}$.

We define the space $\mathcal{A}^c(\mathcal{H})$ of all continuous *$\mathcal{H}$-affine functions* as the space of all continuous functions on K satisfying the following formula:

$$f(x) = \int_K f\, d\mu \quad \text{for each} \quad x \in K \quad \text{and} \quad \mu \in \mathcal{M}_x(\mathcal{H}).$$

Clearly $\mathcal{A}^c(\mathcal{H})$ is a uniformly closed function space with $\mathcal{M}_x(\mathcal{H}) = \mathcal{M}_x(\mathcal{A}^c(\mathcal{H}))$ for every $x \in K$.

Here we recall main examples of function spaces:

(a) *Convex case* - Let X be a compact convex subset of a locally convex space and let $\mathcal{H}$ be the linear space $A(X)$ of all continuous affine functions on X. The Choquet boundary is the set $\operatorname{ext} X$ of all extreme points of X.

(b) *Harmonic case* - Let U be a bounded open subset of the Euclidean space $\mathbb{R}^n$ and let the corresponding function space $H(U)$ be the family of all continuous functions on $\overline{U}$ which are harmonic on U. The Choquet boundary coincides with the set $\partial_{\text{reg}} U$ of all regular points.

An upper bounded Borel function f is called *$\mathcal{H}$-convex* if $f(x) \le \mu(f)$ for any $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Let $\mathcal{K}^c(\mathcal{H})$ denote the family of all continuous $\mathcal{H}$-convex functions on K. Notice that the space $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is uniformly dense in $\mathcal{C}(K)$ due to the lattice version of the Stone-Weierstrass theorem.

The convex cone $\mathcal{K}^c(\mathcal{H})$ determines a partial ordering $\preceq_{\mathcal{H}}$ (called the *Choquet ordering*) on the space $\mathcal{M}^+(K)$:

$$\mu \preceq_{\mathcal{H}} \nu \quad \text{if} \quad \mu(f) \le \nu(f) \quad \text{for each} \quad f \in \mathcal{K}^c(\mathcal{H}).$$

(If the space $\mathcal{H}$ is obvious, we simply write $\mu \preceq \nu$.)

We remark that $\mu \preceq \nu$ if and only if $\mu(f) \le \nu(f)$ for every $f \in \mathcal{W}(\mathcal{H})$, where $\mathcal{W}(\mathcal{H})$ is the smallest family of functions containing $\mathcal{H}$ and closed with respect to taking supremum of finite families.

For any measure $\mu \in \mathcal{M}^+(K)$ there exists a maximal measure ν with $\mu \preceq \nu$. In particular, for every $x \in K$ there exists a maximal $\mathcal{H}$-representing measure. This is the content of the Choquet-Bishop-de Leeuw theorem [1, Theorem I.5.19].

If K is metrizable, then a measure $\mu \in \mathcal{M}^+(K)$ is maximal if and only if $\mu(K \setminus \operatorname{Ch}_{\mathcal{H}} K) = 0$. In nonmetrizable spaces every maximal measure μ satisfies $\mu(G) = 0$ for any G_δ-set disjoint from $\operatorname{Ch}_{\mathcal{H}} K$ (see [1, Proposition I.5.22]).

Theorem 2.1. *Let $\mu \in \mathcal{M}^+(K)$. Then the following assertions are equivalent:*

(i) *μ is maximal,*
(ii) *there exists a set $S \subset \mathcal{C}(K)$ separating points of K such that every function from S is constant on $F_x(\mathcal{H})$ for μ-a.e. $x \in K$,*
(iii) *every function from $\mathcal{C}(K)$ is constant on $F_x(\mathcal{H})$ for μ-a.e. $x \in K$.*

Proof: See [2, Proposition 2]. □

Proposition 2.2. *Let $(K', \mathcal{G})$ be a function space and $\rho : K \to K'$ a continuous mapping such that $F_{\rho(x)}(\mathcal{G}) \subset \rho(F_x(\mathcal{H}))$ for every $x \in \overline{\operatorname{Ch}_{\mathcal{H}} K}$. Then the image measure $\rho\mu$ is a maximal measure on K' for every maximal measure μ on K.*

Proof: See [2, Corollary 3]. □

If for every $x \in K$ the maximal $\mathcal{H}$-representing measure is uniquely determined, we say that $\mathcal{H}$ is *simplicial*. In the convex case it is equivalent to say that X is a *Choquet simplex*. We denote the unique maximal measure representing $x \in K$ by δ_x.

We say that $\mathcal{H}$ has the *weak Riesz interpolation property* (*W.R.I.P.*), if for every $a_1, a_2, b_1, b_2 \in \mathcal{H}$ such that $a_i < b_j$, $i, j = 1, 2$, there exists $c \in \mathcal{H}$ such that $a_i < c < b_j$, $i, j = 1, 2$. It can be shown that $\mathcal{H}$ is simplicial if and only if $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. (see [1, Corollary II.3.11] or [4, Theorem 3.3]).

For a function $f : K \to \mathbb{R}$ we define the *upper envelope* f^* as

$$f^*(x) := \inf\{h(x) : h \ge f,\, h \in \mathcal{H}\}, \quad x \in K,$$

and the *lower envelope* as $f_* := -(-f)^*$. We denote $\widehat{\mathcal{H}} := \{f \in \mathcal{C}(K) : f_* = f^*\}$. It is true that $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$. By [1, Proposition I.5.9 and Corollary I.5.10], we have:

Proposition 2.3. *Let $\mu \in \mathcal{M}^+(K)$. Then the following statements are equivalent:*

(i) *μ is maximal,*
(ii) *$\mu(f) = \mu(f^*)$ for every $f \in \mathcal{C}(K)$,*
(iii) *$\mu(k) = \mu(k^*)$ for every $k \in \mathcal{K}^c(\mathcal{H})$.*

Corollary 2.4. *Let $x \in K$. Then the following statements are equivalent:*

(i) $x \in \operatorname{Ch}_{\mathcal{H}} K$,
(ii) $f(x) = f^*(x)$ *for every* $f \in \mathcal{C}(K)$,
(iii) $k(x) = k^*(x)$ *for every* $k \in \mathcal{K}^c(\mathcal{H})$.

If f and g are functions on K, we write $f \vee g$ for their pointwise maximum and $f \wedge g$ for minimum.

Now we introduce a notation concerning cartesian products: Let $\{E_i\}_{i \in I}$ be a family of topological spaces and let $E := \prod_{i \in I} E_i$ be their cartesian product with the usual topology. We use the convention $\prod_{i \in \emptyset} E_i := \{\emptyset\}$.

Let $J \subset I$. The natural projection from E onto $\prod_{i \in J} E_i$ is denoted by π_J. Let $A \subset E$ and $z \in \prod_{i \in I \setminus J} E_i$. We denote by $\pi_J^z(A)$ the set $\{x \in \prod_{i \in J} E_i : (x, z) \in A\}$.

We use a similar notation for functions. Let $f : E \to \mathbb{R}$ and $y \in \prod_{i \in I \setminus J} E_i$. Then $\pi_J^y(f) : \prod_{i \in J} E_i \to \mathbb{R}$ is defined as

$$\pi_J^y(f)(x) := f(x, y), \quad x \in \prod_{i \in J} E_i.$$

In case f is independent on y, we use notation $\pi_J(f)$.

Finally, for $f_1 : E_1 \to \mathbb{R}$ and $f_2 : E_2 \to \mathbb{R}$ we define $f_1 \otimes f_2 : E_1 \times E_2 \to \mathbb{R}$ by

$$(f_1 \otimes f_2)(x, y) = f_1(x) f_2(y), \quad x \in E_1,\, y \in E_2.$$

We conclude this section with known results on products of Radon measures: Let $\{(K_i, \mathcal{S}_i, \mu_i)\}_{i \in I}$ be a family of compact Hausdorff spaces with Radon probability measures. There exists a unique product measure μ on $\prod_{i \in I} K_i$ with $\mu(\prod_{i \in I} E_i) = \prod_{i \in I} \mu_i(E_i)$, whenever $E_i \in \mathcal{S}_i$ for each $i \in I$ and $E_i \neq K_i$ for finitely many $i \in I$ (see [12, Chapter VI, Theorem 5.3]). By [8, Theorem 417Q], μ can be uniquely extended to a Radon measure $\bigotimes_{i \in I} \mu_i$. We call this measure the *Radon product measure*. Radon products satisfy associative law (see [8, Theorem 417J]) and we can also use Fubini's theorem (see [8, Theorem 417H]). Finally we remark that if two Radon measures coincide on the cylinder sets $\prod_{i \in I} E_i$, where $E_i \subset K_i$ is Borel for each $i \in I$ and $E_i \neq K_i$ for finitely many $i \in I$, then they are equal (see [12, Chapter I, Proposition 5.3] and the proof of [8, Corollary 417F]).

3. Products of function spaces

3.1 Definitions and relations.

Definition 3.1. Let $\{(K_i, \mathcal{H}_i)\}_{i\in I}$ be a family of function spaces and let $K := \prod_{i\in I} K_i$. We define

(a) *algebraic tensor product* $\bigodot_{i\in I} \mathcal{H}_i$ as the linear span of the set

$$\{h_1 \otimes \ldots \otimes h_n \otimes 1_{\prod\{K_i : i\in I\setminus\{i_1,\ldots,i_n\}\}} : h_k \in \mathcal{H}_{i_k},\, i_k \in I,\, 1 \le k \le n,\, n \in \mathbb{N}\},$$

(b) *injective tensor product* $\bigotimes_{i\in I} \mathcal{H}_i$ as the closure of $\bigodot_{i\in I} \mathcal{H}_i$,

(c) *multiaffine product* by

$$\boxtimes_{i\in I} \mathcal{H}_i := \{f \in \mathcal{C}(K) : \pi_j^y(f) \in \mathcal{H}_j \text{ for all } j \in I \text{ and } y \in \prod_{i\in I\setminus\{j\}} K_i\}.$$

We say that a function space $\mathcal{H}$ on K is a *product* of function spaces $\mathcal{H}_i$, $i \in I$, if

$$\bigodot_{i\in I} \mathcal{H}_i \subset \mathcal{H} \subset \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i).$$

In case I is an empty set, we put all products to be equal $\{\emptyset\}$.

Remark 3.2. It can be shown, that $\mathcal{H}_1 \odot \mathcal{H}_2$ is really the 'algebraic tensor product', and if $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed, i.e., Banach spaces, then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is their 'weak (injective) tensor product' (see [19, 20.5.5]). If $\mathcal{H}_i = A(X_i)$ for some compact convex sets X_i, $i \in I$, then $\boxtimes_{i\in I}\mathcal{H}_i$ is the space of all continuous multiaffine functions on K.

Example 3.3. Let $U_1 \subset \mathbb{R}^m$, $U_2 \subset \mathbb{R}^n$, be bounded open sets. We take $\mathcal{H}_i := H(U_i)$, $i = 1, 2$ (see Example (b) in Section 2). If $\mathcal{H}$ is a product of $\mathcal{H}_i$, $i = 1, 2$, then $\mathcal{H} \subset H(U_1 \times U_2)$. Indeed, choose $h \in \mathcal{H} \subset \mathcal{A}^c(\mathcal{H}_1) \boxtimes \mathcal{A}^c(\mathcal{H}_2) = H(U_1) \boxtimes H(U_2)$. Then we have

$$\Delta h(x_1, x_2) = \Delta\pi_1^{x_2}(h)(x_1) + \Delta\pi_2^{x_1}(h)(x_2) = 0, \quad x_1 \in U_1,\ x_2 \in U_2.$$

However, even the largest product does not have to contain all harmonic functions on the cartesian product. Consider $U_i := (0, 1) \subset \mathbb{R}$, $i = 1, 2$. Then $H(U_i) = A(\overline{U_i})$, $i = 1, 2$. So every product consists only of biaffine functions. Now take $f(x, y) := x^2 - y^2$ for $x, y \in [0, 1]$. Clearly, f is harmonic, but not biaffine.

Proposition 3.4. *The following assertions hold.*

(i) $\bigodot_{i\in I} \mathcal{H}_i \subset \boxtimes_{i\in I}\mathcal{H}_i$.

(ii) *If all $\mathcal{H}_i$ are closed, then $\bigodot_{i\in I} \mathcal{H}_i \subset \bigotimes_{i\in I} \mathcal{H}_i \subset \boxtimes_{i\in I}\mathcal{H}_i$. Moreover, $\boxtimes_{i\in I}\mathcal{H}_i$ is closed.*

(iii) *If $\mathcal{H}_j$ is not closed for some $j \in I$, then $\bigodot_{i\in I} \mathcal{H}_i \subsetneq \bigotimes_{i\in I} \mathcal{H}_i$ and $\bigotimes_{i\in I} \mathcal{H}_i \not\subset \boxtimes_{i\in I}\mathcal{H}_i$.*

Proof: Statement (i) and the first inclusion in (ii) are trivial. Since (i) holds, the second inclusion in (ii) will be proved if we show that $\boxtimes_{i\in I}\mathcal{H}_i$ is closed. So let $\{f_n\}_{n\in\mathbb{N}} \subset \boxtimes_{i\in I}\mathcal{H}_i$ be such that $f_n \rightrightarrows f \in \mathcal{C}(K)$. Further, let $j \in I$ and $y \in \prod_{i\in I\setminus\{j\}} K_i$. Then $\pi_j^y(f_n) \rightrightarrows \pi_j^y(f)$, and since $\pi_j^y(f_n) \in \mathcal{H}_j$ for each n and $\mathcal{H}_j$ is closed, we have $\pi_j^y(f) \in \mathcal{H}_j$. Thus $f \in \boxtimes_{i\in I}\mathcal{H}_i$.

Using previous inclusions, it suffices to find $f \in (\bigotimes_{i\in I}\mathcal{H}_i)\setminus(\boxtimes_{i\in I}\mathcal{H}_i)$ to prove (iii). Let $j \in I$ be such that $\mathcal{H}_j$ is not closed and put $K' := \prod_{i\in I\setminus\{j\}} K_i$. There are functions $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_j$ such that $h_n \rightrightarrows h \notin \mathcal{H}_j$. Then also $h_n \otimes 1_{K'} \rightrightarrows h \otimes 1_{K'}$. Since $h_n \otimes 1_{K'} \in \bigodot_{i\in I}\mathcal{H}_i$ for every $n \in \mathbb{N}$, we have $h \otimes 1_{K'} \in \bigotimes_{i\in I}\mathcal{H}_i$. But $\pi_j(h \otimes 1_{K'}) = h \notin \mathcal{H}_j$, therefore $h \otimes 1_{K'} \notin \boxtimes_{i\in I}\mathcal{H}_i$. $\square$

Remark 3.5. Using previous proposition, we can see that all products defined in Definition 3.1 are indeed function spaces, since they are linear spaces and contain algebraic tensor product, which contains constants and separates points.

In the rest of this subsection we will show that the two inclusions in Proposition 3.4(ii) may be proper.

Example 3.6. Let $K_i := [0,1] \subset \mathbb{R}$, $\mathcal{H}_i := \mathcal{C}(K_i)$, $i = 1,2$, and denote $K := K_1 \times K_2$. The functions of $\mathcal{H}_1 \odot \mathcal{H}_2$ are of the form $\sum_{j=1}^n f_1^j \otimes f_2^j$, where $f_i^j \in \mathcal{C}(K_i)$, $i = 1,2$, $j = 1,\dots,n$, $n \in \mathbb{N}$. Since $\mathcal{H}_1 \odot \mathcal{H}_2$ contains all polynomials, we have $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{C}(K)$. However $\mathcal{H}_1 \odot \mathcal{H}_2 \subsetneq \mathcal{C}(K)$, as can be seen by considering the function $f(x,y) := e^{xy}$, $x \in K_1$, $y \in K_2$.

This example also shows that algebraic tensor product of closed function spaces does not have to be closed.

Definition 3.7. A Banach space E is said to have the *approximation property*, if, for every compact set $C \subset E$ and every $\varepsilon > 0$, there is a continuous linear operator $T : E \to E$ of finite rank so that $\|Tx - x\| < \varepsilon$ for every $x \in C$.

(We refer the reader to [14, Chapter 7] for more information on the approximation property.)

Theorem 3.8 (Namioka-Phelps). *The following statements are equivalent.*

(i) *For every two compact convex subsets X_1, X_2 of locally convex Hausdorff spaces is $A(X_1) \otimes A(X_2) = A(X_1) \boxtimes A(X_2)$.*
(ii) *Every Banach space has the approximation property.*

Proof: See [18, Theorem 2.4 and the subsequent remark]. $\square$

Using Theorem 3.8 and Enflo's counterexample [7] of a Banach space not having the approximation property, we may state the following:

Corollary 3.9. *There exist compact convex sets X_1 and X_2 such that*

$$A(X_1) \otimes A(X_2) \subsetneq A(X_1) \boxtimes A(X_2).$$

3.2 Associative laws. In order to be able to use products defined above effectively, we need to establish 'associative laws' for them.

Definition 3.10. We say, that $\{J_\gamma\}_{\gamma\in\Gamma}$ is a *partition* of a set I, if $\bigcup_{\gamma\in\Gamma} J_\gamma = I$ and $J_\alpha \cap J_\beta = \emptyset$ for every $\alpha, \beta \in \Gamma$ such that $\alpha \neq \beta$.

To the end of this subsection, let $\{(K_i, \mathcal{H}_i)\}_{i\in I}$ be a family of function spaces and $\{J_\gamma\}_{\gamma\in\Gamma}$ a partition of I. In the following, we naturally identify spaces $\mathcal{C}(\prod_{i\in I} K_i)$ and $\mathcal{C}(\prod_{\gamma\in\Gamma}(\prod_{i\in J_\gamma} K_i))$.

Proposition 3.11. *The following assertions hold:*

(i) $\bigodot_{i\in I}\mathcal{H}_i = \bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma}\mathcal{H}_i)$,
(ii) $\mathcal{A}^c(\bigodot_{i\in I}\mathcal{H}_i) = \mathcal{A}^c(\bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma}\mathcal{H}_i))$.

Proof: To prove (i), it clearly suffices to show, that the generating functions of both spaces are the same. Function f is a generating function of $\bigodot_{i\in I}\mathcal{H}_i$, if

$$f = h_1^1 \otimes \ldots \otimes h_1^{m_1} \otimes \ldots \otimes h_n^1 \otimes \ldots \otimes h_n^{m_n} \otimes 1_{\prod\{K_i : i\in I\setminus\{i_1^1,\ldots,i_n^{m_n}\}\}},$$

for some $h_k^l \in \mathcal{H}_{i_k^l}$, $i_k^l \in J_{\gamma_k}$, $l = 1,\ldots,m_k$, $k = 1,\ldots,n$. Since

$$f_k := h_k^1 \otimes \ldots \otimes h_k^{m_k} \otimes 1_{\prod\{K_i : i\in J_{\gamma_k}\setminus\{i_k^1,\ldots,i_k^{m_k}\}\}} \in \bigodot_{i\in J_{\gamma_k}}\mathcal{H}_i \quad \text{for each} \quad k = 1,\ldots,n,$$

we have

$$f = f_1 \otimes \ldots \otimes f_n \otimes 1_{\prod\{K_i : i\in I\setminus(J_{\gamma_1}\cup\ldots\cup J_{\gamma_n})\}},$$

which is a generating function of $\bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma}\mathcal{H}_i)$. Reverting the proof we obtain the converse inclusion.

Assertion (ii) follows from (i) and the fact that $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$. $\square$

Proposition 3.12. *The following assertions hold:*

(i) $\bigotimes_{i\in I}\mathcal{H}_i = \bigotimes_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma}\mathcal{H}_i)$,
(ii) $\mathcal{A}^c(\bigotimes_{i\in I}\mathcal{H}_i) = \mathcal{A}^c(\bigotimes_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma}\mathcal{H}_i))$.

Proof: Using Proposition 3.11, we have

$$\bigotimes_{i\in I}\mathcal{H}_i = \overline{\bigodot_{i\in I}\mathcal{H}_i} = \overline{\bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma}\mathcal{H}_i)} \subset \overline{\bigodot_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma}\mathcal{H}_i)} = \bigotimes_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma}\mathcal{H}_i).$$

For the converse inclusion, it suffices to prove $\bigodot_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma}\mathcal{H}_i) \subset \bigotimes_{i\in I}\mathcal{H}_i$, since the latter space is closed. Let f be a generating function of $\bigodot_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma}\mathcal{H}_i)$. We can write

$$f = f_1 \otimes \ldots \otimes f_n \otimes 1_{\prod\{K_i : i\in I\setminus(J_{\gamma_1}\cup\ldots\cup J_{\gamma_n})\}},$$

where $f_i \in \bigotimes_{j\in J_{\gamma_i}} \mathcal{H}_j$, $i = 1, \dots, n$. We may assume that $f_i > 0$, $i = 1, \dots, n$ (otherwise we write $f_i = (\|f_i\| + 1) - (\|f_i\| + 1 - f_i)$ and use distributive law). Denote $M := \max_{i=1,\dots,n} \|f_i\|$. Now choose $0 < \varepsilon < 1$ so that $f_i > \varepsilon$, $i = 1, \dots, n$. For each f_i we can find $h_i \in \bigodot_{j\in J_{\gamma_i}} \mathcal{H}_j$ such that $f_i - \varepsilon < h_i < f_i$. We define

$$h := h_1 \otimes \dots \otimes h_n \otimes 1_{\prod\{K_i : i\in I\setminus(J_{\gamma_1}\cup\dots\cup J_{\gamma_n})\}} \in \bigodot_{i\in I} \mathcal{H}_i,$$

(we used Proposition 3.11) and compute

$$\begin{aligned}
\|f - h\| &= \sup_{x_1\in\prod_{i\in J_{\gamma_1}} K_i} \dots \sup_{x_n\in\prod_{i\in J_{\gamma_n}} K_i} \Big(\prod_{i=1}^n f_i(x_i) - \prod_{i=1}^n h_i(x_i)\Big) \\
&< \sup_{x_1\in\prod_{i\in J_{\gamma_1}} K_i} \dots \sup_{x_n\in\prod_{i\in J_{\gamma_n}} K_i} \Big(\prod_{i=1}^n f_i(x_i) - \prod_{i=1}^n (f_i(x_i) - \varepsilon)\Big) \\
&= \sup_{x_1\in\prod_{i\in J_{\gamma_1}} K_i} \dots \sup_{x_n\in\prod_{i\in J_{\gamma_n}} K_i} \varepsilon\Big(\sum_{k=1}^n (-1)^{k-1}\varepsilon^{k-1} \sum_{|\alpha|=n-k} \prod_{i=1}^{n-k} f_{\alpha_i}(x_{\alpha_i})\Big) \\
&\le \varepsilon\Big(\sum_{k=1}^n \sum_{|\alpha|=n-k} \prod_{i=1}^{n-k} \|f_{\alpha_i}\|\Big) \le \varepsilon\Big(\sum_{k=0}^{n-1} \binom{n}{k} M^k\Big).
\end{aligned}$$

Since ε is arbitrary, we conclude that $f \in \bigotimes_{i\in I} \mathcal{H}_i$.

Assertion (ii) follows from (i) and the fact that $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$. □

Proposition 3.13. *The following assertions hold:*

(i) $\boxtimes_{i\in I}\mathcal{H}_i = \boxtimes_{\gamma\in\Gamma}(\boxtimes_{i\in J_\gamma}\mathcal{H}_i)$,
(ii) $\mathcal{A}^c(\boxtimes_{i\in I}\mathcal{H}_i) = \mathcal{A}^c(\boxtimes_{\gamma\in\Gamma}(\boxtimes_{i\in J_\gamma}\mathcal{H}_i))$.

PROOF: Let $f \in \boxtimes_{i\in I}\mathcal{H}_i$. Pick $\gamma_0 \in \Gamma$ and $k' \in \prod_{i\in I\setminus J_{\gamma_0}} K_i$. We want to prove that $\pi^{k'}_{J_{\gamma_0}}(f) \in \boxtimes_{i\in J_{\gamma_0}}\mathcal{H}_i$, i.e., that $\pi^{k''}_j(\pi^{k'}_{J_{\gamma_0}}(f)) \in \mathcal{H}_j$ for every $j \in J_{\gamma_0}$ and $k'' \in \prod_{i\in J_{\gamma_0}\setminus\{j\}} K_i$. But this is true, since $\pi^{k''}_j(\pi^{k'}_{J_{\gamma_0}}(f)) = \pi^{(k',k'')}_j(f) \in \mathcal{H}_j$. Conversely, let $f \in \boxtimes_{\gamma\in\Gamma}(\boxtimes_{i\in J_\gamma}\mathcal{H}_i)$. Pick $j \in I$ and $k \in \prod_{i\in I\setminus\{j\}} K_i$. Then $j \in J_{\gamma_0}$ for some $\gamma_0 \in \Gamma$. Using the assumption, we have

$$\pi^k_j(f) = \pi_j^{\pi_{J_{\gamma_0}\setminus\{j\}}(k)}(\pi_{J_{\gamma_0}}^{\pi_{I\setminus J_{\gamma_0}}(k)}(f)) \in \mathcal{H}_j.$$

Assertion (ii) follows from (i) and the fact that $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$. □

From now on, we consider $(K, \mathcal{H})$ to be a product of $(K_i, \mathcal{H}_i)$, $i \in I$, unless said otherwise.

3.3 Representing measures.

Notation 3.14. Let $J \subset I$. We denote by $\mathcal{H}_J$ the space of all functions from $\mathcal{H}$ depending on coordinates from J, i.e.,

$$\mathcal{H}_J := \{h \in \mathcal{H} : x, y \in K,\ \pi_J(x) = \pi_J(y) \Rightarrow h(x) = h(y)\},$$

and let $\mathcal{H}_f$ be the space of all functions from $\mathcal{H}$ depending on a finite number of coordinates, i.e.,

$$\mathcal{H}_f := \{h \in \mathcal{H} : \exists\, J \subset I \ \text{ finite, so that } \ h \in \mathcal{H}_J\}.$$

Observation 3.15. *Using the above notation, we observe:*

(a) $I_1 \subset I_2 \subset I,\ h \in \mathcal{H}_{I_1} \ \Rightarrow\ h \in \mathcal{H}_{I_2}$,
(b) $h \in \mathcal{H}_J \ \Leftrightarrow\ h = \pi_J(h) \otimes 1_{\prod\{K_i : i \in I \setminus J\}}$,
(c) $\mu \in \mathcal{M}^+(K),\ h \in \mathcal{H}_J \ \Rightarrow\ \mu(h) = (\pi_J\mu)(\pi_J(h))$,
(d) *$\mathcal{H}_f$ is a product of $\mathcal{H}_i$, $i \in I$.*

Proposition 3.16. *Let us assume either*

(a) $\mathcal{H} \subset \bigotimes_{i\in I} \mathcal{H}_i$, *or*
(b) $\mathcal{H} = \boxtimes_{i\in I} \mathcal{H}_i$.

Then $\mathcal{H}_f$ is dense in $\mathcal{H}$.

Proof: Assuming (a), conclusion is trivial, since $\bigodot_{i\in I} \mathcal{H}_i \subset \mathcal{H}_f$. Assuming (b), we can use the same technique as in the proof of [16, Theorem 3.1] or [6, Lemma 4]. □

Corollary 3.17. *$\mathcal{C}_f(K)$ is dense in $\mathcal{C}(K)$.*

Proof: Notice that $\mathcal{C}(K) = \boxtimes_{i\in I} \mathcal{C}(K_i)$ and use Proposition 3.16(b). □

Example 3.18. The conclusion of Proposition 3.16 does not have to be true for all products. Suppose we have $f \in (\boxtimes_{i\in I} \mathcal{H}_i) \setminus (\bigotimes_{i\in I} \mathcal{H}_i)$, which does not depend on finitely many coordinates. Let $\mathcal{H}$ be the linear span of $\bigodot_{i\in I} \mathcal{H}_i \cup \{f\}$. Then $\mathcal{H}_f = \bigodot_{i\in I} \mathcal{H}_i$, but $f \notin \overline{\mathcal{H}_f}$.

Now we construct such a function f. Let $(K_i, \mathcal{H}_i) := (X_i, A(X_i))$, $i = 1, 2$, be as in Corollary 3.9. Then there is $f_1 \in (\mathcal{H}_1 \boxtimes \mathcal{H}_2) \setminus (\mathcal{H}_1 \otimes \mathcal{H}_2)$. This function is not constant with respect to any of the two coordinates, since $f_1 \notin \mathcal{H}_1 \odot \mathcal{H}_2$. Set $\mathcal{H}_{2n+1} := \mathcal{H}_1$, $\mathcal{H}_{2n+2} := \mathcal{H}_2$, $n \in \mathbb{N}$, and let $f_{n+1} := f_1$ be the function from $(\mathcal{H}_{2n+1} \boxtimes \mathcal{H}_{2n+2}) \setminus (\mathcal{H}_{2n+1} \otimes \mathcal{H}_{2n+2})$ for every $n \in \mathbb{N}$. Set

$$f := \sum_{n=1}^{\infty} 2^{-n+1} f_n \otimes 1_{\prod\{K_i : i \in \mathbb{N} \setminus \{2n-1, 2n\}\}}.$$

Obviously, f does not depend on finite number of coordinates and $f \in \boxtimes_{i\in\mathbb{N}}\mathcal{H}_i$ since this space is closed. Also $f \notin \bigotimes_{i\in\mathbb{N}} \mathcal{H}_i$. Indeed, if we suppose the contrary, then

$$f \in \bigotimes_{i\in\mathbb{N}} \mathcal{H}_i = (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\bigotimes_{i=3}^{\infty} \mathcal{H}_i) \subset (\mathcal{H}_1 \otimes \mathcal{H}_2) \boxtimes (\bigotimes_{i=3}^{\infty} \mathcal{H}_i).$$

Thus, for $y \in \prod_{i=3}^{\infty} K_i$ is $\pi^y_{\{1,2\}}(f) \in \mathcal{H}_1 \otimes \mathcal{H}_2$. But $\pi^y_{\{1,2\}}(f) = f_1 + c$, where c is a constant, which is a contradiction, since $f_1 \notin \mathcal{H}_1 \otimes \mathcal{H}_2$.

Definition 3.19. Let $(K, \mathcal{H})$ be a product of $(K_i, \mathcal{H}_i)$, $i \in I$. For $J \subset I$ we define the projection of $\mathcal{H}$ by

$$\pi_J(\mathcal{H}) := \{f \in \mathcal{C}(\prod_{i\in J} K_i) : f \otimes 1_{\prod\{K_i : i \in I\setminus J\}} \in \mathcal{H}\}.$$

Observation 3.20. *The following assertions hold:*

(a) $\pi_J(\mathcal{H})$ *is a product of* $\mathcal{H}_i$, $i \in J$,
(b) $\pi_J(\bigodot_{i\in I} \mathcal{H}_i) = \bigodot_{i\in J} \mathcal{H}_i$,
(c) $\pi_J(\bigotimes_{i\in I} \mathcal{H}_i) = \bigotimes_{i\in J} \mathcal{H}_i$,
(d) $\pi_J(\boxtimes_{i\in I}\mathcal{H}_i) = \boxtimes_{i\in J}\mathcal{H}_i$.

Proposition 3.21. *Let $x \in K$, $\mu \in \mathcal{M}_x(\mathcal{H})$ and $J \subset I$. Then* $\pi_J\mu \in \mathcal{M}_{\pi_J(x)}(\pi_J(\mathcal{H}))$.

Proof: Let $h_J \in \pi_J(\mathcal{H})$ and define $h := h_J \otimes 1_{\prod\{K_i:\, i\in I\setminus J\}}$. Then $h \in \mathcal{H}$ and

$$h_J(\pi_J(x)) = h(x) = \mu(h) = (\pi_J\mu)(h_J).$$

□

Proposition 3.22. *Let $x = (x_i)_{i\in I} \in K$ and $\mu_i \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for every $i \in I$. Then $\mu := \bigotimes_{i\in I} \mu_i \in \mathcal{M}_x(\mathcal{H})$.*

Proof: It suffices to prove the assertion for $\mathcal{H} = \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$.

(1) First, let $|I| = n \in \mathbb{N}$. Choose $h \in \mathcal{H}$. By Fubini's theorem,

$$\mu(h) = \int_K h\, d\mu = \int_{K_1} \cdots \int_{K_n} h(y_1, \dots, y_n)\, d\mu_n(y_n) \dots d\mu_1(y_1).$$

Since the function $y_n \mapsto h(y_1, \dots, y_n)$ is in $\mathcal{A}^c(\mathcal{H}_n)$ and $\mu_n \in \mathcal{M}_{x_n}(\mathcal{H}_n)$, we have

$$\int_{K_n} h(y_1, \dots, y_{n-1}, y_n)\, d\mu_n(y_n) = h(y_1, \dots, y_{n-1}, x_n)$$

for every $(y_1, \dots, y_{n-1}) \in \prod_{i=1}^{n-1} K_i$. Using induction, we can see that $\mu(h) = h(x_1, \dots, x_n) = h(x)$. Therefore $\mu \in \mathcal{M}_x(\mathcal{H})$.

(2) Now, let I be an arbitrary index set. Choose $h \in \mathcal{H}$ and $\varepsilon > 0$. By Proposition 3.16(b), there is $g \in \mathcal{H}_J$ for some finite $J \subset I$ so that

$$\|g - h\| < \frac{\varepsilon}{2}.$$

Using the first part of the proof, we write

$$\mu(g) = \Big(\bigotimes_{i \in J} \mu_i\Big)(\pi_J(g)) = \pi_J(g)(\pi_J(x)) = g(x).$$

Let us estimate

$$|\mu(h) - h(x)| \le |\mu(h) - \mu(g)| + |\mu(g) - g(x)| + |g(x) - h(x)| < \varepsilon.$$

Since ε is arbitrary, $\mu(h) = h(x)$. Hence $\mu \in \mathcal{M}_x(\mathcal{H})$. $\square$

Notation 3.23. Let $A_i \subset \mathcal{M}^1(K_i)$ for every $i \in I$. We denote $\bigotimes_{i \in I} A_i := \{\bigotimes_{i \in I} \mu_i : \mu_i \in A_i,\ i \in I\}$.

Example 3.24. If $|I| = 2$, Proposition 3.22 yields the inclusion

$$\overline{\mathrm{co}}^{w^*}(\mathcal{M}_{x_1}(\mathcal{H}_1) \otimes \mathcal{M}_{x_2}(\mathcal{H}_2)) \subset \mathcal{M}_x(\mathcal{H}), \quad x = (x_1, x_2) \in K.$$

Now we show that the inclusion may be proper.

Let $K_i := \{r_i, s_i, t_i\}$, $\mathcal{H}_i := \{f \in \mathcal{C}(K_i) : f(s_i) = \frac{1}{2}(f(r_i) + f(t_i))\}$, $i = 1, 2$. Then $\mathcal{M}_{s_i}(\mathcal{H}_i) = \mathrm{co}\{\varepsilon_{s_i}, \frac{\varepsilon_{r_i} + \varepsilon_{t_i}}{2}\}$. Suppose $(K, \mathcal{H})$ is a product of these two spaces. Denote

$$C := \mathrm{co}\left\{\varepsilon_{s_1} \otimes \varepsilon_{s_2}, \varepsilon_{s_1} \otimes \frac{\varepsilon_{r_2} + \varepsilon_{t_2}}{2}, \frac{\varepsilon_{r_1} + \varepsilon_{t_1}}{2} \otimes \varepsilon_{s_2}, \frac{\varepsilon_{r_1} + \varepsilon_{t_1}}{2} \otimes \frac{\varepsilon_{r_2} + \varepsilon_{t_2}}{2}\right\}.$$

We see that $\overline{\mathrm{co}}^{w^*}(\mathcal{M}_{s_1}(\mathcal{H}_1) \otimes \mathcal{M}_{s_2}(\mathcal{H}_2)) = C$. Define

$$\mu := \frac{\varepsilon_{(s_1, t_2)}}{2} + \frac{\varepsilon_{(r_1, r_2)}}{4} + \frac{\varepsilon_{(t_1, r_2)}}{4}.$$

Obviously $\mu \in \mathcal{M}_{(s_1, s_2)}(\mathcal{H})$. For every $x \in K \setminus \{(s_1, t_2), (r_1, r_2), (t_1, r_2)\}$ we have $\mu(\{x\}) = 0$. However, if μ were an element of C, then at least one of the points (s_1, s_2), (s_1, r_2), (r_1, s_2), (r_1, t_2) would have a non-zero measure.

Example 3.25. Let $x \in K$. Denote $\mathcal{M}_x^\pi(\mathcal{H})$ the set of all $\mu \in \mathcal{M}^1(K)$ such that $\pi_i(\mu) \in \mathcal{M}_{\pi_i(x)}(\mathcal{H}_i)$ for every $i \in I$. Proposition 3.21 yields

$$\mathcal{M}_x(\mathcal{H}) \subset \mathcal{M}_x^\pi(\mathcal{H}).$$

Once again, we show that the inclusion may be proper.

Let $(K_i, \mathcal{H}_i)$, $i = 1, 2$, be as in Example 3.24. Consider

$$\mu := \frac{\varepsilon_{(r_1,r_2)}}{2} + \frac{\varepsilon_{(t_1,t_2)}}{2}.$$

We see that $\pi_i(\mu) = \frac{\varepsilon_{r_i}}{2} + \frac{\varepsilon_{t_i}}{2} \in \mathcal{M}_{s_i}(\mathcal{H}_i)$, $i = 1, 2$. Thus $\mu \in \mathcal{M}^\pi_{(s_1,s_2)}(\mathcal{H})$. However $\mu \notin \mathcal{M}_{(s_1,s_2)}(\mathcal{H})$. Indeed, take $f_i \in \mathcal{H}_i$ such that $f_i(r_i) = 0$, $f_i(s_i) = 1$, $f_i(t_i) = 2$, for $i = 1, 2$. Define $f := f_1 \otimes f_2$. Then $f \in \mathcal{H}$, but

$$f(s_1, s_2) = 1 \neq 2 = \mu(f).$$

Question 3.26. Is there a way to characterize $\mathcal{M}_x(\mathcal{H})$ by $\mathcal{M}_{\pi_i(x)}(\mathcal{H}_i)$, $i \in I$?

Proposition 3.27. *Let $x = (x_i)_{i\in I} \in K$. Then $F_x(\mathcal{H}) = \prod_{i\in I} F_{x_i}(\mathcal{H}_i)$.*

PROOF: First we show $F_x(\mathcal{H}) \subset \prod_{i\in I} F_{x_i}(\mathcal{H}_i)$. For each $\mu \in \mathcal{M}_x(\mathcal{H})$ and $i \in I$ we have $\pi_i(\operatorname{spt}\mu) = \operatorname{spt}\pi_i\mu$ and since, by Proposition 3.21, $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$, we get $\pi_i(\operatorname{spt}\mu) \subset F_{x_i}(\mathcal{H}_i)$. Therefore $\pi_i(F_x(\mathcal{H})) \subset F_{x_i}(\mathcal{H}_i)$ for every $i \in I$.

Conversely, let $\mu_i \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for every $i \in I$. Proposition 3.22 yields $\bigotimes_{i\in I}\mu_i \in \mathcal{M}_x(\mathcal{H})$ and thus $\prod_{i\in I}\operatorname{spt}\mu_i = \operatorname{spt}\bigotimes_{i\in I}\mu_i \subset F_x(\mathcal{H})$. □

3.4 $\mathcal{H}$-affine functions.

Proposition 3.28. $\mathcal{A}^c(\mathcal{H}) \subset \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$.

PROOF: Choose $f \in \mathcal{A}^c(\mathcal{H})$, $j \in I$ and $y = (y_i) \in \prod_{i\in I\setminus\{j\}} K_i$. We prove that $f_j := \pi_j^y(f) \in \mathcal{A}^c(\mathcal{H}_j)$. Let $x_j \in K_j$ and $\mu_j \in \mathcal{M}_{x_j}(\mathcal{H}_j)$. Define $x := (x_j, y)$ and $\mu := \mu_j \otimes (\bigotimes_{i\in I\setminus\{j\}}\varepsilon_{y_i})$. According to Proposition 3.22, $\mu \in \mathcal{M}_x(\mathcal{H})$, so we have

$$f_j(x_j) = f(x) = \mu(f) = \mu_j(f_j).$$

Hence $f_j \in \mathcal{A}^c(\mathcal{H}_j)$. □

Lemma 3.29. *Let $|I| = 2$. Then $\mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) \subset \mathcal{A}^c(\mathcal{H})$.*

PROOF: Consider $a_1 \in \mathcal{A}^c(\mathcal{H}_1)$, $a_2 \in \mathcal{A}^c(\mathcal{H}_2)$. We show that $a_1 \otimes a_2 \in \mathcal{A}^c(\mathcal{H})$ by using the characterization $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$.

First suppose that $a_1, a_2 \geq 0$. Choose $x = (x_1, x_2) \in K$ and $\varepsilon > 0$. Find $\delta > 0$ so that

$$\delta(a_1(x_1) + a_2(x_2) + \delta) < \varepsilon.$$

Since $a_i^* = a_i$, $i = 1, 2$, there are $h_1 \in \mathcal{H}_1$, $h_1 \geq a_1$ and $h_2 \in \mathcal{H}_2$, $h_2 \geq a_2$ such that

$$h_1(x_1) < a_1(x_1) + \delta \quad \text{and} \quad h_2(x_2) < a_2(x_2) + \delta.$$

Obviously $h_1 \otimes h_2 \in \mathcal{H}$, $h_1 \otimes h_2 \geq a_1 \otimes a_2$ and

$$\begin{aligned} a_1(x_1)a_2(x_2) &\leq h_1(x_1)h_2(x_2) < (a_1(x_1) + \delta)(a_2(x_2) + \delta) \\ &= a_1(x_1)a_2(x_2) + \delta(a_1(x_1) + a_2(x_2) + \delta) < a_1(x_1)a_2(x_2) + \varepsilon. \end{aligned}$$

Thus $(a_1 \otimes a_2)^* = a_1 \otimes a_2$.

Now suppose $a_1 \geq 0$ and a_2 is arbitrary. Then $a_2 + \|a_2\| \geq 0$. Since $f \mapsto f^*$ is a sublinear functional on $\mathcal{C}(K)$ and $(a_1 \otimes c)^* = a_1 \otimes c$ for every constant function c on K_2, we get

$$\begin{aligned} a_1 \otimes a_2 \leq (a_1 \otimes a_2)^* &= (a_1 \otimes (a_2 + \|a_2\| - \|a_2\|))^* \\ &= (a_1 \otimes (a_2 + \|a_2\|) - a_1 \otimes \|a_2\|)^* \\ &\leq (a_1 \otimes (a_2 + \|a_2\|))^* + (a_1 \otimes (-\|a_2\|))^* \\ &= a_1 \otimes (a_2 + \|a_2\|) + (a_1 \otimes (-\|a_2\|)) = a_1 \otimes a_2. \end{aligned}$$

For the lower envelope we have

$$(a_1 \otimes a_2)_* = -(a_1 \otimes (-a_2))^* = -(a_1 \otimes (-a_2)) = a_1 \otimes a_2.$$

Thus $a_1 \otimes a_2 \in \widehat{\mathcal{H}} = \mathcal{A}^c(\mathcal{H})$.

Finally, let a_1, a_2 be arbitrary. Then

$$a_1 \otimes a_2 = (a_1 + \|a_1\|) \otimes a_2 - \|a_1\| \otimes a_2 \in \mathcal{A}^c(\mathcal{H}).$$

Since $\mathcal{A}^c(\mathcal{H})$ is a closed linear space, the conclusion follows. $\square$

Proposition 3.30. $\bigotimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\mathcal{H})$.

Proof: It suffices to prove $\bigodot_{i \in I} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\mathcal{H})$, since the latter space is closed.

(1) Assume first, that $|I| = n \in \mathbb{N}$ and the assertion holds for $|I| = n - 1$. Using the assumption, previous Lemma 3.29 and the associative law, we get

$$\begin{aligned} \bigodot_{i=1}^{n} \mathcal{A}^c(\mathcal{H}_i) = \Big(\bigodot_{i=1}^{n-1} \mathcal{A}^c(\mathcal{H}_i)\Big) \odot \mathcal{A}^c(\mathcal{H}_n) &\subset \mathcal{A}^c\Big(\bigodot_{i=1}^{n-1} \mathcal{H}_i\Big) \odot \mathcal{A}^c(\mathcal{H}_n) \\ &\subset \mathcal{A}^c\Big(\Big(\bigodot_{i=1}^{n-1} \mathcal{H}_i\Big) \odot \mathcal{H}_n\Big) = \mathcal{A}^c\Big(\bigodot_{i=1}^{n} \mathcal{H}_i\Big) \subset \mathcal{A}^c(\mathcal{H}). \end{aligned}$$

(2) Now, let I be an arbitrary index set. Choose $f \in \bigodot_{i\in I} \mathcal{A}^c(\mathcal{H}_i)$. Then there is a finite $J \subset I$ such that f depends only on coordinates from J. So, according to the first part of the proof, $\pi_J(f) \in \bigodot_{i\in J} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\bigodot_{i\in J} \mathcal{H}_i)$. Since $f = \pi_J(f) \otimes 1_{\prod\{K_i : i\in I\setminus J\}}$, we have

$$\begin{aligned} f &\in \mathcal{A}^c\big(\bigodot_{i\in J} \mathcal{H}_i\big) \odot \mathcal{A}^c\big(\bigodot_{i\in I\setminus J} \mathcal{H}_i\big) \subset \mathcal{A}^c\Big(\big(\bigodot_{i\in J} \mathcal{H}_i\big) \odot \big(\bigodot_{i\in I\setminus J} \mathcal{H}_i\big)\Big) \\ &= \mathcal{A}^c\big(\bigodot_{i\in I} \mathcal{H}_i\big) \subset \mathcal{A}^c(\mathcal{H}). \end{aligned}$$

□

Corollary 3.31. *$\mathcal{A}^c(\mathcal{H})$ is a product of both $\mathcal{H}_i$, $i \in I$, and $\mathcal{A}^c(\mathcal{H}_i)$, $i \in I$.*

Proof: From Proposition 3.30 we have

$$\bigodot_{i\in I} \mathcal{H}_i \subset \bigodot_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\mathcal{H}),$$

and from Proposition 3.28

$$\mathcal{A}^c(\mathcal{H}) \subset \bigboxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \bigboxtimes_{i\in I} \mathcal{A}^c(\mathcal{A}^c(\mathcal{H}_i)).$$

□

Proposition 3.32. *If $\mathcal{A}^c(\mathcal{H}) \subset \boxtimes_{i\in I}\mathcal{H}_i$, then $\mathcal{H}_i = \mathcal{A}^c(\mathcal{H}_i)$ for every $i \in I$.*

Proof: Choose $i \in I$. We prove that $\mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{H}_i$. Pick $f_i \in \mathcal{A}^c(\mathcal{H}_i)$ and define $f := f_i \otimes 1_{\prod\{K_j : j\in I\setminus\{i\}\}}$. Choose $x = (x_j)_{j\in I} \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. From Proposition 3.21 we have $\mu_i := \pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$, which implies

$$f(x) = f_i(x_i) = \mu_i(f_i) = \mu(f).$$

Thus $f \in \mathcal{A}^c(\mathcal{H}) \subset \boxtimes_{i\in I}\mathcal{H}_i$, so $f_i = \pi_i(f) \in \mathcal{H}_i$. □

Proposition 3.33. *Let $\mathcal{H} = \boxtimes_{i\in I}\mathcal{H}_i$. Then $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ if and only if $\mathcal{H}_i = \mathcal{A}^c(\mathcal{H}_i)$ for every $i \in I$.*

Proof: If $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$, we use Proposition 3.32. Conversely, from Proposition 3.28 we have

$$\mathcal{H} \subset \mathcal{A}^c(\mathcal{H}) \subset \bigboxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \bigboxtimes_{i\in I} \mathcal{H}_i = \mathcal{H}.$$

□

Corollary 3.34. *There are function spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that*

$$\mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) \subsetneq \mathcal{A}^c(\mathcal{H}_1 \boxtimes \mathcal{H}_2).$$

Proof: By Corollary 3.9, there are $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \otimes \mathcal{H}_2 \subsetneq \mathcal{H}_1 \boxtimes \mathcal{H}_2$ and $\mathcal{H}_i = \mathcal{A}^c(\mathcal{H}_i)$, $i = 1, 2$. Proposition 3.33 implies $\mathcal{H}_1 \boxtimes \mathcal{H}_2 = \mathcal{A}^c(\mathcal{H}_1 \boxtimes \mathcal{H}_2)$. Thus

$$\mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) = \mathcal{H}_1 \otimes \mathcal{H}_2 \subsetneq \mathcal{H}_1 \boxtimes \mathcal{H}_2 = \mathcal{A}^c(\mathcal{H}_1 \boxtimes \mathcal{H}_2). \qquad \square$$

Example 3.35. Example 3.6 shows there are function spaces such that

$$\mathcal{A}^c(\mathcal{H}_1) \odot \mathcal{A}^c(\mathcal{H}_2) \subsetneq \mathcal{A}^c(\mathcal{H}_1 \odot \mathcal{H}_2).$$

Question 3.36. Is $\bigotimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \mathcal{A}^c(\bigodot_{i\in I} \mathcal{H}_i)$?

Question 3.37. Is $\mathcal{A}^c(\bigodot_{i\in I} \mathcal{H}_i) = \mathcal{A}^c(\boxtimes_{i\in I} \mathcal{H}_i)$?

Question 3.38. Is $\mathcal{A}^c(\boxtimes_{i\in I} \mathcal{H}_i) = \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i)$?

3.5 $\mathcal{H}$-extremal sets.

Proposition 3.39. *Let $E \subset K$ be an $\mathcal{H}$-extremal set. Let $J \subset I$, $y \in \prod_{i\in I\setminus J} K_i$, and let $\mathcal{G}$ be a product of $\mathcal{H}_i$, $i \in J$. Then $\pi_J^y(E)$ is either empty or a $\mathcal{G}$-extremal set.*

Proof: Suppose $E^y := \pi_J^y(E)$ is non-empty and not $\mathcal{G}$-extremal. Then there is $x \in E^y$ and $\mu_J \in \mathcal{M}_x(\mathcal{G})$ so that $\operatorname{spt} \mu_J \not\subset E^y$. According to Proposition 3.21, $\mu_i := \pi_i \mu_J \in \mathcal{M}_{\pi_i(x)}(\mathcal{H}_i)$ for every $i \in J$. Since $\operatorname{spt} \mu_J \subset \operatorname{spt} \bigotimes_{i\in J} \mu_i$, we can see that $\operatorname{spt} \bigotimes_{i\in J} \mu_i \not\subset E^y$. Define

$$\mu := \Big(\bigotimes_{i\in J} \mu_i\Big) \otimes \Big(\bigotimes_{i\in I\setminus J} \varepsilon_{\pi_i(y)}\Big).$$

Hence, we have $(x, y) \in E$ and by Proposition 3.22 also $\mu \in \mathcal{M}_{(x,y)}(\mathcal{H})$. But $\operatorname{spt} \mu \not\subset E$, which is a contradiction. $\square$

The next two propositions are generalizations of Proposition 4.1 and Theorem 4.2 from [16] to function spaces:

Proposition 3.40. *Let $E \subset K$ be an $\mathcal{H}$-extremal set. Let $\emptyset \neq J \subset I$ and let $\mathcal{G}$ be a product of $\mathcal{H}_i$, $i \in J$. Then $\pi_J(E)$ is a $\mathcal{G}$-extremal set.*

Proof: Let $x \in \pi_J(E)$ and $\mu \in \mathcal{M}_x(\mathcal{G})$. Then there is $y \in \prod_{i\in I\setminus J} K_i$ such that $(x, y) \in E$, i.e., $x \in \pi_J^y(E)$. By Proposition 3.39, $\pi_J^y(E)$ is a $\mathcal{G}$-extremal set, therefore $\operatorname{spt} \mu \subset \pi_J^y(E) \subset \pi_J(E)$. $\square$

Proposition 3.41. *Let $E_i \subset K_i$ be an $\mathcal{H}_i$-extremal set for every $i \in I$. Then $E := \prod_{i\in I} E_i$ is an $\mathcal{H}$-extremal set.*

PROOF: Obviously, E is a closed set.

(1) Assume $|I| = 2$. Suppose there is $x = (x_1, x_2) \in E$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ so that $\mu(K \setminus E) > 0$. Denote $\mu_1 := \pi_1\mu$ and $\mu_2 := \pi_2\mu$. Since

$$K \setminus E = ((K_1 \setminus E_1) \times K_2) \cup (K_1 \times (K_2 \setminus E_2)),$$

we have

$$0 < \mu(K \setminus E) \leq \mu_1(K_1 \setminus E_1) + \mu_2(K_2 \setminus E_2).$$

We may assume $\mu_1(K_1 \setminus E_1) > 0$. By Proposition 3.21, $\mu_1 \in \mathcal{M}_{x_1}(\mathcal{H}_1)$. But this is a contradiction, because $x_1 \in E_1$.

We proceed similarly for arbitrary finite products.

(2) Now, let I be infinite. Suppose there is $x = (x_i)_{i\in I} \in E$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ so that $\mu(K \setminus E) > 0$. Then there is some $g \in \mathcal{C}(K)$ such that $g = 0$ on E and $\mu(g) > 0$. Choose $\varepsilon > 0$. According to Corollary 3.17, there is $f \in \mathcal{C}_J(K)$, where $J \subset I$ is finite and $\|g - f\| < \varepsilon$. Then $\pi_J\mu \in \mathcal{M}_{\pi_J(x)}(\pi_J(\mathcal{H}))$ and by the first part of the proof, $\pi_J(x)$ is an element of the $\pi_J(\mathcal{H})$-extremal set $E_J := \prod_{i\in J} E_i$. Thus $\operatorname{spt} \pi_J\mu \subset E_J$ and $|\pi_J(f)| < \varepsilon$ on E_J. Therefore

$$|\mu(f)| = |(\pi_J\mu)(\pi_J(f))| \leq \int_{E_J} |\pi_J(f)|\, d(\pi_J\mu) + \int_{(\prod_{i\in J} K_i)\setminus E_J} |\pi_J(f)|\, d(\pi_J\mu) < \varepsilon.$$

Hence we get

$$0 < |\mu(g)| \leq |\mu(g) - \mu(f)| + |\mu(f)| < 2\varepsilon,$$

which is a contradiction, since ε is arbitrary. □

Using previous results, we can derive the main theorem of this subsection (cf. also [6, Lemma 5], [16, Theorem 3.2] and [10, Lemma 5.11]):

Theorem 3.42. $\operatorname{Ch}_{\mathcal{H}} K = \prod_{i\in I} \operatorname{Ch}_{\mathcal{H}_i} K_i$.

PROOF: Follows immediately from Propositions 3.40 and 3.41. □

Corollary 3.43. $\nabla_{\mathcal{H}} K = \prod_{i\in I} \nabla_{\mathcal{H}_i} K_i$.

PROOF: Using Theorem 3.42 we can write

$$\nabla_{\mathcal{H}} K = \overline{\operatorname{Ch}_{\mathcal{H}} K} = \overline{\prod_{i\in I} \operatorname{Ch}_{\mathcal{H}_i} K_i} = \prod_{i\in I} \overline{\operatorname{Ch}_{\mathcal{H}_i} K_i} = \prod_{i\in I} \nabla_{\mathcal{H}_i} K_i.$$

□

Remarks 3.44. As has been shown by Grossman [9], the characterizations of Choquet and Šilov boundary hold also for the space $\mathcal{H}_1 + \mathcal{H}_2$ defined by

$$\begin{aligned} &\mathcal{H}_1 + \mathcal{H}_2 := \{h_1 + h_2 : h_1 \in \mathcal{H}_1,\, h_2 \in \mathcal{H}_2\}, \quad \text{where} \\ &[h_1 + h_2](x, y) = h_1(x) + h_2(y), \quad (x, y) \in K_1 \times K_2. \end{aligned}$$

It is clear that $\mathcal{H}_1 + \mathcal{H}_2$ does not have to be a product, since the inclusion $\mathcal{H}_1 \odot \mathcal{H}_2 \subset \mathcal{H}_1 + \mathcal{H}_2$ does not have to hold.

Versions of Theorem 3.42 for various tensor products of compact convex sets have been proved by I. Namioka and R.R. Phelps in [18].

Example 3.45. In Example 3.3 we have shown that the space of all harmonic functions on a cartesian product does not have to be a product of harmonic spaces. Moreover, it is not even possible to extend the notion of a product so that the product of harmonic spaces would be a harmonic space and Theorem 3.42 would still hold. Indeed, consider the sets from Example 3.3 and denote $U := U_1 \times U_2$. Then

$$\mathrm{Ch}_{H(U)}\, \overline{U} = \partial_{\mathrm{reg}} U = \partial U \neq \{0, 1\} \times \{0, 1\} = \mathrm{Ch}_{H(U_1)}\, \overline{U}_1 \times \mathrm{Ch}_{H(U_2)}\, \overline{U}_2.$$

3.6 Approximation in product spaces. In the following, we will need some results on approximation of functions in simplicial spaces. So we first state here results that are adaptation of Section 2 from [17].

Definition 3.46. Let $(K, \mathcal{H})$ be a function space. A collection of nonnegative functions $\{\psi_j\}_{j=1}^m \subset \mathcal{H}$ is called a *partition of unity* on K, if $\sum_{j=1}^m \psi_j = 1_K$.

Lemma 3.47. *Let $(K, \mathcal{H})$ be a simplicial function space. Let $\{f_i\}_{i=1}^n \subset \mathcal{A}^c(\mathcal{H})$ and $\varepsilon > 0$. Suppose that $\{\phi_j\}_{j=1}^m$ are nonnegative functions defined on $\mathrm{Ch}_{\mathcal{H}}\, K$, $\{k_l\}_{l=1}^m \subset \mathrm{Ch}_{\mathcal{H}}\, K$ and $\{\alpha_{ij} : 1 \le i \le n,\, 1 \le j \le m\}$ are real numbers such that*

(i) $\sum_{j=1}^m \phi_j = 1$,
(ii) $\phi_j(k_l) = \delta_{jl}, \quad 1 \le j, l \le m$,
(iii) $|f_i(k) - \sum_{j=1}^m \alpha_{ij}\phi_j(k)| \le \varepsilon, \quad k \in \mathrm{Ch}_{\mathcal{H}}\, K,\, 1 \le i \le n.$

Then there exists a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H})$ such that

(iv) $\psi_j(k_l) = \delta_{jl}, \quad 1 \le j, l \le m$,
(v) $|f_i(k) - \sum_{j=1}^m \alpha_{ij}\psi_j(k)| \le \varepsilon, \quad k \in K,\, 1 \le i \le n.$

Proof: See [17, Corollary 2.2]. □

The proof of the next lemma is based on the proof of [17, Lemma 2.4]:

Lemma 3.48. *Let $(K_1,\mathcal{H}_1)$ and $(K_2,\mathcal{H}_2)$ be two function spaces, where $\mathcal{H}_1$ is simplicial. Suppose that $\{f_i\}_{i=1}^n \subset \mathcal{A}^c(\mathcal{H}_1) \boxtimes \mathcal{H}_2$ and $\varepsilon > 0$. Then there is a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_1)$, $\{k_l\}_{l=1}^m \subset \operatorname{Ch}_{\mathcal{H}_1} K_1$ and $\{y_{ij}\} \subset \mathcal{H}_2$, $1 \le i \le n$, $1 \le j \le m$, so that*

(i) $\psi_j(k_l) = \delta_{jl}, \quad 1 \le j,\, l \le m,$

(ii) $\|f_i - \sum_{j=1}^m \psi_j \otimes y_{ij}\| < \varepsilon, \quad 1 \le i \le n.$

Proof: Denote by $\mathcal{H}_2^n$ the n-tuple cartesian product of $\mathcal{H}_2$ with the maximum norm, i.e.,

$$\|y\|_{\max} = \max_{1\le i\le n} \|\pi_i(y)\| \quad \text{for all} \quad y \in \mathcal{H}_2^n,$$

where π_i is the projection to the i-th coordinate. We denote by $B_r(x)$ the open ball with center x and radius $r > 0$.

Let f be a function from K_1 to $\mathcal{H}_2^n$ defined by

$$f(k) := (\pi_2^k(f_1), \dots, \pi_2^k(f_n)), \quad k \in K_1.$$

Since $\pi_i \circ f$ is a continuous function for every $i = 1, \dots, n$ (we use the fact that $\mathcal{C}(K_1 \times K_2)$ is isometric to $\mathcal{C}(K_1, \mathcal{C}(K_2))$), f is also a continuous function on K_1.

For each $y \in \mathcal{H}_2^n$ set

$$U_y := \left\{ k \in K_1 : \|y - f(k)\|_{\max} < \frac{\varepsilon}{3} \right\}. \tag{1}$$

The family $\{U_y\}_{y\in\mathcal{H}_2^n}$ is an open covering of K_1. Let $U_{y_1}, \dots, U_{y_p}$ be a finite subcovering. Define

$$V_{y_j} := U_{y_j} \cap \operatorname{Ch}_{\mathcal{H}_1} K_1, \quad 1 \le j \le p.$$

Without loss of generality we may assume that there is $m \le p$ such that $\{V_{y_l}\}_{l=1}^m$ is an open covering of $\operatorname{Ch}_{\mathcal{H}_1} K_1$ and for every $l \in \{1, \dots, m\}$ there exists $k_l \in V_{y_l}$ such that $k_l \notin V_{y_j}$ for $j \ne l$, $1 \le j \le m$.

Denote

$$\begin{aligned} C &:= \{y_1, \dots, y_p\} - \operatorname{co}(y_1, \dots, y_m),\\ D &:= C + B_{\frac{\varepsilon}{3}}(0). \end{aligned}$$

Choose $i \in \{1, \dots, n\}$. Since C is a compact subset of $\mathcal{H}_2^n$, also $\pi_i(C)$ is a compact subset of $\mathcal{H}_2$. By Arzelà-Ascoli's theorem, the set $\pi_i(C)$ is equicontinuous. Therefore, for each $\xi \in K_2$ we can find its open neighbourhood W_ξ such that $\operatorname{osc}_{W_\xi} h < \frac{\varepsilon}{3}$ for every $h \in \pi_i(C)$. From the open covering $\{W_\xi\}_{\xi\in K_2}$ we choose a finite subcovering $\{W_{\xi_{ir}}\}_{r=1}^{q_i}$. For every $h \in \pi_i(C)$ there is $x_h \in K_2$ such that

$|h(x_h)| = \|h\|$. The point x_h is an element of some $W_{\xi_{ir}}$ and so $\|h\| - \frac{\varepsilon}{3} < |h(\xi_{ir})|$. Thus

$$\|h\| - \frac{\varepsilon}{3} < \max_{1\le r\le q_i} |h(\xi_{ir})| \le \|h\|, \quad h \in \pi_i(C),$$

and since $\pi_i(D) \subset \pi_i(C) + B_{\frac{\varepsilon}{3}}(0)$, also

$$\|h\| - \frac{2}{3}\varepsilon < \max_{1\le r\le q_i} |h(\xi_{ir})| \le \|h\|, \quad h \in \pi_i(D). \tag{2}$$

Let $\Gamma_{ir} \in (\mathcal{H}_2^n)^*$ be a continuous linear functional defined by

$$\Gamma_{ir}(y) := \pi_i(y)(\xi_{ir}), \quad 1 \le i \le n,\ 1 \le r \le q_i,\ y \in \mathcal{H}_2^n.$$

From (2) we can write

$$\|h\|_{\max} - \frac{2}{3}\varepsilon < \max_{\substack{1\le i\le n\\ 1\le r\le q_i}} |\Gamma_{ir}(h)| \le \|h\|_{\max}, \quad h \in D. \tag{3}$$

Set

$$\phi_j(k) := \begin{cases} 1, & \text{if } j = \min\{l : k \in V_{y_l},\ 1 \le l \le m\}, \\ 0, & \text{otherwise,} \end{cases} \qquad 1 \le j \le m,\ k \in \operatorname{Ch}_{\mathcal{H}_1} K_1.$$

Clearly $\phi_j \ge 0$, $\phi_j(k_l) = \delta_{jl}$, $1 \le j, l \le m$, and $\sum_{j=1}^m \phi_j = 1$. Moreover, for every $k \in \operatorname{Ch}_{\mathcal{H}_1} K_1$ there is a unique index j_k so that $\phi_{j_k}(k) \ne 0$. For this index is $k \in V_{y_{j_k}}$. Thus, from (1) we have

$$\|f(k) - y_{j_k}\|_{\max} < \frac{\varepsilon}{3}\,.$$

We can rewrite this inequality as

$$\|f(k) - \sum_{j=1}^m \phi_j(k) y_j\|_{\max} < \frac{\varepsilon}{3}, \quad k \in \operatorname{Ch}_{\mathcal{H}_1} K_1. \tag{4}$$

Since $f(k) - \sum_{j=1}^m \phi_j(k) y_j \in D$, using (3) and (4) we have

$$\begin{aligned} |\Gamma_{ir}(f(k)) - \sum_{j=1}^m \phi_j(k)\Gamma_{ir}(y_j)| &= |\Gamma_{ir}(f(k) - \sum_{j=1}^m \phi_j(k) y_j)| \\ &\le \|f(k) - \sum_{j=1}^m \phi_j(k) y_j\|_{\max} \\ &< \frac{\varepsilon}{3}, \quad 1 \le i \le n,\ 1 \le r \le q_i,\ k \in \operatorname{Ch}_{\mathcal{H}_1} K_1. \end{aligned}$$

Lemma 3.47 yields a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_1)$ such that

$$(5)\qquad |\Gamma_{ir}(f(k)) - \sum_{j=1}^m \psi_j(k)\Gamma_{ir}(y_j)| \le \frac{\varepsilon}{3}, \quad 1 \le i \le n,\ 1 \le r \le q_i,\ k \in K_1,$$

$$(6)\qquad \psi_j(k_l) = \delta_{jl}, \quad 1 \le j, l \le m.$$

Since $f(k) - \sum_{j=1}^m \psi_j(k) y_j \in D$ for every $k \in K_1$, using (3) and (5) we get

$$\begin{aligned}
\|f(k) - \sum_{j=1}^m \psi_j(k)y_j\|_{\max} - \frac{2}{3}\varepsilon &< \max_{\substack{1\le i\le n\\ 1\le r\le q_i}} |\Gamma_{ir}(f(k) - \sum_{j=1}^m \psi_j(k)y_j)| \\
&= \max_{\substack{1\le i\le n\\ 1\le r\le q_i}} |\Gamma_{ir}(f(k)) - \sum_{j=1}^m \psi_j(k)\Gamma_{ir}(y_j)| \\
&\le \frac{\varepsilon}{3}, \quad k \in K_1.
\end{aligned}$$

Hence

$$(7)\qquad \|f(k) - \sum_{j=1}^m \psi_j(k)y_j\|_{\max} < \varepsilon, \quad k \in K_1.$$

Finally, define $y_{ij} := \pi_i(y_j) \in \mathcal{H}_2$, $1 \le i \le n$, $1 \le j \le m$. Assertion (i) then follows from (6) and (ii) follows from (7). $\square$

3.7 Products of simplicial spaces.

Proposition 3.49. *Suppose that at most one of the spaces $\mathcal{H}_i$, $i \in I$, is not simplicial. Then*

$$\bigotimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \mathcal{A}^c(\mathcal{H}) = \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i).$$

Proof: Due to Propositions 3.30 and 3.28, it suffices to prove $\boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i) \subset \bigotimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i)$.

(1) First we prove the assertion for finite products. Let $|I| = n \ge 2$ and suppose that $\mathcal{H}_1, \dots, \mathcal{H}_{n-1}$ are simplicial. We repeatedly use associative laws and Lemma 3.48 to get

$$\begin{aligned}
\boxtimes_{i=1}^n \mathcal{A}^c(\mathcal{H}_i) &= \mathcal{A}^c(\mathcal{H}_1) \boxtimes (\mathcal{A}^c(\mathcal{H}_2) \boxtimes (\dots(\mathcal{A}^c(\mathcal{H}_{n-1}) \boxtimes \mathcal{A}^c(\mathcal{H}_n))\dots)) \\
&\subset \mathcal{A}^c(\mathcal{H}_1) \otimes (\mathcal{A}^c(\mathcal{H}_2) \boxtimes (\dots(\mathcal{A}^c(\mathcal{H}_{n-1}) \boxtimes \mathcal{A}^c(\mathcal{H}_n))\dots)) \\
&\subset \dots \subset \mathcal{A}^c(\mathcal{H}_1) \otimes (\mathcal{A}^c(\mathcal{H}_2) \otimes (\dots(\mathcal{A}^c(\mathcal{H}_{n-1}) \otimes \mathcal{A}^c(\mathcal{H}_n))\dots)) \\
&= \bigotimes_{i=1}^n \mathcal{A}^c(\mathcal{H}_i).
\end{aligned}$$

(2) Now, let I be infinite. Choose $f \in \boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$ and $\varepsilon > 0$. According to Proposition 3.16(b), there is some $h \in \boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$ depending on finitely many coordinates J such that $\|f - h\| < \varepsilon$. From the first part of the proof we have $\pi_J(h) \in \boxtimes_{i \in J} \mathcal{A}^c(\mathcal{H}_i) \subset \bigotimes_{i \in J} \mathcal{A}^c(\mathcal{H}_i)$. Thus, $h \in \bigotimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$. Since the space is closed, we get $f \in \bigotimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$. □

Example 3.50. The assumption on the number of simplicial spaces in Proposition 3.49 may not be weakened. We show that for every index set I with $|I| \geq 2$ there is a family of function spaces $\mathcal{H}_i$, $i \in I$, with two non-simplicial spaces, which does not satisfy the equality in Proposition 3.49. Once again, we use Corollary 3.9 to construct a counterexample.

Let $\mathcal{H}_i$, $i \in I$, be a family of function spaces such that there are $i_1, i_2 \in I$ so that $\mathcal{H}_{i_1}$, $\mathcal{H}_{i_2}$ are as in Corollary 3.9. Thus, there is $f' \in (\mathcal{A}^c(\mathcal{H}_{i_1}) \boxtimes \mathcal{A}^c(\mathcal{H}_{i_2})) \setminus (\mathcal{A}^c(\mathcal{H}_{i_1}) \otimes \mathcal{A}^c(\mathcal{H}_{i_2}))$. Using Proposition 3.49, we can see that neither of the two spaces is simplicial. Now, define

$$f := f' \otimes 1_{\prod\{K_i : i \in I \setminus \{i_1, i_2\}\}}.$$

We have $f \in \boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$, but $f \notin \bigotimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$.

Lemma 3.51. *Let $|I| = 2$ and suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are simplicial. Then $\mathcal{H}$ is simplicial.*

Proof: It is sufficient to show that $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. Let a, b, c, d be functions from $\mathcal{A}^c(\mathcal{H}) = \mathcal{A}^c(\mathcal{H}_1) \boxtimes \mathcal{A}^c(\mathcal{H}_2)$ such that $a \vee b < c \wedge d$. By Lemma 3.48, there is a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_1)$, $\{k_l\}_{l=1}^m \subset \operatorname{Ch}_{\mathcal{H}_1} K_1$ and functions $\{a_j, b_j, c_j, d_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_2)$ so that

$$\psi_j(k_l) = \delta_{jl}, \quad 1 \leq j, l \leq m, \tag{8}$$

and for

$$\begin{aligned} a' &:= \sum_{j=1}^m \psi_j \otimes a_j, \quad & b' &:= \sum_{j=1}^m \psi_j \otimes b_j, \\ c' &:= \sum_{j=1}^m \psi_j \otimes c_j, \quad & d' &:= \sum_{j=1}^m \psi_j \otimes d_j, \end{aligned} \tag{9}$$

it holds

$$a \vee b < a' \vee b' < c' \wedge d' < c \wedge d. \tag{10}$$

Then also

$$\pi_2^k(a') \vee \pi_2^k(b') < \pi_2^k(c') \wedge \pi_2^k(d'), \quad k \in K_1. \tag{11}$$

For every $j = 1, \dots, m$, we get from (8) and (9)

$$\pi_2^{k_j}(a') = a_j, \quad \pi_2^{k_j}(b') = b_j, \quad \pi_2^{k_j}(c') = c_j, \quad \pi_2^{k_j}(d') = d_j,$$

and from (11)

$$a_j \vee b_j < c_j \wedge d_j.$$

Since $\mathcal{A}^c(\mathcal{H}_2)$ has W.R.I.P., there are $h_j \in \mathcal{A}^c(\mathcal{H}_2)$, $j = 1, \dots, m$, such that

$$a_j \vee b_j < h_j < c_j \wedge d_j. \tag{12}$$

Define $h := \sum_{j=1}^m \psi_j \otimes h_j \in \mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) = \mathcal{A}^c(\mathcal{H})$. The non-negativity of $\{\psi_j\}_{j=1}^m$ and inequalities (12) and (10) imply

$$a \vee b < h < c \wedge d.$$

Hence $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. and the proof is complete. $\square$

Now we may prove the theorem, which is a generalization of [6, Theorem 11] and [16, Theorem 3.1]:

Theorem 3.52. *Suppose that $\mathcal{H}_i$ is simplicial for each $i \in I$. Then $\mathcal{H}$ is simplicial.*

Proof: First we prove the theorem for finite I. By Lemma 3.51, the theorem holds for $|I| = 2$. Suppose that $|I| = n > 2$ and the theorem holds for $|I| < n$. Clearly $\boxtimes_{i=1}^n \mathcal{H}_i = (\boxtimes_{i=1}^{n-1} \mathcal{H}_i) \boxtimes \mathcal{H}_n$ is simplicial and $\mathcal{A}^c(\mathcal{H}) = \mathcal{A}^c(\boxtimes_{i=1}^n \mathcal{H}_i)$ has W.R.I.P. Therefore $\mathcal{H}$ is simplicial.

Now, let I be infinite. Choose a, b, c, d from $\mathcal{A}^c(\mathcal{H}) = \boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$ such that $a \vee b < c \wedge d$. According to Proposition 3.16(b), there are

$$a' \in \left[\bigboxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_a}, \quad b' \in \left[\bigboxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_b},$$
$$c' \in \left[\bigboxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_c}, \quad d' \in \left[\bigboxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_d},$$

so that

$$a \vee b < a' \vee b' < c' \wedge d' < c \wedge d$$

and $J := I_a \cup I_b \cup I_c \cup I_d$ is a finite subset of I. Then also

$$\pi_J(a') \vee \pi_J(b') < \pi_J(c') \wedge \pi_J(d').$$

From the first part of the proof we know that $\boxtimes_{i\in J}\mathcal{H}_i$ is simplicial, so we can find $h' \in \mathcal{A}^c(\boxtimes_{i\in J}\mathcal{H}_i) = \boxtimes_{i\in J}\mathcal{A}^c(\mathcal{H}_i)$ such that

$$\pi_J(a') \vee \pi_J(b') < h' < \pi_J(c') \wedge \pi_J(d').$$

The function $h := h' \otimes 1_{\prod\{K_i : i\in I\setminus J\}} \in \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i) = \mathcal{A}^c(\mathcal{H})$ clearly satisfies

$$a \vee b < h < c \wedge d.$$

Hence $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. and $\mathcal{H}$ is simplicial. $\square$

The converse, whose special case has been proved in [18, Proposition 2.10], is also valid:

Theorem 3.53. *Suppose that $\mathcal{H}$ is simplicial. Then $\mathcal{H}_i$ is simplicial for each $i \in I$.*

PROOF: We use the W.R.I.P. property of simplicial spaces again. Choose $j \in I$. Let $a_j, b_j, c_j, d_j \in \mathcal{A}^c(\mathcal{H}_j)$ be such that $a_j \vee b_j < c_j \wedge d_j$. Denote $K' := \prod_{i\in I\setminus\{j\}} K_i$. According to Proposition 3.30,

$$a := a_j \otimes 1_{K'}, \quad b := b_j \otimes 1_{K'}, \quad c := c_j \otimes 1_{K'}, \quad d := d_j \otimes 1_{K'},$$

are elements of $\mathcal{A}^c(\mathcal{H})$. Moreover, $a \vee b < c \wedge d$. Using simpliciality of $\mathcal{H}$, there exists $h \in \mathcal{A}^c(\mathcal{H})$ so that

$$a \vee b < h < c \wedge d.$$

Pick $y \in K'$. By Proposition 3.28, $h \in \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$, therefore $\pi_j^y(h) \in \mathcal{A}^c(\mathcal{H}_j)$. Since

$$a_j \vee b_j < \pi_j^y(h) < c_j \wedge d_j,$$

we conclude that the space $\mathcal{A}^c(\mathcal{H}_j)$ has W.R.I.P. $\square$

Example 3.54. The space $\mathcal{H}_1 + \mathcal{H}_2$, defined by Grossman (see Remarks 3.44), does not have to be simplicial, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are simplicial. Indeed, let $K_1 = K_2 = [0,1] \subset \mathbb{R}$ and $\mathcal{H}_1 = \mathcal{H}_2 = A([0,1])$. Obviously, $\mathcal{H}_1$ and $\mathcal{H}_2$ are simplicial spaces. Denote $K := K_1 \times K_2$. It is easy to prove that $\mathcal{H}_1 + \mathcal{H}_2 = A(K)$. However, K is not a simplex, which is the sought contradiction.

3.8 Maximal measures. We start with two propositions, which are analogies of [2, Theorem 4]:

Proposition 3.55. *Let $\mu \in \mathcal{M}^+(K)$ be $\mathcal{H}$-maximal. Let $J \subset I$ and let $\mathcal{G}$ be a product of $\mathcal{H}_i$, $i \in J$. Then $\pi_J\mu$ is a $\mathcal{G}$-maximal measure.*

PROOF: According to Proposition 2.2, it suffices to show $F_{\pi_J(x)}(\mathcal{G}) \subset \pi_J(F_x(\mathcal{H}))$ for every $x \in K$. Using Proposition 3.27, we have

$$F_{\pi_J(x)}(\mathcal{G}) = \prod_{i\in J} F_{x_i}(\mathcal{H}_i) = \pi_J(\prod_{i\in I} F_{x_i}(\mathcal{H}_i)) = \pi_J(F_x(\mathcal{H})),$$

for every $x = (x_i)_{i\in I} \in K$. $\square$

Lemma 3.56. *Let $\mu, \nu \in \mathcal{M}^+(K)$ be such that $\mu \preceq_{\mathcal{H}} \nu$. Then for every $J \subset I$ is $\pi_J\mu \preceq_{\pi_J(\mathcal{H})} \pi_J\nu$.*

Proof: Choose $w_J \in \mathcal{W}(\pi_J(\mathcal{H}))$. Then $w := w_J \otimes 1_{\prod\{K_i : i \in I\setminus J\}} \in \mathcal{W}(\mathcal{H})$. Thus $\mu(w) \le \nu(w)$, and we get

$$(\pi_J\mu)(w_J) = \mu(w) \le \nu(w) = (\pi_J\nu)(w_J).$$

Since w_J is arbitrary, we have $\pi_J\mu \preceq \pi_J\nu$. □

Proposition 3.57. *Let $|I| = 2$ and let $\mu \in \mathcal{M}^+(K)$ be such that $\pi_i\mu$ is an $\mathcal{H}_i$-maximal measure for $i = 1, 2$. Then μ is $\mathcal{H}$-maximal.*

In particular, if $\mu_i \in \mathcal{M}^1(K_i)$ is an $\mathcal{H}_i$-maximal measure for $i = 1, 2$, then $\mu_1 \otimes \mu_2$ is $\mathcal{H}$-maximal.

Proof: We may proceed exactly as in the second part of the proof of [2, Theorem 4] to show that for every $h \in \mathcal{H}$ and μ-almost all $x \in K$ is

$$h(x_1, x_2) = h(\pi_1(x), \pi_2(x)), \quad x_1 \in F_{\pi_1(x)}(\mathcal{H}_1),\ x_2 \in F_{\pi_2(x)}(\mathcal{H}_2).$$

According to Proposition 3.27, $F_x(\mathcal{H}) = F_{\pi_1(x)}(\mathcal{H}_1) \times F_{\pi_2(x)}(\mathcal{H}_2)$ for every $x \in K$. Therefore h is constant on $F_x(\mathcal{H})$ for μ-almost all $x \in K$. As follows from Theorem 2.1, μ is an $\mathcal{H}$-maximal measure. □

Theorem 3.58. *Let $\mu \in \mathcal{M}^+(K)$ be such that $\pi_i\mu$ is an $\mathcal{H}_i$-maximal measure for every $i \in I$. Then μ is $\mathcal{H}$-maximal.*

In particular, if $\mu_i \in \mathcal{M}^1(K_i)$ is an $\mathcal{H}_i$-maximal measure for every $i \in I$, then $\bigotimes_{i\in I} \mu_i$ is $\mathcal{H}$-maximal.

Proof: It suffices to show that μ is a $(\bigodot_{i\in I} \mathcal{H}_i)$-maximal measure.

First we prove the assertion for finite products. Suppose that it holds for $|I| \le n$ and let $|I| = n+1$. We know that $\pi_{n+1}\mu$ is an $\mathcal{H}_{n+1}$-maximal measure. Since $\pi_i(\pi_{\{1,\dots,n\}}\mu) = \pi_i\mu$ is an $\mathcal{H}_i$-maximal measure for every $i = 1, \dots, n$, the induction hypothesis implies that $\pi_{\{1,\dots,n\}}\mu$ is a $(\bigodot_{i=1}^n \mathcal{H}_i)$-maximal measure. Thus, both projections are maximal measures and Proposition 3.57 implies that μ is a $((\bigodot_{i=1}^n \mathcal{H}_i) \odot \mathcal{H}_{n+1})$-maximal measure, therefore also $(\bigodot_{i=1}^{n+1} \mathcal{H}_i)$-maximal measure.

Now, let I be infinite. According to Choquet-Bishop-de Leeuw's theorem, there exists a $(\bigodot_{i\in I} \mathcal{H}_i)$-maximal measure $\nu \in \mathcal{M}^+(K)$ such that $\mu \preceq_{\bigodot_{i\in I} \mathcal{H}_i} \nu$. Suppose $J \subset I$ is finite. By Lemma 3.56, $\pi_J\mu \preceq_{\bigodot_{i\in J} \mathcal{H}_i} \pi_J\nu$. From the first part of the proof is $\pi_J\mu$ a $(\bigodot_{i\in J} \mathcal{H}_i)$-maximal measure and therefore $\pi_J\mu = \pi_J\nu$. Hence, for every finite subset $J \subset I$ and every $E = \prod_{i\in I} E_i$, where E_i is a Borel subset of K_i for each $i \in I$ and $E_i = K_i$ for $i \in I \setminus J$,

$$\mu(E) = (\pi_J\mu)(\prod_{i\in J} E_i) = (\pi_J\nu)(\prod_{i\in J} E_i) = \nu(E).$$

Since μ and ν coincide on the Borel cylinder sets, they must coincide as Radon measures. Therefore μ is a $(\bigodot_{i\in I} \mathcal{H}_i)$-maximal measure. □

Theorem 3.59. *Suppose that $\mathcal{H}_i$ is simplicial for each $i \in I$. Then $\delta_x = \bigotimes_{i\in I} \delta_{x_i}$ for every $x = (x_i)_{i\in I} \in K$.*

Proof: From Proposition 3.22 we have $\bigotimes_{i\in I} \delta_{x_i} \in \mathcal{M}_x(\mathcal{H})$ and by Theorem 3.58, this measure is $\mathcal{H}$-maximal. Since $\mathcal{H}$ is simplicial, according to Theorem 3.52, we get $\delta_x = \bigotimes_{i\in I} \delta_{x_i}$. □

At the end of this section we investigate relationship between maximal measures in product spaces and Radon products of maximal measures. We denote by $\mathcal{Z}^1(\mathcal{H})$ the set of $\mathcal{H}$-maximal measures from $\mathcal{M}^1(K)$. Let $\varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K} := \{\varepsilon_x : x \in \mathrm{Ch}_{\mathcal{H}}\, K\}$ and let $\mathcal{D}(\mathcal{H})$ denote the linear span of $\mathcal{C}(K) \cup \{f^* : f \in \mathcal{C}(K)\}$. We denote by τ the weak topology on $\mathcal{M}^1(K)$ generated by $\mathcal{D}(\mathcal{H})$. Then we have:

Proposition 3.60. *The following assertions hold:*

(a) $\mathrm{co}\, \varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K} \subset \mathcal{Z}^1(\mathcal{H}) \subset \overline{\mathrm{co}}^{w^*} \varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K}$,

(b) $\mathcal{Z}^1(\mathcal{H}) = \overline{\mathrm{co}}^{\tau} \varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K}$.

Proof: (a) The first inclusion is obvious. The second follows from the fact that

$$\overline{\mathrm{co}}^{w^*} \varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K} = \mathcal{M}^1(\overline{\mathrm{Ch}_{\mathcal{H}}\, K})$$

and all maximal measures are supported by $\overline{\mathrm{Ch}_{\mathcal{H}}\, K}$.

(b) We may proceed as in the proof of [1, Theorem I.6.14] to show that $\mathcal{Z}^1(\mathcal{H})$ is a τ-closed set and that for every $\mu \in \mathcal{Z}^1(\mathcal{H}) \setminus \overline{\mathrm{co}}^{\tau} \varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K}$, there are $f \in \mathcal{C}(K)$ and $\alpha \in \mathbb{R}$ such that

$$\sup_{x\in \mathrm{Ch}_{\mathcal{H}}\, K} \varepsilon_x(f) = \alpha < \mu(f).$$

Therefore $f(x) \le \alpha$ for every $x \in \overline{\mathrm{Ch}_{\mathcal{H}}\, K}$. But since $\mathrm{spt}\, \mu \subset \overline{\mathrm{Ch}_{\mathcal{H}}\, K}$, also $\mu(f) \le \alpha$, which is a contradiction. □

Example 3.61. From Theorem 3.58 we have $\mathrm{co} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i) \subset \mathcal{Z}^1(\mathcal{H})$. By Proposition 3.60(a), $\mathcal{Z}^1(\mathcal{H}) \subset \overline{\mathrm{co}}^{w^*} \varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K}$. Since $\varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K} = \bigotimes_{i\in I} \varepsilon_{\mathrm{Ch}_{\mathcal{H}_i}\, K_i} \subset \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i)$, we get

$$\mathrm{co} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i) \subset \mathcal{Z}^1(\mathcal{H}) \subset \overline{\mathrm{co}}^{w^*} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i).$$

Now we show that both inclusions may be proper:

Let $K_i := [0,2] \subset \mathbb{R}$, $\mathcal{H}_i := \{f \in \mathcal{C}(K_i) : f(1) = \frac{f(0)+f(2)}{2}\}$, $i = 1,2$. Then $\mathrm{Ch}_{\mathcal{H}_i}\, K_i = [0,1) \cup (1,2]$, $i = 1,2$. Choose $\{x_n\}_{n\in\mathbb{N}} \subset [0,1) \cup (1,2]$ so that $x_n \to 1$ and let $(K,\mathcal{H})$ be a product of $(K_i,\mathcal{H}_i)$, $i = 1,2$.

(a) Define $\mu := \sum_{n=1}^{\infty} 2^{-n} \varepsilon_{(x_n,x_n)}$. Clearly $\mu \in \mathcal{Z}^1(\mathcal{H})$, since it is supported by $\mathrm{Ch}_{\mathcal{H}}\, K$. However, $\mu \notin \mathrm{co}(\mathcal{Z}^1(\mathcal{H}_1) \otimes \mathcal{Z}^1(\mathcal{H}_2))$. Indeed, μ is supported by the diagonal Δ of K, but the only measures of $\mathcal{Z}^1(\mathcal{H}_1) \otimes \mathcal{Z}^1(\mathcal{H}_2)$ supported by Δ are ε_x, $x \in \Delta$. Thus, μ would be supported by a finite set.

(b) Obviously $\varepsilon_{(1,1)} \notin \mathcal{Z}^1(\mathcal{H})$. However, $\varepsilon_{(x_n,x_n)} \overset{w^*}{\to} \varepsilon_{(1,1)}$. Thus, $\varepsilon_{(1,1)} \in \overline{\mathrm{co}}^{w^*}(\mathcal{Z}^1(\mathcal{H}_1) \otimes \mathcal{Z}^1(\mathcal{H}_2))$.

Proposition 3.62. $\mathcal{Z}^1(\mathcal{H}) = \overline{\mathrm{co}}^{\tau} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i)$.

Proof: Using Proposition 3.60(b) and Theorems 3.42 and 3.58, we can write

$$\mathcal{Z}^1(\mathcal{H}) = \overline{\mathrm{co}}^{\tau} \varepsilon_{\mathrm{Ch}_{\mathcal{H}}\, K} = \overline{\mathrm{co}}^{\tau} \bigotimes_{i\in I} \varepsilon_{\mathrm{Ch}_{\mathcal{H}_i}\, K_i} \subset \overline{\mathrm{co}}^{\tau} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i) \subset \mathcal{Z}^1(\mathcal{H}).$$

□

4. Projective limits of function spaces

Definition 4.1. Let $(K_1, \mathcal{H}_1)$ and $(K_2, \mathcal{H}_2)$ be function spaces. We say that a continuous surjection $\varphi : K_2 \to K_1$ is an *admissible map*, if $\mathcal{H}_1 \circ \varphi := \{h \circ \varphi : h \in \mathcal{H}_1\} \subset \mathcal{H}_2$.

Let I be an up-directed index set. We say that $((K_i, \mathcal{H}_i), \pi_{ij})_{i,j\in I}$ is a *projective system* of function spaces, if every $\pi_{ij} : K_j \to K_i$, $i \le j$, is an admissible map such that

(i) π_{ii} is the identity on K_i for each i,
(ii) $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ for all $i \le j \le k$.

Projective limit, denoted by $\varprojlim ((K_i, \mathcal{H}_i), \pi_{ij})_{i,j\in I}$, of this projective system is the function space $(K, \mathcal{H})$, where

$$K := \{(x_i)_{i\in I} \in \prod_{i\in I} K_i : x_i = \pi_{ij}(x_j) \ \text{ for every } \ i \le j,\, i,j \in I\}$$

and $\mathcal{H}$ is the restriction to K of the function space $\bigcup_{i\in I} \mathcal{H}_i \circ \pi_i$ with π_i the i-th projection map.

It follows from standard results on projective limits of compact Hausdorff spaces (see e.g. [5]), that K is a non-empty compact Hausdorff space, if K_i is non-empty for every $i \in I$, and that each π_i is a surjection. Notice that $\pi_{ij} \circ \pi_j = \pi_i$ for every $i \le j$. Clearly, $\mathcal{H}$ contains constant functions and separates points of K. If $h = h_i \circ \pi_i \in \mathcal{H}$ for some $i \in I$, then also $\alpha h \in \mathcal{H}$ for every $\alpha \in \mathbb{R}$, since $\alpha h_i \in \mathcal{H}_i$. Now, let $h_1, h_2 \in \mathcal{H}$. Suppose $h_1 = h_{i_1} \circ \pi_{i_1}$, $h_2 = h_{i_2} \circ \pi_{i_2}$ for some $i_1, i_2 \in I$ and $h_{i_1} \in \mathcal{H}_{i_1}$, $h_{i_2} \in \mathcal{H}_{i_2}$. Let $j \in I$ be such that $i_1, i_2 \le j$. Then $h_1 = h_{i_1} \circ \pi_{i_1 j} \circ \pi_j$ and $h_2 = h_{i_2} \circ \pi_{i_2 j} \circ \pi_j$ where $h_{i_1} \circ \pi_{i_1 j}, h_{i_2} \circ \pi_{i_2 j} \in \mathcal{H}_j$. Now it is easy to see that $h_1 + h_2 \in \mathcal{H}$, since $\mathcal{H}_j$ is a linear space. Thus $\mathcal{H}$ is a function space with each π_i being an admissible map.

Remark 4.2. If $(K_i, \mathcal{H}_i) = (X_i, A(X_i))$ where X_i is a compact convex set for every $i \in I$, then the projective limit defined above is dense in $A(K)$ as shown in [13].

Lemma 4.3. *Let $(K_i, \mathcal{H}_i)$, $i = 1, 2$, be function spaces, $\varphi : K_2 \to K_1$ admissible map and $x \in K_2$. If $\mu \in \mathcal{M}_x(\mathcal{H}_2)$, then $\varphi\mu \in \mathcal{M}_{\varphi(x)}(\mathcal{H}_1)$.*

PROOF: Choose $h \in \mathcal{H}_1$. Then

$$(\varphi\mu)(h) = \mu(h \circ \varphi) = (h \circ \varphi)(x) = h(\varphi(x)),$$

since $h \circ \varphi \in \mathcal{H}_2$. □

Observation 4.4. *If $\mu \in \mathcal{M}^+(K)$, then $(\pi_i\mu, \pi_{ij})_{i,j\in I}$ forms a projective system of measures.*

Theorem 4.5. *Let $(\mu_i, \pi_{ij})_{i,j\in I}$ be a projective system of measures with $\mu_i \in \mathcal{M}^1(K_i)$ for each $i \in I$. Then there is a unique measure $\mu = \varprojlim \mu_i \in \mathcal{M}^1(K)$ such that $\pi_i\mu = \mu_i$ for every $i \in I$.*

PROOF: See [8, Theorem 418M and Proposition 418O]. □

Proposition 4.6. *Let $x = (x_i)_{i\in I} \in K$ and $\mu \in \mathcal{M}^1(K)$. Then $\mu \in \mathcal{M}_x(\mathcal{H})$ if and only if $(\pi_i\mu, \pi_{ij})_{i,j\in I}$ is a projective system of measures with $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$.*

PROOF: First assume $\mu \in \mathcal{M}_x(\mathcal{H})$. It follows from Lemma 4.3 that $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$, since each π_i is admissible, and from Observation 4.4 that this system is projective.

On the contrary, suppose $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$, $i \in I$. Let $h \in \mathcal{H}$. Then $h = h_j \circ \pi_j$ with $h_j \in \mathcal{H}_j$ for some $j \in I$. Thus

$$\mu(h) = \mu(h_j \circ \pi_j) = (\pi_j\mu)(h_j) = h_j(x_j) = h(x).$$

□

Corollary 4.7. *Let $x = (x_i)_{i\in I} \in K$ and let $(\mu_i, \pi_{ij})_{i,j\in I}$ be a projective system of measures with $\mu_i \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$. Then $\mu := \varprojlim \mu_i \in \mathcal{M}_x(\mathcal{H})$.*

Lemma 4.8. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map. Then $\mathcal{K}^c(\mathcal{H}_1) \circ \varphi \subset \mathcal{K}^c(\mathcal{H}_2)$.*

PROOF: Let $k \in \mathcal{K}^c(\mathcal{H}_1)$. Choose $x \in K_2$ and $\mu \in \mathcal{M}_x(\mathcal{H}_2)$. Since $\varphi\mu \in \mathcal{M}_{\varphi(x)}(\mathcal{H}_1)$, we have

$$(k \circ \varphi)(x) = k(\varphi(x)) \leq (\varphi\mu)(k) = \mu(k \circ \varphi).$$

Thus $k \circ \varphi \in \mathcal{K}^c(\mathcal{H}_2)$. □

Lemma 4.9. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map. Then $\mathcal{A}^c(\mathcal{H}_1) \circ \varphi \subset \mathcal{A}^c(\mathcal{H}_2)$. In particular, $\bigcup_{i \in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i \subset \mathcal{A}^c(\mathcal{H})$.*

PROOF: Follows from Lemma 4.8, since $\mathcal{A}^c(\mathcal{H}_i) = \mathcal{K}^c(\mathcal{H}_i) \cap (-\mathcal{K}^c(\mathcal{H}_i))$, $i = 1, 2$. □

Proposition 4.10. *If $\mathcal{H}_i$ is simplicial for every $i \in I$, then $\bigcup_{i \in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i$ is dense in $\mathcal{A}^c(\mathcal{H})$.*

PROOF: Let $a \in \mathcal{A}^c(\mathcal{H})$ and $\varepsilon > 0$. Since $a \in \widehat{\mathcal{H}}$, for every $x \in K$ there are $h_x^-, h_x^+ \in \mathcal{H}$ such that $h_x^- < a < h_x^+$ and

$$a(x) - \varepsilon < h_x^-(x) < a(x) < h_x^+(x) < a(x) + \varepsilon.$$

These inequalities hold on some open neighbourhood U_x of x. By compactness, we can choose $U_{x_1}, \dots, U_{x_n}$ covering K. Suppose that $h_{x_m}^-$ and $h_{x_m}^+$ depend on coordinates $i_m^-, i_m^+ \in I$, respectively, for $m = 1, \dots, n$. Let $j \in I$ be an upper bound of the set $\{i_m^-, i_m^+\}_{m=1}^n$. Denote $h^- := h_{x_1}^- \vee \dots \vee h_{x_n}^-$ and $h^+ := h_{x_1}^+ \wedge \dots \wedge h_{x_n}^+$. Now we have $h^- < a < h^+$ and $\|a - h^-\|, \|a - h^+\| < \varepsilon$. Since both h^-, h^+ depend on coordinate j, using W.R.I.P. for $\mathcal{H}_j$ we find $a_j \in \mathcal{A}^c(\mathcal{H}_j)$ such that $h^- < a_j \circ \pi_j < h^+$. Hence $a_j \circ \pi_j \in \bigcup_{i \in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i$ and $\|a - a_j \circ \pi_j\| < \varepsilon$. □

Theorem 4.11. *If $\mathcal{H}_i$ is simplicial for every $i \in I$, then $\mathcal{H}$ is simplicial.*

PROOF: We show that $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. Let $a_1, \dots, a_4 \in \mathcal{A}^c(\mathcal{H})$ be such that $a_1 \vee a_2 < a_3 \wedge a_4$. By Proposition 4.10, we may assume that $a_1, \dots, a_4 \in \bigcup_{i \in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i$ with a_m depending on coordinate i_m, $m = 1, \dots, 4$. Let $j \in I$ be an upper bound of $i_1, \dots, i_4$. Since a_m, $m = 1, \dots, 4$, depend on coordinate j, from W.R.I.P. for $\mathcal{H}_j$ there is $a_j \in \mathcal{A}^c(\mathcal{H}_j)$ such that

$$a_1 \vee a_2 < a_j \circ \pi_j < a_3 \wedge a_4.$$

By Lemma 4.9, $a_j \circ \pi_j \in \mathcal{A}^c(\mathcal{H})$, which completes the proof. □

Proposition 4.12. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map. Then $\varphi(\mathrm{Ch}_{\mathcal{H}_2} K_2) \supset \mathrm{Ch}_{\mathcal{H}_1} K_1$.*

PROOF: See [1, Proposition I.5.20]. □

Proposition 4.13. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map, where $\mathcal{H}_1$ is simplicial. Then the following assertions are equivalent:*

(i) $\varphi(\mathrm{Ch}_{\mathcal{H}_2} K_2) = \mathrm{Ch}_{\mathcal{H}_1} K_1$,
(ii) $(k \circ \varphi)^* = k^* \circ \varphi$ *for every* $k \in \mathcal{K}^c(\mathcal{H}_1)$,
(iii) φ *maps $\mathcal{H}_2$-maximal measures onto $\mathcal{H}_1$-maximal measures.*

PROOF: The proof of (i) ⇒ (ii) is included in the proof of [15, Theorem 1.3]. Moreover, the proof mentioned above shows that (ii) is a sufficient condition for φ to map maximal measures onto maximal measures. The last implication (iii) ⇒ (i) is immediate. □

A convex versions of the next theorem can be found in [6, Theorem 14] and [13, Theorem 2]. A proof for closed function spaces has been given in [10, Corollary 4.13]. For the sake of completeness we include the proof using different approach:

Theorem 4.14. *Let $x = (x_i)_{i\in I} \in K$. The following assertions hold.*

(i) *If $x_i \in \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \in I$, then $x \in \mathrm{Ch}_{\mathcal{H}} K$.*
(ii) *Suppose that $\mathcal{H}_i$ is simplicial for every $i \in I$ and $\pi_{ij}(\mathrm{Ch}_{\mathcal{H}_j} K_j) \subset \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \le j$, $i, j \in I$. Then $x \in \mathrm{Ch}_{\mathcal{H}} K$ if and only if $x_i \in \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \in I$.*

PROOF: First assume $x_i \in \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \in I$. Let $\mu \in \mathcal{M}_x(\mathcal{H})$. According to Proposition 4.6, $(\pi_i\mu, \pi_{ij})_{i,j\in I}$ is a projective system of measures with $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$. Thus $\pi_i\mu = \varepsilon_{x_i}$ for each $i \in I$ and from the uniqueness of the projective limit of measures we see that $\mu = \varprojlim (\varepsilon_{x_i}, \pi_{ij})_{i,j\in I} = \varepsilon_x$.

Now assume $x \in \mathrm{Ch}_{\mathcal{H}} K$ and the conditions of (ii) are satisfied. Choose $i \in I$. According to Corollary 2.4, it is enough to prove that $k_i(x_i) = k_i^*(x_i)$ for every $k_i \in \mathcal{K}^c(\mathcal{H}_i)$. So let $k_i \in \mathcal{K}^c(\mathcal{H}_i)$ and $\varepsilon > 0$. Denote $k := k_i \circ \pi_i \in \mathcal{K}^c(\mathcal{H})$. Since $x \in \mathrm{Ch}_{\mathcal{H}} K$, there is some $h \in \mathcal{H}$ such that $k \le h$ and $k(x) \le h(x) < k(x) + \varepsilon$. Without loss of generality suppose that $h = h_j \circ \pi_j$ for some $j \ge i$, $j \in I$, and $h_j \in \mathcal{H}_j$. Then $(k_i \circ \pi_{ij})(x_j) \le h_j(x_j) < (k_i \circ \pi_{ij})(x_j) + \varepsilon$. Using these inequalities and Proposition 4.13 we get

$$(k_i^* \circ \pi_{ij})(x_j) = (k_i \circ \pi_{ij})^*(x_j) \le (k_i \circ \pi_{ij})(x_j) + \varepsilon.$$

Hence $k_i^*(x_i) \le k_i(x_i) + \varepsilon$. Since ε is arbitrary, we conclude that $k_i(x_i) = k_i^*(x_i)$. □

Example 4.15. This example shows that the characterization in Theorem 4.14(ii) does not have to hold, if we omit the assumption of simpliciality, and also that the converse to Theorem 4.11 is not valid.

Choose a sequence $\{q_n\}_{n\in\mathbb{N}} \subset (0, 1)$ of real numbers such that $q_n \to 0$. For every $i \in \mathbb{N}$ set $K_i := \{0\} \cup \{-q_n, q_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and

$$\mathcal{H}_i := \{f \in \mathcal{C}(K_i) : f(0) = \frac{f(-q_n) + f(q_n)}{2}, n \ge i, n \in \mathbb{N}\}.$$

Let $(K, \mathcal{H}) := \varprojlim((K_i, \mathcal{H}_i), \mathrm{Id}_{ij})_{i,j\in\mathbb{N}}$, where $\mathrm{Id}_{ij} : K_j \to K_i$ denotes the identity map. Clearly $\mathrm{Ch}_{\mathcal{H}_i} K_i = K_i \setminus \{0\}$ for every $i \in \mathbb{N}$. We claim that $x := (0, 0, \dots) \in \mathrm{Ch}_{\mathcal{H}} K$. Indeed, choose $\mu \in \mathcal{M}_x(\mathcal{H})$. By Proposition 4.6, $\pi_i\mu \in \mathcal{M}_0(\mathcal{H}_i)$ for every

$i \in \mathbb{N}$ and $(\pi_i\mu, \mathrm{Id}_{ij})_{i,j\in\mathbb{N}}$ is a projective system, so $\pi_i\mu = \pi_j\mu$ for every $i, j \in \mathbb{N}$. But the only measure representing 0 in all spaces $(K_i, \mathcal{H}_i)$, $i \in \mathbb{N}$, is ε_0. Hence $\mu = \varprojlim (\varepsilon_0, \mathrm{Id}_{ij})_{i,j\in\mathbb{N}} = \varepsilon_x$, which proves the claim. Using Theorem 4.14(i) we conclude that $\mathrm{Ch}_{\mathcal{H}}\ K = K$.

Therefore the conclusion of Theorem 4.14(ii) does not hold and we also see that the projective limit of non-simplicial spaces may be simplicial.

Example 4.16. Now we show that we cannot take the restriction of a product space from Section 3 as the definition of the projective limit of function spaces, if we want Theorem 4.14 to hold.

Let $K_i := [-1, 1] \subset \mathbb{R}$ and $\mathcal{H}_i := A(K_i)$ for $i = 1, 2$. Let K stand for the topological projective limit of the projective system $(K_i, \mathrm{Id}_{ij})_{i,j=1,2}$ (i.e., the diagonal of $K_1 \times K_2$) and define $\mathcal{H} := (\mathcal{H}_1 \odot \mathcal{H}_2)\restriction_K$.

Clearly all conditions of Theorem 4.14(ii) are satisfied. However, we can see that $0 \notin \mathrm{Ch}_{\mathcal{H}_i}\ K_i$, $i = 1, 2$, but $(0, 0) \in \mathrm{Ch}_{\mathcal{H}}\ K$, since $f_1 \otimes f_2 \in \mathcal{H}$ is an exposing function of $(0, 0)$, where $f_i(x) = x$, $x \in K_i$, $i = 1, 2$. The point $(0, 0)$ is also in the Choquet boundary of the restriction of any other product space, since $\mathcal{H}_1 \odot \mathcal{H}_2$ is the smallest product.

Lemma 4.17. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map and let $\mu, \nu \in \mathcal{M}^+(K_2)$ be such that $\mu \preceq \nu$. Then $\varphi\mu \preceq \varphi\nu$.*

PROOF: Let $k \in \mathcal{K}^c(\mathcal{H}_1)$. Since $k \circ \varphi \in \mathcal{K}^c(\mathcal{H}_2)$, we have

$$(\varphi\mu)(k) = \mu(k \circ \varphi) \le \nu(k \circ \varphi) = (\varphi\nu)(k).$$

□

Proposition 4.18. *Suppose $\mathcal{H}_i$ is simplicial for every $i \in I$ and $\pi_{ij}(\mathrm{Ch}_{\mathcal{H}_j}\ K_j) \subset \mathrm{Ch}_{\mathcal{H}_i}\ K_i$ for every $i \le j$, $i, j \in I$. Let $\mu \in \mathcal{M}^1(K)$. Then μ is $\mathcal{H}$-maximal if and only if $\pi_i\mu$ is $\mathcal{H}_i$-maximal for every $i \in I$.*

PROOF: First assume that μ is maximal and choose $i \in I$. According to Theorem 4.14, $\pi_i(\mathrm{Ch}_{\mathcal{H}}\ K) = \mathrm{Ch}_{\mathcal{H}_i}\ K_i$. Using Proposition 4.13 we conclude that $\pi_i\mu$ is maximal.

Conversely, let $\pi_i\mu$ be maximal for every $i \in I$. Let $\nu \in \mathcal{M}^1(K)$ be such that $\mu \preceq \nu$. By Lemma 4.17, $\pi_i\mu \preceq \pi_i\nu$ for every $i \in I$. Therefore $\pi_i\mu = \pi_i\nu$ for every $i \in I$ and from the uniqueness of the projective limit $\mu = \nu$. □

Definition 4.19. We say that $J \subset I$ is *cofinal*, if for every $i \in I$ there is $j \in J$ such that $i \le j$.

Proposition 4.20. *Let $J \subset I$ be cofinal and let $(K', \mathcal{H}') := \varprojlim ((K_i, \mathcal{H}_i), \pi_{ij})_{i,j\in J}$. Then*

(a) *there is a homeomorphism $\phi : K \to K'$,*
(b) *$\mathcal{H}$ is isometrically isomorphic to $\mathcal{H}'$,*

(c) $\phi(\mathrm{Ch}_{\mathcal{H}}\, K) = \mathrm{Ch}_{\mathcal{H}'}\, K'$,
(d) $\mu \in \mathcal{M}^+(K)$ *is maximal if and only if* $\phi\mu$ *is maximal,*
(e) $\mathcal{H}$ *is simplicial if and only if* $\mathcal{H}'$ *is simplicial.*

In particular, if there is the greatest element $m \in I$*, then previous statements hold with* $(K_m, \mathcal{H}_m)$ *in place of* $(K', \mathcal{H}')$*.*

Proof: (a) The canonical bijection $\phi : (x_i)_{i\in I} \mapsto (x_i)_{i\in J}$ is a homeomorphism by standard results (see e.g. [5]).

(b) Mapping $\Phi : f \mapsto f \circ \phi$ is an isometrical isomorphism of $\mathcal{C}(K')$ onto $\mathcal{C}(K)$. Let us denote by π_i projections on K and by π_i' projections on K'. Suppose $h = h_j \circ \pi_j' \in \mathcal{H}'$ for some $h_j \in \mathcal{H}_j$ and $j \in J$. Then $\Phi(h) = h \circ \phi = h_j \circ \pi_j' \circ \phi = h_j \circ \pi_j \in \mathcal{H}$. Conversely, let $h = h_i \circ \pi_i \in \mathcal{H}$ for some $h_i \in \mathcal{H}_i$ and $i \in I$. Choose $j \in J$ such that $i \le j$ and denote $h_j := h_i \circ \pi_{ij} \in \mathcal{H}_j$. Then $\Phi^{-1}(h) = h \circ \phi^{-1} = h_j \circ \pi_j \circ \phi^{-1} = h_j \circ \pi_j' \in \mathcal{H}'$.

(c) Notice that the mapping Φ above is also order preserving. The statement follows easily from the characterization of the Choquet boundary (Corollary 2.4) and properties of Φ.

(d) Since ϕ is a homeomorphism, $\phi : \mathcal{M}^+(K) \to \mathcal{M}^+(K')$ is a bijection. Now we use Proposition 2.3. Suppose μ is $\mathcal{H}$-maximal and let $k \in \mathcal{K}^c(\mathcal{H}')$. From the proof of (b) we can see that ϕ is admissible map and $(k \circ \phi)^* = k^* \circ \phi$. Thus

$$(\phi\mu)(k) = \mu(k \circ \phi) = \mu((k \circ \phi)^*) = \mu(k^* \circ \phi) = (\phi\mu)(k^*).$$

Since k is arbitrary, maximality of $\phi\mu$ follows. The converse is analogical.

(e) Let $x \in K$. We claim that ϕ maps $\mathcal{M}_x(\mathcal{H})$ onto $\mathcal{M}_{\phi(x)}(\mathcal{H}')$. Indeed, suppose $\mu \in \mathcal{M}_x(\mathcal{H})$ and let $h \in \mathcal{H}'$ be arbitrary. Now $(\phi\mu)(h) = \mu(\Phi(h)) = \Phi(h)(x) = h(\phi(x))$. Therefore $\phi\mu \in \mathcal{M}_{\phi(x)}(\mathcal{H}')$. The converse is analogical. Hence, using statement (d), ϕ maps maximal representing measures onto maximal representing measures and the conclusion follows. □

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