Addition theorems, D-spaces and dually discrete spaces

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Abstract. A *neighbourhood assignment* in a space X is a family $\mathcal{O} = \{O_x : x \in X\}$ of open subsets of X such that $x \in O_x$ for any $x \in X$. A set $Y \subseteq X$ is *a kernel of* $\mathcal{O}$ if $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\} = X$. If every neighbourhood assignment in X has a closed and discrete (respectively, discrete) kernel, then X is said to be a D-space (respectively a dually discrete space). In this paper we show among other things that every GO-space is dually discrete, every subparacompact scattered space and every continuous image of a Lindelöf P-space is a D-space and we prove an addition theorem for metalindelöf spaces which answers a question of Arhangel'skii and Buzyakova.

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1. Introduction

A *neighbourhood assignment* in a space X is a family $\mathcal{O} = \{O_x : x \in X\}$ of open subsets of X such that $x \in O_x$ for any $x \in X$. A set $Y \subseteq X$ is *a kernel of* $\mathcal{O}$ if $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\} = X$.

For any class (or property) $\mathcal{P}$ we define a dual class $\mathcal{P}^d$ which consists of spaces X such that, for any neighbourhood assignment $\mathcal{O}$ in the space X there exists a subspace $Y \subseteq X$ such that $\mathcal{O}(Y) = X$ and $Y \in \mathcal{P}$; the spaces from $\mathcal{P}^d$ are called *dually* $\mathcal{P}$. Thus a space is *dually discrete* if every neighbourhood assignment in X has a discrete kernel and is a *D-space* if it has a closed and discrete kernel. It is an immediate consequence of the definition, that if X is dually discrete, then $L(X) \le s(X)$ (where $L(X)$ is the *Lindelöf number* of X and $s(X)$ is the *spread* of X; definitions can be found in [12]).

The concept of a D-space was introduced in [9] and has attracted a great deal of attention recently (see for example [4], [5] and [11]). Possibly the first mention of dually discrete spaces can be found in [16] and their study was continued in [3] and [7] and most recently [1]. On consulting these papers it is immediately obvious that the class of dually discrete spaces is "very large" — in some sense it is difficult to construct spaces which are not dually discrete. However, in [7],

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examples of (Hausdorff, some even Tychonoff) spaces which are not dually discrete were constructed in ZFC but all the known examples depend on the existence of spaces X in which $hd(X) < hL(X)$ (where $hd(X)$ denotes the *hereditary density* of X and $hL(X)$ the *hereditary Lindelöf number* of X).

All spaces are assumed to be T_1 and all undefined notation and terminology is taken from [12].

2. Addition theorems

In this section we consider the conditions under which the properties of being a D-space, being dually discrete and being metalindelöf are preserved under finite unions. The main result of this section (Theorem 2.11) answers a question posed in [5].

Theorem 2.1. *If (X, τ) is a T_1-space and $F \subseteq X$ is the union of a σ-locally finite family of closed (in X) D-subspaces (respectively, dually discrete subspaces), then $(F, \tau|F)$ is a D-space (respectively, a dually discrete space).*

Proof: We prove the theorem for D-subspaces, the proof for dually discrete subspaces is virtually identical. So, assume that $F = \bigcup\{\bigcup \mathcal{F}_n : n \in \omega\}$, where each $\mathcal{F}_n$ is a locally finite family of closed (in X), D-subspaces (in the relative topology) and $\mathcal{O} = \{O_x : x \in F\}$ is a neighbourhood assignment in F. Note first that for each $n \in \omega$, $C_n = \bigcup \mathcal{F}_n$ is a D-space since for each $C \in \mathcal{F}_n$ we can choose a closed and discrete set $D_C \subseteq C$ such that $\mathcal{O}(D_C) \supseteq C$. It is immediate that $\bigcup\{D_C : C \in \mathcal{F}_n\}$ is a closed discrete kernel of $\mathcal{O}$.

To complete the proof it is clearly sufficient to prove that a countable union of closed D-subspaces is a D-space. To this end, suppose that $F = \bigcup\{C_n : n \in \omega\}$, where each set C_n is a closed D-subspace of X and $\{O_x : x \in F\}$ is a neighbourhood assignment in F; then since C_0 is a D-space, it follows that there is some closed and discrete set $D_0 \subseteq C_0$ such that $\bigcup\{O_x : x \in D_0\} \supseteq C_0$.

Having chosen closed discrete sets $\{D_0, D_1, \ldots, D_{n-1}\}$ so that

$$D_k \subseteq C_k \setminus \bigcup\{O_x : x \in \bigcup\{D_j : 0 \le j < k\}\} \subseteq \bigcup\{O_x : x \in D_k\}$$

for each $k \le n-1$, it follows that $C_n \setminus \bigcup\{O_x : x \in \bigcup\{D_j : 0 \le j \le n-1\}\}$ is a closed subset of C_n and hence is a D-space. Thus we can choose a closed discrete subset $D_n \subseteq X$ such that

$$D_n \subseteq C_n \setminus \bigcup\{O_x : x \in \bigcup\{D_j : 0 \le j < n\}\} \subseteq \bigcup\{O_x : x \in D_n\}.$$

Let $D = \bigcup\{D_k : k \in \omega\}$; it is clear that $\bigcup\{O_x : x \in D\} \supseteq F$ and we claim that D is closed and discrete in F. To see this, suppose that $z \in F$ and let $m \in \omega$ be the minimal integer such that $z \in \mathcal{O}(D_m)$. Clearly $z \notin \mathrm{cl}(\bigcup\{D_k : 1 \le k \le m-1\})$, and since $z \in \mathcal{O}(D_m)$ and $\mathcal{O}(D_m) \cap D_k = \emptyset$ for each $k > m$, it follows from the fact that D_m is closed and discrete that z is not an accumulation point of D. $\square$

Corollary 2.2. *If F is an F_σ-set in a D-space (respectively, a dually discrete space) (X,τ), then $(F,\tau|F)$ is a D-space (respectively, a dually discrete space).*

Corollary 2.3. *The product of a σ-compact space and a dually discrete space is dually discrete.*

PROOF: It is an immediate consequence of Theorem 2.7 of [7] that the product of a compact T_1-space and a dually discrete T_1-space is dually discrete. The result now follows from Theorem 2.1. $\square$

Theorem 2.4. *If a space X is the union of two dually discrete subspaces Y and Z where Z is closed in X, then X is dually discrete.*

PROOF: Let $\mathcal{O} = \{O_x : x \in X\}$ be a neighbourhood assignment in X. Then $\mathcal{O}_Z = \{O_x \cap Z : x \in Z\}$ is a neighbourhood assignment in Z and hence has a discrete kernel, D_Z. Now $W = Y \setminus \bigcup\{O_x : x \in D_Z\}$ is a closed subspace of the dually discrete space Y and hence is dually discrete. Thus the neighbourhood assignment in W, $\mathcal{O}_W = \{O_x \cap W : x \in W\}$ has a discrete kernel D_Y, say and it is straightforward to check that $D_Y \cup D_Z$ is a discrete kernel of $\mathcal{O}$. $\square$

Corollary 2.5. *If a space X is the finite union of dually discrete spaces $\{Z_1, \dots, Z_n\}$ where, for each $1 \le j \le n-1$, the subspace Z_j is closed, then X is dually discrete.*

We say that a topological space is *adequate* if every closed subspace with countable extent is Lindelöf. It is easy to see that a D-space is adequate.

Theorem 2.6. *Let $X = Y \cup Z$ be a space of countable extent. If Y is adequate and Z is a D-space, then X is linearly Lindelöf.*

PROOF: Suppose to the contrary that X is not linearly Lindelöf; then there is some strictly increasing open cover $\{U_\alpha : \alpha \in \kappa\}$ of uncountable regular cardinality which has no countable subcover. Define $f : X \to \kappa$ by $f(x) = \min\{\alpha \in \kappa : x \in U_\alpha\}$ and a neighbourhood assignment $\mathcal{O}$ by $O_x = U_{f(x)}$.

Since Z is a D-space, there is some closed (in Z) discrete set $D \subseteq Z$ such that

$$\bigcup\{O_x : x \in D\} \supseteq Z.$$

Now $F = \mathrm{cl}_X(D) \setminus D$ is a (possibly empty) closed subset of X which is contained in Y. It follows that F has countable extent and since X is adequate, F is Lindelöf. Thus there is a countable set $S \subseteq X$ such that $F \subseteq \bigcup\{O_x : x \in S\}$; now $D \setminus \bigcup\{O_x : x \in S\}$ is closed and discrete in X, hence is countable, and so there is a countable set $T \subseteq X$ such that $\mathrm{cl}_X(D) \subseteq \bigcup\{O_t : t \in T\}$. Let $\gamma = \sup\{f(t) : t \in T\} < \kappa$ and $z \in Z$; then there is $d \in D$ such that $z \in O_d$ and $t \in T$ such that $d \in O_t$. Hence $f(d) \le f(t) \le \gamma$ and $z \in U_{f(d)}$.

The set $X \setminus U_\gamma$ is closed in X, is contained in Y and has countable extent, so again, since Y is adequate, $X \setminus U_\gamma$ is Lindelöf; thus there is a countable $Q \subseteq X$ such that $X \setminus U_\gamma \subseteq \bigcup\{O_q : q \in Q\}$. Let $\delta = \sup\{f(q) : q \in Q\}$ and $\eta = \max\{\gamma, \delta\} + 1$. Since κ has uncountable cofinality, we have $\eta < \kappa$, but $X = \bigcup\{U_\alpha : \alpha < \eta\} \subseteq U_\eta$, a contradiction. □

Recall that a space X is *metalindelöf* if every open cover of X has a point-countable open refinement.

The following lemma and its corollaries, each having easy proofs, are part of the folklore.

Lemma 2.7. *For each open cover $\mathcal{U}$ of a topological space X, there is a closed discrete set $D \subseteq X$ such that $\bigcup\{\mathrm{St}(d, \mathcal{U}) : d \in D\} = X$.*

Corollary 2.8. *If X is a metalindelöf space then $L(X) = e(X)$.*

Corollary 2.9. *A metalindelöf space of countable extent is Lindelöf, hence linearly Lindelöf.*

Recall that a cover $\mathcal{V} = \{V_\alpha : \alpha \in I\}$ is a *shrinking* of a cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ if $V_\alpha \subseteq U_\alpha$ for all $\alpha \in I$ ($V_\alpha = \emptyset$ is not excluded).

In [14], Gruenhage proved that if a space X has countable extent and is a finite union of D-spaces, then it is linearly Lindelöf. Below we prove a analogous theorem, involving a finite union of metalindelöf subspaces, which allows us to answer a question of Arhangel'skii and Buzyakova. First we need a simple lemma.

Lemma 2.10. *If an open cover of a space X has a point-countable open refinement, then it has a point-countable open shrinking.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of X and $\mathcal{C}$ a point-countable open refinement of $\mathcal{U}$. For each $C \in \mathcal{C}$, choose $\alpha(C) \in I$ so that $C \subseteq U_{\alpha(C)}$ and define

$$W_\alpha = \bigcup\{C \in \mathcal{C} : \alpha(C) = \alpha\}.$$

Clearly $W_\alpha \subseteq U_\alpha$ for each $\alpha \in I$ and $\bigcup\{W_\alpha : \alpha \in I\} = X$; hence to complete the proof we must show that $\mathcal{W} = \{W_\alpha : \alpha \in I\}$ is a point-countable family. To this end, we fix $x \in X$ and enumerate the countable set $\{C \in \mathcal{C} : x \in C\}$ as $\{C_n : n \in \omega\}$. It is then clear that $x \in W_\beta$ if and only if $\beta \in \{\alpha(C_n) : n \in \omega\}$, which completes the proof. □

Theorem 2.11. *If a space X of countable extent is the finite union of metalindelöf spaces, then it is linearly Lindelöf.*

Proof: Suppose that X is a space of countable extent which is a finite union of metalindelöf subspaces. The proof is by induction on the number n of such subspaces. It follows from Corollary 2.9 that the theorem is true if $n = 1$. So suppose that the theorem is true for any union of n metalindelöf subspaces and assume that $X = \bigcup\{X_k : 1 \le k \le n+1\}$ where each subspace X_k is metalindelöf.

We suppose to the contrary that X is not linearly Lindelöf; then there is some uncountable regular cardinal κ and a strictly increasing open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ which has no countable subcover. Without loss of generality we may assume that the open cover $\mathcal{V} = \{U_\alpha \cap X_{n+1} : \alpha \in \kappa\}$ of X_{n+1} has no countable subcover. Since X_{n+1} is metalindelöf, it follows from Lemma 2.10 that the open cover $\mathcal{V}$ of X_{n+1} has a point-countable open (in X_{n+1}) shrinking $\{W_\alpha : \alpha < \kappa\}$. For each $\alpha \in \kappa$ we may then find open sets Y_α in X such that $Y_\alpha \cap X_{n+1} = W_\alpha$ and $Y_\alpha \subseteq U_\alpha$; let $Y = \bigcup\{Y_\alpha : \alpha \in \kappa\}$. Then Y is an open subset of X which contains X_{n+1} and so $X \setminus Y = \bigcup\{X_k \setminus Y : 1 \le k \le n\}$ is a closed subspace of a space of countable extent which is the union of at most n metalindelöf subspaces and hence by the induction hypothesis it is linearly Lindelöf. Now $\{U_\alpha \cap (X \setminus Y) : \alpha \in \kappa\}$ is a strictly increasing open cover of $X \setminus Y$ and since κ is regular and uncountable, for some $\lambda < \kappa$, $U_\lambda \supseteq X \setminus Y$.

We now consider the open cover $\mathcal{F} = \{U_\lambda\} \cup \{Y_\alpha : \alpha \in \kappa\}$ of X. Fix $x_0 \in X_{n+1}$; since each point of X_{n+1} is contained in at most countably many sets Y_α, $\mathcal{V}$ has no countable subcover and $Y_\alpha \subseteq U_\alpha$ for each $\alpha \in \kappa$, it follows that $\mathrm{St}(x_0, \mathcal{F}) \not\supseteq X_{n+1}$ and we may find $x_1 \in X_{n+1} \setminus \mathrm{St}(x_0, \mathcal{F})$. Now suppose for some $\alpha < \omega_1 \le \kappa$ and for each $\beta < \alpha$ we have chosen $x_\beta \in X_{n+1} \setminus \bigcup\{\mathrm{St}(x_\gamma, \mathcal{F}) : \gamma < \beta\}$, then since $\{F \in \mathcal{F} : x_\gamma \in F \text{ for some } \gamma < \alpha\}$ is countable, it follows that $X_{n+1} \setminus \bigcup\{\mathrm{St}(x_\gamma, \mathcal{F}) : \gamma < \alpha\} \ne \emptyset$ and we may choose $x_\alpha \in X_{n+1} \setminus \bigcup\{\mathrm{St}(x_\gamma, \mathcal{F}) : \gamma < \alpha\}$. Thus we construct a closed (in X_{n+1}) discrete subset $D = \{x_\alpha : \alpha \in \omega_1\}$ of X_{n+1} with the property that no countable subcollection of $\mathcal{F}$ covers D. Since X has countable extent, D cannot be closed in X and so the set $\mathrm{cl}_X(D) \setminus D$ is a closed non-empty subspace of $\bigcup\{X_k : 1 \le k \le n\}$ which by the induction hypothesis must be linearly Lindelöf. Thus there is a countable subset $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathrm{cl}_X(D) \setminus D \subseteq \bigcup \mathcal{G} = U$. Now $D \setminus U$ is a closed and discrete subset of X and hence is countable. But then, $D \subseteq \mathrm{cl}_X(D)$ is contained in a countable subcollection of $\mathcal{F}$, which is a contradiction; thus X is linearly Lindelöf. □

The next result gives a positive answer to Question 21 of [5].

Corollary 2.12. *If X has countable extent and is the union of finitely many paracompact subspaces, then X is linearly Lindelöf.*

PROOF: A paracompact space is metalindelöf. □

3. Scattered spaces

Recall that a T_1-space is *scattered* if every non-empty subspace has an isolated point. Given a scattered T_1-space X, for each ordinal number γ, the *γ-th derived set of* X, X_γ, is defined recursively as follows: $X_0 = X$, $X_{\gamma+1}$ is the derived set of X_γ, and if γ is limit then $X_\gamma = \bigcap\{X_\beta : \beta < \gamma\}$. The minimal ordinal μ such that $X_\mu = \emptyset$ is called the *Cantor-Bendixson height of* X (or more simply in the sequel, *the height of* X) and will be denoted by $\mathrm{ht}(X)$. The family of subspaces $\{X_\gamma : \gamma < \mathrm{ht}(X)\}$ is called the *Cantor-Bendixson decomposition of* X.

It is known from [9] that every left-separated T_1-space is a D-space. Since every scattered space of finite height is left-separated, the following result is immediate (and a direct proof is an easy exercise).

Theorem 3.1. *Each scattered space of finite height is a D-space.*

Corollary 3.2. *The product of a dually discrete space and a scattered space of finite height is dually discrete.*

Proof: Suppose that Y is dually discrete and X is a scattered space of height $m \in \omega$. If $m = 1$, then $X \times Y$ is the topological union of dually discrete spaces and hence is dually discrete. The proof proceeds by induction on the height m of X. If the result is true for each scattered space X of height $m - 1$, then we write $X = (X \setminus X_1) \cup X_1$. The set $X \setminus X_1$ is discrete and X_1 is a scattered space of height $m - 1$. Thus $X \times Y$ is the union of two dually discrete subspaces, one of which, X_1, is closed, and the result follows from Theorem 2.4. □

As is well-known, the space ω_1 with its order topology is not a D-space and so not every scattered T_1-space is a D-space. Our next result gives a large class of scattered spaces which are D-spaces.

Recall that a space is *subparacompact* if every open cover has a closed σ-discrete refinement (we do not assume any separation axiom stronger than T_1). It is well known that every paracompact Hausdorff space is subparacompact.

Theorem 3.3. *A subparacompact scattered space is a D-space.*

Proof: Assume that X is a non-empty subparacompact scattered space; if $\mathrm{ht}(X) = 1$, then X being discrete, is a D-space. Proceeding inductively assume that α is an ordinal and that any subparacompact space Y with $\mathrm{ht}(Y) < \alpha$ is a D-space. Now suppose that a space X has height α and let $\{X_\beta : \beta < \alpha\}$ be the Cantor-Bendixson decomposition of X. Take an arbitrary neighbourhood assignment $\mathcal{O} = \{O_x : x \in X\}$ in the space X.

If α is a successor then $\alpha = \beta + 1$ and X_β is a closed discrete subspace of X; let $U = \mathcal{O}(X_\beta)$. The set $F = X \setminus U$ is closed in X and it follows from $F \cap X_\beta = \emptyset$ that $\mathrm{ht}(F) < \alpha$ and hence F is a D-space by the induction hypothesis. Choose a closed discrete set $D \subseteq F$ such that $\mathcal{O}(D) \supseteq F$. It is evident that $D \cup X_\beta$ is a closed discrete kernel of $\mathcal{O}$ so X is a D-space.

Next assume that α is a limit ordinal and hence $\bigcap\{X_\beta : \beta < \alpha\} = \emptyset$. For any point $x \in X$ there exists $\beta < \alpha$ such that $x \notin X_\beta$; we can find an open neighbourhood U_x of the point x such that $U_x \cap X_\beta = \emptyset$ and hence the height of the space U_x is strictly less than α. Since X is subparacompact, there exists a σ-discrete closed refinement of the cover $\{U_x : x \in X\}$ which we denote by $\mathcal{K} = \bigcup\{\mathcal{K}_n : n \in \omega\}$, where for each $n \in \omega$, $\mathcal{K}_n$ is a discrete family of closed sets. It is clear that for each $n \in \omega$ and each $K \in \mathcal{K}_n$, the height of the subspace K is strictly less than α so the induction hypothesis implies that K is a D-space. It remains only to apply Theorem 2.1 to conclude that X is a D-space. □

Corollary 3.4. *Each regular Lindelöf scattered space is a D-space.*

Recall that F. Galvin [14] and R. Telgársky [17] introduced *the point-open game* $\mathcal{PO}$ in which at the n-th move the first player I picks a point $x_n \in X$ while the second player II replies by choosing an open set $U_n \subseteq X$ with $x_n \in U_n$. The game is finished after ω moves and I is deemed to be the winner if $\bigcup\{U_n : n \in \omega\} = X$; otherwise player II wins the game $\{(x_n, U_n) : n \in \omega\}$. A space X is called *I-favorable* (*II-favorable*) for the point-open game if the first (second) player has a winning strategy on X.

It is easy to see that any space which fails to be Lindelöf, is II-favorable for the point-open game. Therefore every space which is not II-favorable (in particular each I-favorable space) is Lindelöf.

The class of (regular) spaces which are I-favorable or II-favorable for the point-open game has received a lot of attention recently. Telgársky proved in [17] that a regular Lindelöf scattered space is I-favorable for the point-open game and it is easy to see that not every I-favorable space is scattered. Therefore the following result is stronger than Corollary 3.4.

Theorem 3.5. *If a regular space X is not II-favorable for the point-open game then X is a D-space. In particular, any I-favorable space is a D-space.*

PROOF: Given a neighbourhood assignment $\mathcal{O} = \{O_x : x \in X\}$ in the space X define a strategy σ of the second player as follows: if x_0 is the first move of I then let $U_0 = \sigma(x_0) = O_{x_0}$. Assume that $n \in \omega$ and moves $x_0, U_0, \dots, x_n, U_n$ have been made in the point-open game on X. If I selects x_{n+1} for his move $(n+1)$ then let $\sigma(x_0, \dots, x_n, x_{n+1}) = U_0 \cup \dots \cup U_n$ if $x_{n+1} \in U_0 \cup \dots \cup U_n$; if not, then let $\sigma(x_0, \dots, x_n, x_{n+1}) = O_{x_{n+1}}$.

By our assumption the strategy σ is not winning for the second player so there is a play $\{x_i, U_i : i \in \omega\}$ on the space X in which II applies the strategy σ and loses, that is, $\bigcup_{n\in\omega} U_n = X$. Let $A = \{n \in \omega : x_{n+1} \in U_0 \cup \dots \cup U_n\}$ and enumerate the set $\omega \setminus A$ as $\{n_i : i < \alpha\}$ for some ordinal $\alpha \le \omega$ in such a way that $i < j$ implies $n_i < n_j$. It takes a trivial induction to see that $U_{n_i} = O_{x_{n_i}}$ and $x_{n_{i+1}} \notin O_{x_{n_0}} \cup \dots \cup O_{x_{n_i}}$ for any $i < \alpha$ while $\bigcup_{n\in\omega} U_n = \bigcup_{i\in\omega} O_{x_{n_i}} = X$. It is immediate that $D = \{x_{n_i} : i < \alpha\}$ is a closed discrete kernel of $\mathcal{O}$ so X is a D-space as promised. □

Corollary 3.6. *Every continuous image of a regular Lindelöf P-space is a D-space.*

PROOF: It is well-known (and easy to prove) that the property of not being II-favorable for the first player in the point-open game is preserved by continuous images. Since each Lindelöf P-space is not II-favorable for the point-open game (see Theorem 6.10 of [18]), Theorem 3.5 applies. □

Corollary 3.7. *Every continuous image of a regular Lindelöf scattered space is a D-space.*

PROOF: If X is a Lindelöf scattered space then let Y be the set X with the topology generated by all G_δ-subsets of X. It is clear that X is a continuous image of Y and Y is a P-space. By Proposition 1 of [19], the space Y is also Lindelöf[1], and so every continuous image of X is a continuous image of a Lindelöf P-space; Corollary 3.6 now completes the proof. □

Question 3.8. *Is every metacompact scattered Hausdorff space dually discrete* (*or even a D-space*)*?*

Recall that a *submaximal space* (respectively, *nodec space*) is a dense-in-itself space in which every dense set is open (respectively, every nowhere dense set is closed); again we assume no separation axiom beyond T_1. Clearly a submaximal space is nodec. From Corollary 3.4 of [2], under $V = L$, every submaximal Hausdorff space is strongly σ-discrete and hence from Theorem 2.1 every Hausdorff submaximal space is dually discrete. In fact an even stronger result is true in ZFC.

Theorem 3.9. *Every nodec space is a D-space.*

PROOF: Suppose that X is a nodec space and $\mathcal{O} = \{O_x : x \in X\}$ is a neighbourhood assignment in X. It was proved in Proposition 2.1 of [7] that every space is dually scattered so we can find a scattered kernel $F \subseteq X$ for the assignment $\mathcal{O}$. However, every scattered subspace of a dense-in-itself space is nowhere dense. Since X is nodec, F is a closed and discrete kernel of $\mathcal{O}$. □

The space Γ of [10] is a locally compact, scattered Hausdorff space of height ω, which is not a D-space and so we are led to ask:

Question 3.10. *Is Γ dually discrete? More generally, is every scattered Hausdorff space* (*or even T_1-space*) *of countable height, dually discrete?*

A related result is the following:

Theorem 3.11. *A countably compact, scattered T_1-space of countable height is compact.*

We omit the simple proof which is by induction on the scattering height.

4. Dual discreteness of generalized ordered spaces

Let $(X, \tau, <)$ be a GO-space and C its Dedekind compactification, that is to say, the minimal ordered compactification of X. By the term *left pseudogap* of X,

[1]The referee has pointed out to us that this result was known to Paul R. Meyer in 1966, but was apparently never published.

we mean a pair (A, B) of open subsets of X such that $a < b$ for all $a \in A$ and $b \in B$, $A \cup B = X$ and A has no maximum element. A *right pseudogap* is defined analogously. The pair (A, B) is called a gap of X if it is both a right and a left pseudogap. If $(\emptyset, X)$ (respectively, $(X, \emptyset)$) is a gap then it is called the *left end gap* (respectively, *right end gap*) of X.

Recall that a left pseudogap (A, B) of X is a *left Q-pseudogap* if for some regular cardinal κ there is a strictly increasing transfinite sequence $\{d_\alpha : \alpha < \kappa\}$ in A which is closed and discrete as a subspace of X and cofinal in A, that is to say, $\sup_C(A) = \sup_C(D)$. *Right Q-pseudogaps* are defined analogously. For simplicity, we say that a left (respectively, right) pseudogap which is not a left Q-pseudogap (respectively, not a right Q-pseudogap) is a *left* (respectively, *right*) *N-pseudogap*.

We define an ordered compactification K of X as follows: For each non-end gap (A, B) of X, add two points a^*, b^* such that $a < a^* < b^* < b$ for all $a \in A$ and $b \in B$ and for each left pseudogap (A, B) which is not a gap (respectively, right pseudogap (C, D) which is not a gap) add a point p_A (respectively, p_D) such that $a < p_A < b$ for all $a \in A$ and $b \in B$ (such that $c < p_D < d$ for all $c \in C$ and $d \in D$). Also add a minimal point m if X has a left end gap and a maximal point M if X has a right end gap. In the sequel, we identify the points $m, M, a^*, b^*, p_A, p_D \in K$ with the left and/or right pseudogaps of X. In [15], Lutzer showed that a GO-space is paracompact if and only if each of its pseudogaps is a Q-pseudogap.

We denote the set of left (respectively, right) Q-pseudogaps of X (considered as subsets of K) by L_Q (respectively R_Q) and the set of left (respectively, right) N-pseudogaps by L_N (respectively R_N).

It was shown in [8] that a GO-space is a D-space if and only if it is paracompact and in [7] that a GO-space of countable extent is dually discrete. It turns out that the requirement of countable extent can be omitted; the following theorem answers Problems 4.1 and 4.2 from [7].

Theorem 4.1. *Each GO-space is dually discrete.*

Proof: Suppose that X is a GO-space and K is the ordered compactification of X as defined in the preceding paragraphs. We consider the subspace $Y \subseteq K$ defined by $Y = X \cup L_N \cup R_N$. We first show that every pseudogap of Y is a Q-pseudogap and hence by Theorem E of [15], Y is paracompact. To this end, suppose that $p \in K \setminus Y$ is a pseudogap of Y and hence is a Q-pseudogap of X; we assume without loss of generality that p is a left Q-pseudogap of X. Then for some regular cardinal κ, there is a closed (in X) and discrete, strictly increasing transfinite sequence $D = \{d_\alpha : \alpha < \kappa\} \subseteq (\leftarrow, p)_K \cap X$, such that $p = \sup_K(D)$. Since D is closed in X, it follows that for each limit ordinal $\lambda < \kappa$, $q_\lambda = \sup_K\{d_\alpha : \alpha < \lambda\} \notin X$ and hence is a pseudogap of X; furthermore, q_λ is a Q-pseudogap of X since $\{d_\alpha : \alpha < \lambda\}$ is a strictly increasing transfinite sequence which is closed and discrete in X and hence $q_\lambda = \sup_K\{d_\alpha : \alpha < \lambda\} \in K\backslash Y$. Thus $\{d_\alpha : \alpha < \kappa\}$ is also closed and discrete in Y, showing that p is a Q-pseudogap of Y, completing the proof that Y is paracompact.

Let $\mathcal{O} = \{O_x : x \in X\}$ be an arbitrary neighbourhood assignment in X where, without loss of generality, we assume that each set O_x is convex. We will extend the family $\mathcal{O}$ to a neighbourhood assignment in Y. To this end, suppose that $y \in Y\backslash X$; the point y corresponds to an N-pseudogap of X and again without loss of generality we assume that y is a left N-pseudogap and hence $y \notin \mathrm{cl}_K((y, \rightarrow)_K)$.

We claim that there is a point $a_y \in (\leftarrow, y)_X$ and a discrete cofinal subset $D_y \subseteq (\leftarrow, y)_X$ such that $(a_y, z] \subseteq O_z$ for all $z \in D_y$. For if to the contrary, no such a_y and D_y exist then, since each member of $\mathcal{O}$ is convex, for any $x \in (\leftarrow, y)_X$ there is a point $b \in (x, y)_X$ such that $(x, z) \nsubseteq O_z$ (that is $O_z \subseteq (x, \rightarrow))$ for each $z \in (b, y)_X$.

Now, since y is a left N-pseudogap of X, $\chi(y, (\leftarrow, y)_X \cup \{y\})) > \omega$ and hence no countable set is cofinal in $(\leftarrow, y)_X$; thus for some cardinal κ we can construct recursively a strictly increasing transfinite sequence $B = \{b_\alpha : \alpha < \kappa\} \subseteq (\leftarrow, y)_X$ such that $O_z \subseteq (b_\alpha, \rightarrow)_X$ for each $\alpha < \kappa$ for any $z \in (b_\beta, y)_X$. Now since y is a left N-pseudogap of X, there is no strictly increasing, transfinite sequence which is closed and discrete subset of $(\leftarrow, y)_K \cap X$ whose supremum in K is y. Thus the set B must have a cofinal set of cluster points B^d in $(\leftarrow, y)_K \cap X$. Now if $x \in B^d$, then since B is a strictly increasing sequence, $x \in \mathrm{cl}_X(\rightarrow, x)_X$ and hence there are $\alpha < \beta < \kappa$ such that $\{b_\alpha, b_\beta\} \subseteq O_x$. However, by the recursive hypothesis, $O_x \subseteq (b_\alpha, \rightarrow)_X$, which is a contradiction.

Analogously, if the point y is a right N-pseudogap, then we can choose a discrete subspace $E_y \subseteq (y, \rightarrow)_X$ and $b_y \in (y, \rightarrow)_X$ such that y is the infimum of E_y and $[x, b_y) \subseteq O_x$ for each $x \in E_y$.

The proof now proceeds exactly as in Theorem 2.23 of [7] using the fact that Y is paracompact and hence is a D-space (see [8]). $\square$

5. Open problems

The problem of whether the union of two D-spaces is a D-space has been posed previously. Neither is it known whether the union of two dually discrete spaces is dually discrete. (If one of the subspaces is closed, then a positive answer is provided by Theorem 2.4.)

Problem 5.1. *Suppose that $X = X_0 \cup X_1$ and X_i is dually discrete for $i = 0, 1$. Must X be dually discrete? What happens if both sets X_0 and X_1 are dense in X?*

If X is a Lindelöf P-space then any countable subset of X is closed and discrete; this clearly implies that X is a D-space. The following problems involving continuous images of Lindelöf spaces show how little is known of this topic and point to possible future lines of research.

Problem 5.2. *Is any continuous image of a Lindelöf GO-space, dually discrete? Must it be a D-space?*

Problem 5.3. *Is any continuous image of a Lindelöf LOTS, dually discrete? Must it be a D-space?*

Problem 5.4. *Suppose that X is a Lindelöf space such that every second countable continuous image of X is countable. Must X be dually discrete? Must it be a D-space?*

Problem 5.5. *Is it true that every Lindelöf space is a continuous image of a Lindelöf GO-space?*

Problem 5.6. *Is it true that every Lindelöf space is a continuous image of a Lindelöf LOTS?*

Problem 5.7. *Is it true that every compact space is a continuous image of a Lindelöf GO-space?*

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