A note on perfect matchings in uniform hypergraphs with large minimum collective degree

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Abstract. For an integer $k \geq 2$ and a k-uniform hypergraph H, let $\delta_{k-1}(H)$ be the largest integer d such that every $(k-1)$-element set of vertices of H belongs to at least d edges of H. Further, let $t(k,n)$ be the smallest integer t such that every k-uniform hypergraph on n vertices and with $\delta_{k-1}(H) \geq t$ contains a perfect matching. The parameter $t(k,n)$ has been completely determined for all k and large n divisible by k by Rödl, Ruciński, and Szemerédi in [*Perfect matchings in large uniform hypergraphs with large minimum collective degree*, submitted]. The values of $t(k,n)$ are very close to $n/2-k$. In fact, the function $t(k,n) = n/2 - k + c_{n,k}$, where $c_{n,k} \in \{3/2, 2, 5/2, 3\}$ depends on the parity of k and n. The aim of this short note is to present a simple proof of an only slightly weaker bound: $t(k,n) \leq n/2 + k/4$. Our argument is based on an idea used in a recent paper of Aharoni, Georgakopoulos, and Sprüssel.

Keywords: hypergraph, perfect matching

Classification: Primary 05C70; Secondary 05C65

1. Introduction

A k-uniform hypergraph is a pair $H = (V, E)$, where $V := V(H)$ is a finite set of vertices and $E := E(H) \subseteq \binom{V}{k}$ is a family of k-element subsets of V. Whenever convenient we will identify H with $E(H)$. *A matching* in H is a set of pairwise disjoint edges of H.

Given a k-uniform hypergraph H and r vertices $v_1, \ldots, v_r \in V(H)$, $1 \leq r \leq k-1$, we denote by $\deg_H(v_1, \ldots, v_r)$ the number of edges of H which contain $v_1, \ldots, v_r$. Let $\delta_r(H) := \delta_r$ be the minimum of $\deg_H(v_1, \ldots, v_r)$ over all r-element sets of vertices of H.

Definition 1. For all integers $k \geq 2$ and $n \geq k$ divisible by k, denote by $t(k,n)$ the smallest integer t such that every k-uniform hypergraph on n vertices with $\delta_{k-1} \geq t$ contains a perfect matching, that is, a matching of size n/k.

The first author was supported by NSF grant DMS 0300529.

The second author was supported by grant N201036 32/2546 and part of research was performed at Emory University, Atlanta.

The fourth author was supported by NSF grants DMS-0100784 and DMS-0603745.

For graphs, an easy argument shows that $t(2,n) = n/2$. It follows from [3] that $t(k,n) \leq n/2 + o(n)$. In [2], Kühn and Osthus proved that $t(k,n) \leq n/2 + 3k^2\sqrt{n \log n}$. This was further improved in [5] to $t(k,n) \leq n/2 + C \log n$. Finally, the precise result was proved in [4], where it was shown that $t(k,n) = n/2 - k + c_{n,k}$, where $c_{n,k} \in \{3/2, 2, 5/2, 3\}$ depends on the parity of k and n. The aim of this short note is to present a simple proof of an only slightly weaker bound.

Theorem 2. *For all $k \geq 3$ and n divisible by k, $t(k,n) \leq n/2 + k/4$.*

Our argument is based on an idea used in a recent paper of Aharoni, Georgakopoulos, and Sprüssel [1]. Answering a question from [2], those authors proved in [1] a similar result for k-partite, k-uniform hypergraphs. Their result says that if $V(H) = V_1 \cup \cdots \cup V_k$, $|V_1| = \cdots = |V_k| = n$, and for every $(k-1)$-tuple of vertices $(v_1, \ldots, v_{k-1}) \in V_1 \times \cdots \times V_{k-1}$ we have $\deg_H(v_1, \ldots, v_{k-1}) > n/2$, while for every $(v_2, \ldots, v_k) \in V_2 \times \cdots \times V_k$ we have $\deg_H(v_2, \ldots, v_k) \geq n/2$, then H has a perfect matching. While their simple and elegant approach does not seem to readily yield the precise function $t(n,k)$, it can be modified to prove Theorem 2.

2. Proof of Theorem 2

Let H be a k uniform hypergraph on n vertices, where n is divisible by k, such that $\delta_{k-1}(H) \geq n/2 + k/4$. Further, let M be a largest matching in H. Suppose to the contrary that $|M| \leq n/k - 1$, that is, M is not perfect. By adding fake edges if necessary, without loss of generality we may assume that $|M| = n/k - 1$. (Alternatively, one could apply Proposition 2.1 from [4] — see Remark 2.1 there, which says that H contains a matching of size at least $n/k - 1$, if $\delta_{k-1}(H) \geq n/k + O(\log n)$.) Let $x_1, \ldots, x_k$ be the vertices of H not covered by M.

For every $u \in V(M)$, let e_u be the edge of M containing u. For every vertex v of H, let $T_M(v)$ be the set of vertices $u \in V(M)$ such that $(e_u \setminus \{u\}) \cup \{v\}$ is an edge of H. Set $t_M(v) = |T_M(v)|$.

Observation 1. *For each $i = 1, \ldots, k$, $t_M(x_i) \leq n/2 - 5k/4$.*

Proof: If, say, $t_M(x_k) > n/2 - 5k/4$, then $\deg_H(x_1, \ldots, x_{k-1}) + t_M(x_k) > n - k = |V(M)|$, so $N(x_1, \ldots, x_{k-1}) \cap T_M(x_k) \neq \emptyset$. Let $u \in N(x_1, \ldots, x_{k-1}) \cap T_M(x_k)$. Then, setting $e' = \{u, x_1, \ldots, x_{k-1}\}$ and $e'' = (e_u \setminus \{u\}) \cup \{x_k\}$, we see that $M' = (M \setminus \{e_u\}) \cup \{e', e''\}$ is a perfect matching in H — a contradiction. □

Observation 2. *There exists $w \in V(M)$ with $t_M(w) > n/2 - k/4$.*

Proof: Let $B = (X \dot\cup Y, E_B)$ be an auxiliary bipartite graph where $X = V(M)$, $Y = V(H)$, and $uv \in E_B$ if and only if $u \in X$, $v \in Y$, and $u \in T_M(v)$. In view of the assumption on $\delta_{k-1}(H)$, for each of the $n - k$ vertices $u \in X$ we have

$\deg_B(u) \geq n/2 + k/4$. Let $Y' = Y \setminus \{x_1, \ldots, x_k\}$. Then, in view of Observation 1, the number of edges in the induced subgraph $B' = B[X \cup Y']$ is at least

$$(n-k)\left(\frac{n}{2}+\frac{k}{4}\right) - k\left(\frac{n}{2}-\frac{5k}{4}\right).$$

Hence, by averaging, there exists $w \in Y' = V(M)$ such that

$$t_M(w) = \deg_{B'}(w) \geq \frac{e(B')}{n-k} \geq \left(\frac{n}{2}+\frac{k}{4}\right) - \frac{k(n/2-5k/4)}{n-k} > \frac{n}{2} - \frac{k}{4}.$$

□

Fix w as in Observation 2.

Observation 3. *There exists two vertices v_1 and v_2 and an edge $e \in M \setminus \{e_w\}$ such that $\{v_1, v_2\} \subseteq e$, $v_1 \in N_H(e_w \setminus \{w\})$, and $v_2 \in N_H(x_1, \ldots, x_{k-1})$.*

Proof: Together, the $(k-1)$-tuples $S_1 = e_w \setminus \{w\}$ and $S_2 = \{x_1, \ldots, x_{k-1}\}$ have at most $2(k+1)-1 = 2k+1$ neighbors in $e_w \cup \{x_1, \ldots, x_k\}$. Thus, the total number of pairs (v, i), where $v \in N_H(S_i)$, $v \notin e_w \cup \{x_1, \ldots, x_k\}$, and $i = 1, 2$, is at least $2(n/2 + k/4) - 2k - 1$, and, by averaging, there exists $e \in M \setminus \{e_w\}$ for which

$$|\{(v,i)\colon v \in N_H(S_i) \cap e, i = 1,2\}| \geq \frac{n + k/2 - 2k - 1}{n/k - 2} > k.$$

Consequently, there exist $v_1, v_2 \in e$, $v_1 \neq v_2$, such that $v_i \in N_H(S_i)$, $i = 1, 2$. □

By Observation 3, setting $e' = (e_w \setminus \{w\}) \cup \{v_1\}$ and $e'' = \{x_1, \ldots, x_{k-1}, v_2\}$, one can replace M with another matching $M' = (M \setminus \{e_w, e\}) \cup \{e', e''\}$ of the same size, but such that $w \notin V(M')$. Note that $T_M(w) \setminus T_{M'}(w) \subseteq e$, and so,

$$t_{M'}(w) \geq t_M(w) - k > n/2 - 5k/4.$$

This is, however, a contradiction to Observation 1 (applied to M'). This completes the proof of Theorem 2.

Remark 3. We believe that the bound on $t(n, k)$ from Theorem 2 can be improved slightly, with a more cumbersome case analysis. However, for a clearer presentation we avoided those details.

Acknowledgments. We are grateful to the authors of [1] for providing us with an early version of their manuscript.

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(*Received* January 8, 2008, *revised* March 26, 2008)