Functional calculus for a class of unbounded linear operators on some non-archimedean Banach spaces

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Abstract. This paper is mainly concerned with extensions of the so-called Vishik functional calculus for analytic bounded linear operators to a class of unbounded linear operators on c_0. For that, our first task consists of introducing a new class of linear operators denoted $W(c_0(J, \omega, \mathbb{K}))$ and next we make extensive use of such a new class along with the concept of convergence in the sense of resolvents to construct a functional calculus for a large class of unbounded linear operators.

Keywords: non-archimedean Banach space, Shnirelman integral, spectrum, unbounded linear operator, functional calculus

Classification: Primary 47S10, 46S10; Secondary 12G25, 26E30

1. Introduction

Many applications arising in mathematics can be described explicitly or implicitly through bounded or unbounded linear operators on Banach or Hilbert spaces. It is also well-known that the best way to analyze those linear operators is either through their spectral theory or through their functional calculus. In particular, the spectral theory of linear operators in the classical (archimedean) setting plays an important role in several fields such as quantum mechanics, mathematical physics, pure and applied mathematics, and has its roots not only in matrix theory but also in the theory of integral equations.

Let A be a linear operator on a Banach space $\mathbb{X}$ and let I stand for the identity operator of $\mathbb{X}$. The spectrum of A denoted σ_A, is the set of all scalars λ such that the linear operator $\lambda I - A$ fails to be invertible. Consequently, if f is an analytic function defined in a neighborhood of σ_A, one then defines the linear operator $f(A)$ through the Cauchy integral of the operator-valued function $f(\lambda)(\lambda I - A)^{-1}$ on a chosen contour, which encloses σ_A. It is then clear that the mapping $f \mapsto f(A)$ is an homomorphism that maps 1 to I and maps λ to A. Such a mapping, in the literature, is called the Riesz functional calculus or the Riesz-Dunford functional calculus of the linear operator A. For more on these and related topics, we refer the reader to Conway [4], Davies [5], and Kato [14].

In the non-archimedean world, things are far from being settled. In particular, there are a few papers on the spectral theory as well as the functional calculus of linear operators. Moreover, to the best of our knowledge there are no papers on either the spectral theory or the functional calculus of unbounded linear operators in the non-archimedean setting. This in fact is the main motivation of this paper, that is, constructing a functional calculus for a class of unbounded operators.

Recall the pioneer work of Vishik [23] who developed a spectral theory together with a functional calculus for the class of the so-called analytic bounded linear operators with compact spectrum on c_0. Moreover, some interesting perturbation results on those analytic bounded linear operators were established. A function defined over a quasiconnected set Ω [17] is said to be 'Krasner analytic' if it can be uniformly approximated by a sequence of rational functions whose poles do belong to the complement of Ω. To deal with these spectral theory and functional calculus issues, Vishik showed that Krasner analyticity in the sense of Krasner was more appropriate than the classical analycity. Consequently, instead of the use of the Cauchy integral as in the classical context, Vishik made extensive use of a 'line' integral in the non-archimedean context known as the Shnirelman's integral [21] to construct both a spectral theory and a functional calculus for those analytic bounded linear operators with compact support. Indeed, the Shnirelman's integral is compatible with the above-mentioned notion of local analyticity. In this paper, the above-mentioned work of Vishik will be our starting point.

In Diarra-Ludkovsky [10], a spectral integration for some non-archimedean Banach algebras was also initiated and studied. In particular, a spectral measure is constructed. Moreover, a non-archimedean counterpart of Stone theorem is obtained. Finally, a spectral theory for commuting bounded linear operators on a non-archimedean Banach space is constructed.

In Baker [3], an extension of Vishik spectral theory to a certain Banach algebra is also studied.

In this paper, it goes back to studying a functional calculus for a class of unbounded linear operators on c_0 paralleling that constructed for the class of analytic bounded linear operators with compact support. For that, our first task consists of introducing a new class of linear operators denoted $W(c_0(J, \omega, \mathbb{K}))$. Next, we make extensive use of $W(c_0(J, \omega, \mathbb{K}))$ along with the concept of convergence in the sense of resolvents to construct a functional calculus for a large class of unbounded linear operators.

This paper is organized as follows. In Section 2, we review some of the basic results on unbounded linear operators on c_0, Shnirelman's integral, and distributions with compact supports. Section 3 is devoted to the so-called Cauchy-Stieltjes and Vishik transforms. Section 4 studies analytic bounded linear operators on c_0. Sections 5 contain our main results as well as a few examples.

2. Preliminaries

2.1 Unbounded linear operators on $c_0(J, \omega, \mathbb{K})$. Let $(\mathbb{K}, |\cdot|)$ be a complete non-archimedean valued field, which is also algebraically closed. Basic examples of such a field include $\mathbb{C}_p$, the field of complex p-adic numbers equipped with its p-adic valuation [19], and $\mathcal{C}$, the so-called Levi-Civita field equipped with its absolute value [22].

Let J be a set. Fix once and for all a family $\omega = (\omega_j)_{j\in J} \subset \mathbb{K}$ of nonzero terms. The space $c_0(J, \omega, \mathbb{K})$ is defined as the set of all families $u = (u_j)_{j\in J}$, $u_j \in \mathbb{K}$ for all $j \in J$ such that $\omega_j u_j^2$ tends to 0 with respect to the filter of complements of finite subsets. This is will be denoted by $\lim_{j\in J} \omega_j u_j^2 = 0$. In other words,

$$c_0(J, \omega, \mathbb{K}) = \left\{ u = (u_j)_{j\in J} : u_j \in \mathbb{K},\ \forall\, j \in J,\ \ \lim_{j\in J} |u_j||\omega_j|^{1/2} = 0 \right\}.$$

The space $c_0(J, \omega, \mathbb{K})$ equipped with the norm defined for each $u = (u_j)_{i\in J}$ by

$$\|u\| = \sup_{j\in J} |u_j||\omega_j|^{1/2}$$

is a non-archimedean Banach space.

Similarly, an inner product (symmetric, non-degenerate, bilinear form) is also defined on $c_0(J, \omega, \mathbb{K})$ for all $u = (u_j)_{j\in J}$, $v = (v_j)_{j\in J} \in c_0(J, \omega, \mathbb{K})$ by

$$\langle u, v\rangle := \sum_{j\in J}^{\infty} \omega_j\, u_j\, v_j.$$

It is then easy to check that this series converges for all $u, v \in c_0(J, \omega, \mathbb{K})$ and that the Cauchy-Schwartz inequality is satisfied, that is, $|\langle u, v\rangle| \le \|u\| \cdot \|v\|$ for all $u, v \in c_0(J, \omega, \mathbb{K})$. The non-archimedean Banach space $c_0(J, \omega, \mathbb{K})$ has a special base, denoted $(e_i)_{i\in j}$ where e_i is the sequence whose j-th term is 0 if $i \ne j$, and the i-th term is 1.

Define $e_i' \in c_0(J, \omega, \mathbb{K})^*$ by $u = \sum_{i\in J} u_i e_i$, $e_i'(u) = u_i$. It turns out that

$$\|e_i'\| = \frac{1}{\|e_i\|} = \frac{1}{|\omega_i|^{1/2}}$$

for all $i \in J$. Furthermore, every $u' \in c_0(J, \omega, \mathbb{K})^*$ can be expressed as a pointwise convergent series: $u' = \sum_{i\in J} \langle u', e_i\rangle e_i'$. Moreover,

$$\|u'\| := \sup_{i\in J} \frac{|\langle u', e_i\rangle|}{\|e_i\|}.$$

In view of the above, the space $c_0(J, \omega, \mathbb{K})$ equipped with its above-mentioned norm and inner product is a free Banach space according to the terminology of

Diarra [11], [12] and it is also called a non-archimedean Hilbert space [8], [9], [11], [12].

Following the work of Diarra [8], [9], [11], [12], it is well-known that a linear operator A on $c_0(J, \omega, \mathbb{K})$ can be expressed as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j) \in J \times J}$ with coefficients in $\mathbb{K}$ such that:

$$A = \sum_{i,j \in J} a_{ij}(e'_j \otimes e_i), \quad \text{and for any } \; j \in J, \quad \lim_{i \in J} |a_{ij}||\omega_i|^{1/2} = 0,$$

where for all $i, j \in J$, $e'_j \otimes e_i$ is the bounded linear operator defined by

$$(e'_j \otimes e_i)u = \langle e'_j, u \rangle \cdot e_i = u_j e_i$$

for each $u = (u_j)_{j \in J} \in c_0(J, \omega, \mathbb{K})$.

Throughout this paper we will be using the so-called stable unbounded linear operators on $c_0(J, \omega, \mathbb{K})$ due to Diagana [8], [9] and whose definition is given as follows:

Definition 2.1. A stable unbounded linear operator A from $c_0(J, \omega, \mathbb{K})$ into itself is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset c_0(J, \omega, \mathbb{K})$ (the domain of A) and a (possibly not continuous) linear transformation A such that $(e_i)_{i \in J} \subset D(A)$, that is, Ae_j is well-defined in $c_0(J, \omega, \mathbb{K})$ with $\lim_{j \in J} \|Ae_j\| = 0$, and

$$(2.1) \qquad \begin{cases} D(A) := \left\{ u = (u_j)_{j \in J} \in c_0(J, \omega, \mathbb{K}) : \quad \lim\limits_{j \in J} |u_j| \|Ae_j\| = 0 \right\}, \\[2ex] Au = \sum\limits_{i,j \in J} a_{ij} e'_j(u) e_i \;\; \text{for all} \;\; u \in D(A). \end{cases}$$

The collection of all stable unbounded linear operators on $c_0(J, \omega, \mathbb{K})$ will be denoted by $S(c_0(J, \omega, \mathbb{K}))$.

In this paper, only stable unbounded linear operators will be considered.

It can be easily shown that if the norm $\|A\|$ of A defined by

$$\|A\| := \sup_{u \neq 0} \frac{\|Au\|}{\|u\|} = \sup_{i,j \in J} \frac{|a_{ij}| \|e_i\|}{\|e_j\|}$$

is finite, then $D(A) = c_0(J, \omega, \mathbb{K})$. In that case, the linear operator is said to be bounded. In what follows, the collection of all bounded linear operators on $c_0(J, \omega, \mathbb{K})$ will be denoted by $B(c_0(J, \omega, \mathbb{K}))$. Let us note that from Definition 2.1, $B(c_0(J, \omega, \mathbb{K})) \subset S(c_0(J, \omega, \mathbb{K}))$.

2.2 Properties of Shnirelman integral. This subsection is devoted to an important tool for the construction of Vishik spectral theory and functional calculus, that is, the so-called Shnirelman integral [3], [16], [23]. Important applications of such an integral are in transcendental number theory [23]. Moreover, it can be utilized to prove a non-archimedean version of the Cauchy integral theorem, the residue theorem and the maximum principle. The proofs of all these important results will be given throughout the paper, as they are not familiar results or cannot be easily found in the literature.

In this paper, we will make extensive use of Shnirelman integral to study functional calculus of unbounded linear operators as Vishik did in [23]. For that, we require that the non-archimedean valued field $(\mathbb{K}, |\cdot|)$ be complete and algebraically closed (and hence its valuation is dense in $\mathbb{R}_+$).

Definition 2.2. Let $\sigma \subset \mathbb{K}$ be a subset and let $r > 0$. The sets $D(\sigma, r)$ and $D(\sigma, r^-)$ are defined respectively as follows:

$$D(\sigma, r) := \big\{x \in \mathbb{K} : \operatorname{dist}(x, \sigma) \leq r\big\}$$

and

$$D(\sigma, r^-) := \big\{x \in \mathbb{K} : \operatorname{dist}(x, \sigma) < r\big\},$$

where $\operatorname{dist}(x, \sigma) = \inf_{y \in \sigma} |x - y|$.

Additionally, for $a \in \mathbb{K}$, we define $D(a, r^-)$ and $D(a, r)$ respectively by

$$D(a, r^-) := \big\{x \in \mathbb{K} : |x - a| < r\big\}$$

and

$$D(a, r) := \big\{x \in \mathbb{K} : |x - a| \leq r\big\}.$$

Lemma 2.3 ([3])**.** *Let $\sigma \subset \mathbb{K}$ be a nonempty compact subset. Then for every $s > 0$, there exist $0 < r \in |\mathbb{K}|$ and $a_1, \cdots, a_N \in \sigma$ such that $r < s$ and*

$$D(\sigma, r) = \bigsqcup_{i=1}^{N} D(a_i, r) \quad \text{and} \quad \sigma \subset \bigsqcup_{i=1}^{N} D(a_i, r^-),$$

where the symbol $\bigsqcup$ denotes disjoint unions.

One can generalize Lemma 1.3 from Baker [3] as follows:

Lemma 2.4. *Let $\emptyset \neq \sigma \subset \mathbb{K}$ and let $r > 0$. Then if I is a nonempty set and if $\{b_i : i \in I\} \subset \mathbb{K}$ is a subset such that*

$$\sigma \subset \bigsqcup_{i \in I} D(b_i, r^-),$$

then there exist subsets $J \subset I$ and $\{a_j : j \in J\} \subset \sigma$ such that

$$D(\sigma, r^-) = \bigsqcup_{j \in J} D(a_j, r^-) = \bigsqcup_{j \in L} D(b_j, r^-), \quad \text{and}$$
$$D(\sigma, r) = \bigsqcup_{j \in L} D(a_j, r) = \bigsqcup_{j \in J} D(b_j, r).$$

Proof: Set $J = \{j \in I : D(b_j, r^-) \cap \sigma \neq \emptyset\}$ and rewrite J as $J = \{i_j : j \in J\}$. For all $j \in J$, choose $a_j \in D(b_{i_j}, r^-)$. Then, $D(b_{i_j}, r^-) = D(a_j, r^-)$ and thus,

$$\sigma \subset \bigsqcup_{j \in J} D(a_j, r^-) = \bigsqcup_{j \in J} D(b_{i_j}, r^-) = \bigsqcup_{j \in J} D(b_j, r^-).$$

Obviously,

(i) $\bigsqcup_{j \in J} D(a_j, r^-) \subset D(\sigma, r^-)$;
(ii) $\bigsqcup_{j \in J} D(a_j, r) \subset D(\sigma, r)$.

Now we show the reverse inclusions. Let $x \in D(\sigma, r^-)$. Then $\operatorname{dist}(x, \sigma) < r$ and hence, there exists $a \in \sigma$ such that $|x - a| < r$. Since $a \in \sigma = \bigsqcup_{j \in J} D(a_j, r^-)$, there exists $a_{j_0} \in \sigma$ with $j_0 \in J$ such that $|a - a_{j_0}| < r$.

Now
$$|x - a_{j_0}| \leq \max\{|x - a|, |a - a_{j_0}|\} < r,$$
and hence $x \in \bigsqcup_{j \in J} D(a_j, r^-)$. Therefore, $D(\sigma, r^-) \subset \bigsqcup_{j \in J} D(a_j, r^-)$.

Finally, let $x \in D(\sigma, r)$, that is, $\operatorname{dist}(x, \sigma) \leq r$ and there exists $a \in \sigma$ such that $|x - a| \leq r$. Again, since $a \in \sigma$, there exists $a_{j_1} \in \sigma$ such that $|a - a_{j_1}| < r$.

Now
$$|x - a_{j_1}| \leq \max\{|x - a|, |a - a_{j_1}|\} \leq r,$$
and hence $x \in \bigsqcup_{j \in J} D(a_j, r)$ and therefore, $D(\sigma, r) \subset \bigsqcup_{j \in J} D(a_j, r)$. □

The following corollary is then immediate and hence its proof is omitted.

Corollary 2.5. *Let $\emptyset \neq \sigma \subset \mathbb{K}$ and let $r > 0$. Let $b_1, \cdots, b_M$ be in $\mathbb{K}$ with*

$$\sigma \subset \bigsqcup_{i=1}^{M} D(b_i, r^-).$$

Then there exist $a_1, \cdots, a_N$ in σ and $\emptyset \neq J \subset \{1, \cdots, M\}$ such that the $D(a_i, r)$ are disjoint,

$$D(\sigma, r^-) = \bigsqcup_{i=1}^{N} D(a_i, r^-) = \bigsqcup_{i \in J} D(b_i, r^-)$$

and

$$D(\sigma, r) = \bigsqcup_{i=1}^{N} D(a_i, r) = \bigsqcup_{i \in J} D(b_i, r).$$

The notion of (local) analyticity in the next definition plays a crucial role throughout the paper.

Definition 2.6. Let $a \in \mathbb{K}$ and let $r > 0$. A function $f : D(a, r) \mapsto \mathbb{K}$ is said to be analytic if f can be represented by a power series on $D(a, r)$, that is,

$$f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k \quad \text{with} \quad \lim_{k \to \infty} r^k |c_k| = 0.$$

Remark 2.7. Let $a \in \mathbb{K}$ and let $r > 0$. A function $f : D(a, r) \mapsto \mathbb{K}$ is said to be 'Krasner analytic' if it is a uniform limit of rational functions with poles belong to the complement of $D(a, r)$. In fact if $r \in |\mathbb{K}|$, it can be shown that a function analytic over $D(a, r)$ in the sense of Krasner is also analytic in the sense of Definition 2.6, see, e.g., [16].

Definition 2.8. Let $\emptyset \neq \sigma \subset \mathbb{K}$ and let $r > 0$. Let $B_r(\sigma)$ be the collection of all functions $f : D(\sigma, r) \to \mathbb{K}$ such that f is analytic on $D(a, r)$ whenever $a \in \mathbb{K}$ and $D(a, r) \subset D(\sigma, r)$. If f is bounded on $D(\sigma, r)$, we then set

$$\|f\|_r = \max_{x \in D(\sigma, r)} |f(x)|.$$

One should point out that the notion of local analyticity appearing in Definition 2.9 is new and due to Baker [3]. Additional comments on this new notion can be found in Remark 2.11.

Definition 2.9 ($L(\sigma)$)**.** Let $\emptyset \neq \sigma \subset \mathbb{K}$. Define $L(\sigma)$ to be the collection of all $\mathbb{K}$-valued functions f for which there exist $a_1, \cdots, a_N$ in $\mathbb{K}$ and $0 < r \in |\mathbb{K}|$ such that

$$\sigma \subset \bigsqcup_{i=1}^{N} D(a_i, r^-),$$

where the $D(a_i, r)$ are disjoint and f is analytic on each $D(a_i, r)$.

The class of functions $L(\sigma)$ will be called the set of locally analytic functions on σ. Note that in view of Definition 2.9, $\mathrm{Dom}(f)$, the domain of $f \in L(\sigma)$ is

$$\mathrm{Dom}(f) \subset \bigsqcup_{i=1}^{N} D(a_i, r).$$

Moreover, $L(\sigma) \neq \emptyset$, as polynomials belong to it.

Theorem 2.10 ([3])**.** *Let $\emptyset \neq \sigma \subset \mathbb{K}$ be a compact subset. Then*

$$L(\sigma) = \bigcup_{r>0} B_r(\sigma). \tag{2.2}$$

Remark 2.11. It is worth mentioning that the concept of local analyticity given in Definition 2.9 generalizes that of Koblitz [16, p. 136], in which the local analyticity on compact $\emptyset \neq \sigma \subset \mathbb{K}$ was defined as $L(\sigma) = \bigcup_{r>0} B_r(\sigma)$.

Definition 2.12 (Shnirelman integral)**.** Let κ be the residue field of $\mathbb{K}$ and let $f(x)$ be a $\mathbb{K}$-valued function defined for all $x \in \mathbb{K}$ such that $|x-a| = r$ where $a \in \mathbb{K}$ and $r > 0$ with $r \in |\mathbb{K}|$. Let $\Gamma \in \mathbb{K}$ be such that $|\Gamma| = r$. Then the Shnirelman integral of f is defined as the following limit, if it exists,

$$\int_{a,\Gamma} f(x)\,dx := \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\eta^n=1} f(a+\eta\Gamma), \tag{2.3}$$

where $\lim'$ indicates that the limit is taken over n such that $\gcd(\mathrm{char}(\kappa), n) = 1$.

Lemma 2.13. (i) *Suppose that f is bounded on the circle $|x-a| = r$. If $\int_{a,\Gamma} f(x)\,dx$ exists, then $|\int_{a,\Gamma} f(x)\,dx| \leq \max_{|x-a|=r} |f(x)|$.*

(ii) *The integral $\int_{a,\Gamma}$ commutes with limits of functions which are uniform limits on $\{x \in \mathbb{K} : |x-a| = r\}$.*

(iii) *If $r_1 \leq r \leq r_2$ and $f(x)$ is given by a convergent Laurent series $\sum_{k\in\mathbb{Z}} c_k(x-a)^k$ in the annulus $r_1 \leq |x-a| \leq r_2$, then*

$$\int_{a,\Gamma} f(x)\,dx = c_0$$

and is independent of the choice of Γ with $|\Gamma| = r$, as long as $r_1 \leq r \leq r_2$. More generally,

$$\int_{a,\Gamma} \frac{f(x)}{(x-a)^k}\,dx = c_k.$$

Proof: The proof of statements (i) and (ii) follow directly from the definition of the Shnirelman integral. To prove (iii), note that for $k \neq 0$ and $n > |k|$,

$$\sum_{\eta^n=1} \eta^k = 0, \tag{2.4}$$

and hence

$$f(a+\Gamma\eta) = c_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k\Gamma^k\eta^k.$$

The result is now a consequence of (2.4), (2.3) and the fact that $\lim_{k\to\infty} c_k\Gamma^k = 0$. $\square$

Lemma 2.14. *Fix $x_0 \in \mathbb{K}$ and $m > 0$. Then, the following holds:*

$$\int_{a,\Gamma} \frac{dx}{(x-x_0)^m} = \begin{cases} 0 & \text{if } |a-x_0| < r; \\ (a-x_0)^{-m} & \text{if } |a-x_0| > r. \end{cases}$$

Corollary 2.15. *Fix $x_0 \in \mathbb{K}$ and $m > 1$. Then,*

$$\int_{a,\Gamma} \frac{x-a}{(x-x_0)}\, dx = \begin{cases} 1 & \text{if } |a-x_0| < r; \\ 0 & \text{if } |a-x_0| > r. \end{cases}$$

and

$$\int_{a,\Gamma} \frac{x-a}{(x-x_0)^m}\, dx = \begin{cases} 0 & \text{if } |a-x_0| < r; \\ 0 & \text{if } |a-x_0| > r. \end{cases}$$

Lemma 2.16 (Non-archimedean Cauchy integral formula [23]). *If f is analytic on $D(a,r)$ and if $|\Gamma| = r \in |\mathbb{K}|$, then*

$$\int_{a,\Gamma} \frac{f(x)(x-a)}{(x-x_0)}\, dx = \begin{cases} f(x_0) & \text{if } |a-x_0| < r, \\ 0 & \text{if } |a-x_0| > r. \end{cases} \tag{2.5}$$

In particular, this integral does not depend on the choice of a, Γ or r as long as $|x_0 - x|$ is either less than or greater than r.

Theorem 2.17 (Non-archimedean residue theorem [23]). *Let f be a rational function over $\mathbb{K}$ and suppose none of the poles x_0 of f satisfy $|x_0 - a| = |\Gamma|$, where $\Gamma \in \mathbb{K} - \{0\}$. Then*

$$\int_{a,\Gamma} f(x)(x-a)\, dx = \sum_{|x_0 - a| < |\Gamma|} \mathrm{res}_{x=x_0} f(x), \tag{2.6}$$

where $\mathrm{res}_{x=x_0} f(x)$ is the coefficient of $(x-x_0)^{-1}$ in the Laurent expansion of f about x_0.

We next state some interesting results on the Shnirelman integral.

2.3 Distributions with compact support. Let $\sigma \subset \mathbb{K}$ be a compact subset and let $r > 0$. It follows that there is a finite set I with $a_i \in \mathbb{K}$ for $i \in I$ with

$$\sigma \subset \bigsqcup_{i \in I} D(a_i, r^-), \quad \text{and} \quad D(\sigma, r^-) = \bigsqcup_{i \in I} D(a_i, r^-). \tag{2.7}$$

Let $f \in B_r(\sigma)$. Then $f : D(\sigma, r^-) \to \mathbb{K}$ is clearly analytic and hence satisfies:

(i) $f(x) = \sum_{j \in \mathbb{N}} f_{ij} (x - a_i)^j$ for $x \in D(a_i, r^-)$;

(ii) for all $i \in I$, $|f_{ij}| r^j \to 0$ as $j \to \infty$;

(iii) the norm of f is defined as $\|f\|_r = \sup_{i \in I, j \in \mathbb{N}} |f_{ij}| r^j$.

It is not hard to check that $(B_r(\sigma), \|.\|_r)$ is a non-archimedean Banach space. Moreover, the following embedding is continuous

$$B_r(\sigma) \hookrightarrow B_{r_1}(\sigma)$$

with $0 < r_1 < r$.

Recall that (from Theorem 2.10) the following holds

$$L(\sigma) = \bigcup_{r>0} B_r(\sigma).$$

Definition 2.18. The space $L^*(\sigma) := L(\sigma)^*$ (topological dual of $L(\sigma)$) is called the space of distributions with support σ.

For all $\mu \in L^*(\sigma)$ and $f \in L(\sigma)$, we represent the canonical pairing between μ and f as

$$(\mu, f) = (\mu(x), f(x)) = \mu(f).$$

Moreover, it is easy to see that for $\mu \in L^*(\sigma)$, $\mu_{|B_r(\sigma)}$ is a continuous linear functional whose norm is denoted by

$$|||\mu|||_r := \sup_{f \in B_r(\sigma), f \neq 0} \frac{|\mu(f)|}{\|f\|_r}.$$

In particular, if $0 < r_1 < r$, then

$$|||\mu|||_{r_1} \geq |||\mu|||_r.$$

For $r > 0$, $i \in I$, $j \in \mathbb{N}$ and $x \in \mathbb{K}$, we define

$$\chi(r, i, j; x) = \begin{cases} (x - a_i)^j & |x - a_i| < r, \\ 0 & |x - a_i| \geq 0. \end{cases}$$

Obviously, $\chi(r, i, j; \cdot) \in B_r(\sigma)$.

It can also be shown (see [16]) that the weak topology on $L^*(\sigma)$ whose basis is the neighborhoods of zero given by

$$U_{f,\varepsilon} := \{\mu \in L^*(\sigma) : |\mu(f)| < \varepsilon\}$$

and the stronger topology on $L^*(\sigma)$ whose basis is the neighborhoods of zero given by

$$U(r, \varepsilon) := \{\mu \in L^*(\sigma) : \|\mu\|_r < \varepsilon\}$$

have the same convergent sequences.

Throughout, we set $\overline{\sigma} := \mathbb{K} - \sigma$ and

$$\overline{D}(\sigma, r) = \mathbb{K} - D(\sigma, r^-) = \{x \in \mathbb{K} : \quad \operatorname{dist}(x, \sigma) \geq r\}.$$

Definition 2.19. The collection of all functions $\varphi : \overline{\sigma} \to \mathbb{K}$ which are Krasner analytic and vanish at infinity, that is:

(i) φ is a limit of rational functions whose poles are contained in σ, the limit being uniform in any set of the form $\overline{D}(\sigma, r)$;

(ii) $\lim_{|z|\to\infty} \varphi(z) = 0$;

is denoted $H_0(\overline{\sigma})$.

For $\varphi \in H_0(\overline{\sigma})$, we define

$$\|\phi\|_r := \max_{z\in\overline{D}(\sigma,r)} |\phi(z)| = \max_{\mathrm{dist}(z,\sigma)=r} |\phi(z)|.$$

In particular, if $0 < r_1 < r$ then

$$\|\phi\|_r \leq \|\phi\|_{r_1}.$$

As a topology on $H_0(\overline{\sigma})$, we take as a basis the open neighborhoods of zero given by

$$U_0(r, \varepsilon) = \{\phi : \|\phi\|_r < \varepsilon\}.$$

3. Cauchy-Stieltjes and Vishik transforms

Definition 3.1 (Cauchy-Stieltjes transform)**.** Let $\sigma \subset \mathbb{K}$ be a compact subset and let $\mu \in L^*(\sigma)$. The Cauchy-Stieltjes transform of μ is the function

$$\begin{aligned} \varphi = S\mu : \overline{\sigma} &\to \mathbb{K} \\ z &\mapsto \left(\mu(x), \frac{1}{z-x}\right). \end{aligned}$$

Let $f \in B_r(\sigma)$ and suppose $\sigma \subset \bigsqcup_{i\in I} D(a_i, r^-)$ where I is a finite index set. Fix $\Gamma \in \mathbb{K}$ such that for all $i \in I$,

$$\sup_{b\in D(a_i,r^-)\bigcap\sigma} |a_i - b| < |\Gamma| < r. \tag{3.1}$$

Definition 3.2. We define the Vishik transform V (under the assumptions leading to (3.1)) by

$$\begin{aligned} V\varphi : B_r(\sigma) &\to \mathbb{K} \\ f &\mapsto \sum_{i\in I} \int_{a_i,\Gamma} (z - a_i)\varphi(z) f(z)\, dz. \end{aligned}$$

Lemma 3.3 ([23])**.** *Let $\mu \in L^*(\sigma)$ be a distribution with compact support. Then $S\mu \in H_0(\overline{\sigma})$ and $S : L^*(\sigma) \to H_0(\overline{\sigma})$ is continuous.*

Lemma 3.4 ([23])**.** *Let $\varphi \in H_0(\overline{\sigma})$. Then,*

$$V\varphi(f) := \sum_{i\in I} \int_{a_i,\Gamma} (z - a_i)\varphi(z)f(z)\, dz, \quad f \in B_r(\sigma),$$

does not depend on the choice of a_i and Γ satisfying (3.1)*. Furthermore, it is compatible with the inclusion*

$$B_r(\sigma) \hookrightarrow B_{r_1}(\sigma) \ \ \textit{for} \ \ r_1 < r.$$

In addition, both $V\varphi : B_r(\sigma) \to \mathbb{K}$ and $V : H_0(\overline{\sigma}) \to L^*(\sigma)$ are continuous.

Lemma 3.5. *We have $VS = SV = Id$.*

4. Analytic bounded linear operators on $c_0(J, \omega, \mathbb{K})$

Recall that for a linear operator A on $c_0(J, \omega, \mathbb{K})$ or more generally on a Banach space over a non-archimedean valued field $\mathbb{K}$, one defines its resolvent set $\rho(A)$ to be the set of all $\lambda \in \mathbb{K}$ such that the operator $R_A(\lambda) := (\lambda I - A)^{-1}$ is one-to-one and bounded. We then define the spectrum of A, σ_A as the complement of $\rho(A)$ in $\mathbb{K}$, that is $\sigma_A = \mathbb{K} - \rho(A)$.

Definition 4.1. An operator $A \in B(c_0(J, \omega, \mathbb{K}))$ is said to be analytic with compact spectrum if

(i) $\sigma_A \subset \mathbb{K}$ is compact, and
(ii) for all $h \in c_0(J, \omega, \mathbb{K})^*$ and $u \in c_0(J, \omega, \mathbb{K})$, the function defined by

$$z \mapsto \langle h, R_A(z)u\rangle$$

belongs to $H_0(\overline{\sigma}_A)$.

Classical examples of analytic linear operators with compact support include among others those completely continuous linear operators on a Banach space due to Serre [20] as well as non-archimedean analogues of normal operators on Hilbert spaces. (A bounded linear operator N on a Hilbert space $\mathbb{H}$ is said to be normal if $NN^* = N^*N$, where N^* is the adjoint of N.) An example of an analytic linear operator with compact support on a Banach space other than $c_0(J, \omega, \mathbb{K})$ is given below.

Example 4.2 (Position operator)**.** Let $\mathbb{K} = \mathbb{Q}_p$ equipped with its p-adic topology and let $C(\mathbb{Z}_p, \mathbb{Q}_p)$ be the Banach space of all continuous functions from

$$\mathbb{Z}_p = \big\{z \in \mathbb{Q}_p : |z| \le 1\big\}$$

into $\mathbb{Q}_p$ equipped with the sup norm defined by

$$\|\varphi\|_\infty = \sup_{z\in\mathbb{Z}_p} |\varphi(z)|$$

for each $\varphi \in C(\mathbb{Z}_p, \mathbb{Q}_p)$.

Consider the (bounded) position operator, $A : C(\mathbb{Z}_p, \mathbb{Q}_p) \to C(\mathbb{Z}_p, \mathbb{Q}_p)$ defined by

$$A\varphi(x) = x\varphi(x)$$

for all $\varphi \in C(\mathbb{Z}_p, \mathbb{Q}_p)$.

It can be shown that the spectrum $\sigma_A = \mathbb{Z}_p$ and that for all $\xi \in C(\mathbb{Z}_p, \mathbb{Q}_p)^*$ and $u \in C(\mathbb{Z}_p, \mathbb{Q}_p)$, the function $z \mapsto \langle \xi, R_A(z)u\rangle$ belongs to $H_0(\overline{\mathbb{Z}}_p) = H_0(\mathbb{Q}_p - \mathbb{Z}_p)$. Therefore, A is an analytic linear operator on $C(\mathbb{Z}_p, \mathbb{Q}_p)$ with compact support.

Define for $0 < r_1 < r_2$, the set $D(a; r_1, r_2)$ as follows:

$$D(a; r_1, r_2) = \{b \in \mathbb{K} : r_1 \le |a - b| \le r_2\}.$$

Definition 4.3 ([23]). A function $F : D(a; r_1, r_2) \to B(c_0(J, \omega, \mathbb{K}))$ is called an analytic operator valued function if for all $h \in c_0(J, \omega, \mathbb{K})^*$ and $u \in c_0(J, \omega, \mathbb{K})$, the function

$$z \mapsto \langle h, F(z)u\rangle = \sum_{j\in\mathbb{Z}} F_j \cdot (z - a)^j,$$

with $\lim_{j\to-\infty} |F_j| r_1^j = 0$ and $\lim_{j\to\infty} |F_j| r_2^j = 0$.

Lemma 4.4 ([23]). *Let $F : D(a; r_1, r_2) \to B(c_0(J, \omega, \mathbb{K}))$ be an analytic operator valued function. Then the sequence $(S_n)_n \subset B(c_0(J, \omega, \mathbb{K}))$ defined by*

$$S_n := \frac{1}{n} \sum_{\eta^n=1} F(a + \Gamma\eta)$$

converges strongly as $n \to \infty$ (the limit is taken assuming that $(n, \mathrm{char}(\mathbb{K})) = 1$ when $\mathrm{char}(\mathbb{K}) \neq 0$) to a bounded linear operator. More precisely,

$$\lim_{n\to\infty} S_n := \int_{a,\Gamma} F(z)\, dz.$$

Corollary 4.5 ([23]). *We have*

$$\Big\langle h, \Big(\int_{a,\Gamma} F(z)\, dz\Big) u\Big\rangle = \int_{a,\Gamma} \langle h, F(z)u\rangle\, dz.$$

4.1 Vishik spectral theorem. Let $\sigma \subset \mathbb{K}$ be a compact subset. An operator valued distribution is a $\mathbb{K}$-linear continuous mapping $L(\sigma) \to B(c_0(J,\omega,\mathbb{K}))$.

Theorem 4.6 ([23]). *Let $A \in B(c_0(J,\omega,\mathbb{K}))$ be an analytic linear operator with compact spectrum. Then the following operator valued distribution $\mu_A := VR_A$ ($\mu_A(f) := f(A)$ for all $f \in B_r(\sigma)$) with support σ is well-defined for all $f \in B_r(\sigma)$ by*

$$\mu_A(f) = \sum_{i\in J}\int_{a_i,\Gamma}(z-a_i)R_A(z)f(z)\,dz = \sum_{i\in J}\int_{a_i,\Gamma}(z-a_i)(zI-A)^{-1}f(z)\,dz.$$

Lemma 4.7 ([23]). *The mapping $\mu_A : L(\sigma_A) \to B(c_0(J,\omega,\mathbb{K}))$ is continuous. In addition,*

$$R_A(z) = \left(\mu_A, \frac{1}{z-x}\right), \quad z \in \overline{\sigma}_A,$$

and

$$A^j = (\mu_A(x), x^j).$$

5. Main results and examples

5.1 Resolvent approximating sequences.

Definition 5.1. Let A be an unbounded linear operator on $c_0(J,\omega,\mathbb{K})$ such that $\rho(A) \neq \emptyset$. A sequence of bounded linear operators $(A_n)_{n\in\mathbb{N}}$ on $c_0(J,\omega,\mathbb{K})$ is said to converge to an operator A in the sense of resolvent if whenever $\lambda \in \rho(A)$, then $\lambda \in \rho(A_n)$ for n sufficiently large, and

$$\lim_{n\to\infty}\|R_{A_n}(\lambda) - R_A(\lambda)\| = 0, \quad \text{for all} \quad \lambda \in \rho(A). \tag{5.1}$$

The sequence $(A_n)_{n\in\mathbb{N}}$ will then be called a resolvent approximating sequence (r.a.s. for short) for the (unbounded) linear operator A.

Lemma 5.2. *If* (5.1) *holds for some $\lambda_0 \in \rho(A)$, then it is true for all $\lambda \in \rho(A)$.*

Proof: Suppose (5.1) holds for some $\lambda_0 \in \rho(A)$. Proceeding as in the classical case (see Davies [5, Lemma 2.6.1]), one can show that for all $\lambda \in \rho(A)$,

$$(\lambda I - A)^{-1} - (\lambda I - A_n)^{-1} = \Delta_1\big\{(\lambda_0 I - A)^{-1} - (\lambda_0 I - A_n)^{-1}\big\}\Delta_2,$$

where $\Delta_1 = (A_n - \lambda_0 I)(A_n - \lambda I)^{-1}$ and $\Delta_2 = (A - \lambda_0 I)(A - \lambda I)^{-1}$.

Now

$$\begin{aligned}
\|\Delta_1\| &= \|1 + (\lambda - \lambda_0)(A_n - \lambda I)^{-1}\| \\
&\le \max\left\{1, \max(|\lambda|, |\lambda_0|)\|(A_n - \lambda I)^{-1}\|\right\} \\
&\le \max\left\{1, \max(|\lambda|, |\lambda_0|)\sup_{n\in\mathbb{N}}\|(A_n - \lambda I)^{-1}\|\right\} \\
&= A_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\Delta_2\| &= \|1 + (\lambda - \lambda_0)(A - \lambda I)^{-1}\| \\
&\le \max\left\{1, \max(|\lambda|, |\lambda_0|)\|(A - \lambda I)^{-1}\|\right\} \\
&\le \max\left\{1, \max(|\lambda|, |\lambda_0|)\|(A - \lambda)^{-1}\|\right\} \\
&= A_2,
\end{aligned}$$

and hence

$$\|(\lambda I - A_n)^{-1} - (\lambda I - A)^{-1}\| \le A_1 A_2 \|(\lambda_0 I - A_n)^{-1} - (\lambda_0 I - A)^{-1}\|.$$

Therefore, letting $n \to \infty$ in the previous inequality, we obtain the desired result. $\square$

Example 5.3. Take $J = \mathbb{N}$ and let $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ be a sequence satisfying $\lambda_0 = 0$ and

$$\lim_{i\to\infty} |\lambda_i| = \infty.$$

Let A be the diagonal operator associated with the sequence $(\lambda_i)_{i\in\mathbb{N}}$ and defined on $c_0(\mathbb{N}, \omega, \mathbb{K})$ by

$$Au = \sum_{i,j\in\mathbb{N}} (\lambda_i \delta_j^i e'_j \otimes e_i)(u), \quad u \in D(A),$$

where

$$D(A) = \left\{u = (u_i)_{i\in\mathbb{N}} \in c_0(\mathbb{N}, \omega, \mathbb{K}) : \lim_{i\to\infty} |\lambda_i||u_i||\omega_i|^{1/2} = 0\right\}.$$

It is easy to see that A is a well defined unbounded linear operator whose spectrum and resolvent are respectively given by

$$\sigma_A = \{0\} \cup \{\lambda_i\}_{i\ge 1} \quad \text{and} \quad \rho(A) = \mathbb{K} - \{0\} \cup \{\lambda_i\}_{i\ge 1}.$$

Indeed, solving the equation

$$(A - zI)u = v \quad \text{for} \quad v = (v_i)_{i\in\mathbb{N}} \in c_0(\mathbb{N}, \omega, \mathbb{K})$$

for $u = (u_i)_{i\in\mathbb{N}}$, we obtain

$$\begin{aligned} \langle (A - zI)u, e_i\rangle &= \langle v, e_i\rangle \\ (\lambda_i - z)u_i &= v_i \\ u_i &= \frac{v_i}{\lambda_i - z}, \quad z \neq 0, \; z \neq \lambda_i, \quad i \geq 1, \end{aligned}$$

and hence

$$(zI - A)^{-1}v = \sum_{i\in\mathbb{N}} \frac{v_i}{z - \lambda_i} e_i \quad \text{for} \quad z \in \mathbb{K} - \{0\} \cup \{\lambda_i\}_{i\geq 1}.$$

Using the assumption that $|\lambda_i| \to \infty$ as $i \to \infty$, it is easy to see that $(zI - A)^{-1}$ is a bounded linear operator for all $z \in \mathbb{K} - \{0\} \cup \{\lambda_i\}_{i\geq 1}$.

Define the sequence of linear operators $(A_n)_{n\in\mathbb{N}}$ by setting $A_n = \sum_{i,j\in\mathbb{N}} a_{ij}^{(n)} e'_j \otimes e_i$, where

$$a_{ij}^{(n)} = \begin{cases} \lambda_i & \text{if } \; i = j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\|A_n\| = \max_{0\leq i\leq n} |\lambda_i| < \infty$. Moreover, the spectrum and resolvent of A_n, respectively, are

$$\sigma_{A_n} = \big\{0, \lambda_1, \cdots, \lambda_n\big\} \quad \text{and} \quad \rho(A_n) = \mathbb{K} - \big\{0, \lambda_1, \cdots, \lambda_n\big\}.$$

Note that

$$(zI - A_n)^{-1}v = \sum_{i=0}^{n} \frac{v_i}{z - \lambda_i} e_i \quad \text{for} \quad z \in \mathbb{K} - \big\{0, \lambda_1, \cdots, \lambda_n\big\}.$$

In addition, for $z \notin \sigma_{A_n}$ and $h \in c_0(J, \omega, \mathbb{K})^*$,

$$z \mapsto \langle h, R_{A_n}(z)u\rangle = \sum_{i\in\mathbb{N}} \frac{u_i}{z - \lambda_i} h(e_i)$$

and hence the function $z \mapsto \langle h, R_{A_n}(z)u\rangle$ belongs to $H_0(\overline{\sigma}_{A_n})$ showing that A_n are analytic. Furthermore,

$$\|(zI - A)^{-1} - (zI - A_n)^{-1}\| \to 0$$

as $n \to \infty$ for all $\lambda \in \rho(A)$, and hence $(A_n)_{n\in\mathbb{N}}$ is a r.a.s. for A.

5.2 Extension of Vishik functional calculus. This setting requires the introduction of the relation $\bowtie$, which we define as follows:

Definition 5.4. Let $\sigma_1 \subset \sigma_2 \subset \mathbb{K}$ be two compact subsets. If $f \in B_r(\sigma_2)$, then it is easy to see that $f^{(1)} := f\chi_{D(\sigma_1, r^-)} \in B_r(\sigma_1)$, where χ_S is the characteristic function of the set S. Similarly, if $g \in B_r(\sigma_1)$ then $\tilde{g} : D(\sigma_2, r^-) \to \mathbb{K}$ is an element of $B_r(\sigma_2)$ where $\tilde{g}$ is defined by:

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in D(\sigma_1, r^-), \\ 0 & \text{if } x \in D(\sigma_2, r^-) - D(\sigma_1, r^-). \end{cases}$$

Throughout the rest of the paper, we will express the above-mentioned relationship between $B_r(\sigma_1)$ and $B_r(\sigma_2)$ by $B_r(\sigma_1) \bowtie B_r(\sigma_2)$.

The following is immediate.

Proposition 5.5. *Given a sequence of compact sets $(\sigma_n)_n$ whose union σ is also compact, then for $r > 0$,*

$$B_r(\sigma_n) \bowtie B_r(\sigma) \quad \textit{for all} \quad n \in \mathbb{N}.$$

We also introduce the set (space) $\widehat{B}_r(\sigma)$ in the next definition, which in fact is a natural generalization of $B_r(\sigma)$.

Now, let σ be an arbitrary subset of $\mathbb{K}$ such that for all $r > 0$,

$$\sigma \subset \bigsqcup_{i \in I} D(a_i, r^-) \tag{5.2}$$

where the index set I is *not necessarily finite* and

$$\sup_{b \in \sigma \bigcap D(a_i, r^-)} |b - a_i| < r. \tag{5.3}$$

In view of Lemma 2.4, we assume that for all $i \in I,\ a_i \in \sigma$.

Definition 5.6. Let σ satisfy (5.2) and (5.3). Define $\widehat{B}_r(\sigma)$ as the collection of all functions

$$f : D(\sigma, r) = \bigsqcup_{i \in I} D(a_i, r) \to \mathbb{K}$$

satisfying

(i) $f_{|D(a_i, r^-)}(x) = \sum_{j \in \mathbb{N}} f_{ij} \cdot (x - a)^j$ with $|f_{ij}| r^j \to 0$ as $j \to \infty$, for all $i \in I$;

(ii) $\|f\|_r := \sup_{i \in I, j \in \mathbb{N}} |f_{ij}| r^j < \infty$.

Definition 5.7. If $\tilde{\sigma}_1 \subset \tilde{\sigma}_2$, both satisfy (5.2) and (5.3), we then define the relation $\bowtie$ between $\widehat{B}_r(\sigma_1)$ and $\widehat{B}_r(\sigma_2)$, in a similar way as in Definition 5.4.

The following proposition is immediate.

Proposition 5.8. *If σ is compact, then $B_r(\sigma) = \widehat{B_r}(\sigma)$. In addition, if $\sigma \subset \tilde{\sigma}$ and both, σ and $\tilde{\sigma}$ satisfy* (5.2) *and* (5.3), *then $\widehat{B_r}(\sigma) \bowtie \widehat{B}_r(\tilde{\sigma})$.*

Given that $\tilde{\sigma}$ satisfies (5.2) and (5.3), then for $\varphi \in H_0(\overline{\tilde{\sigma}})$, and $f \in \widehat{B}_r(\tilde{\sigma})$, we extend the Vishik's transform to $\bigcup_{r>0} \widehat{B}_r(\tilde{\sigma})$ as follows:

$$V\varphi(f) = \sum_{i\in I} \int_{a_i,\Gamma} (z - a_i)\varphi(z) f(z)\, dz, \tag{5.4}$$

if it exists.

If the right hand side of equation (5.4) exists, it is then well-defined and does not depend on all $\Gamma \in \mathbb{K}$ satisfying

$$\sup_{b\in\sigma \bigcap D(a_i, r^-)} |b - a_i| < |\Gamma| < r.$$

This follows directly from the proof of Lemma 3.4. Sufficient conditions for the existence of $V\varphi(f)$ above include:

(i) $f \in B_r(\sigma) \bowtie B_r(\tilde{\sigma})$ where σ is compact,

(ii) if in general, $f \in B_r(\tilde{\sigma})$, then $\sum_{\rho\in D(\tilde{\sigma}, |\Gamma|^-)} |\operatorname{res}_{z=\rho}(\varphi(z) f(z))| < \infty$.

Definition 5.9. Let $(A_n)_n$ be a sequence of bounded linear operators on the space $c_0(J, \omega, \mathbb{K})$. We say that the sequence of operators $(A_n)_{n\in\mathbb{N}}$ belongs to the class $W(c_0(J, \omega, \mathbb{K}))$ if

$$\lim_{n\to\infty} \mu_{A_n}(f)$$

exists for any $f \in \bigcup_{n\in\mathbb{N}} B_r(\sigma_{A_n})$.

Theorem 5.10. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of analytic bounded linear operators on $c_0(J, \omega, \mathbb{K})$ with compact spectrum. Suppose that $(A_n)_{n\in\mathbb{N}}$ is also a r.a.s. for an unbounded linear operator $A : D(A) \subset c_0(J, \omega, \mathbb{K}) \mapsto c_0(J, \omega, \mathbb{K})$ whose resolvent is not empty. Then $(A_n)_{n\in\mathbb{N}}$ belongs to the class $W(c_0(J, \omega, \mathbb{K}))$.*

The proof of Theorem 5.10 requires the next lemma whose proof is straightforward and hence omitted.

Lemma 5.11. *Let $F : S_{a,\Gamma} = \{z \in \mathbb{K} : |z - a| = |\Gamma| = r\} \mapsto B(c_0(J, \omega, \mathbb{K}))$ be an analytic operator-valued function such that $\int_{a,\Gamma} F(z)\, dz$ exists. Then*

$$\Big\| \int_{a,\Gamma} F(z)\, dz \Big\| \le \max_{|z-a|=r} \|F(z)\|.$$

Proof: Let $f \in \widehat{B}_r(\sigma_A) \bowtie \bigcup_{n\in\mathbb{N}} B_r(\sigma_{A_n})$. From Vishik spectral theorem, that is, Theorem 4.6, $\mu_{A_n}(f)$ exists for each $n \in \mathbb{N}$. Moreover, the integral defined by

$$\mu_{A_n}(f) - \widehat{\mu}_A(f) = \sum_{i\in I} \int_{a_i,\Gamma} (z-a_i) f(z) [R_{A_n}(z) - R_A(z)]\, dz$$

is well-defined.

Set

$$M := \sup_{i\in I}\big(\max_{|z-a_i|=|\Gamma|} |(z-a_i)f(z)|\big) \leq |\Gamma| \sup_{i\in I}\big(\max_{|z-a_i|=|\Gamma|} |f(z)|\big) < \infty.$$

Since $(A_n)_{n\in\mathbb{N}}$ is a r.a.s. for A, it follows that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that (see Lemma 5.2) if $\lambda \in \rho(A)$, then $\lambda \in \rho(A_n)$ and

$$\|R_{A_n}(\lambda) - R_A(\lambda)\| \leq \frac{\varepsilon}{M}$$

whenever $n \geq N_0$.

Now, using Lemma 5.11, it follows that

$$\begin{aligned}
\|\mu_{A_n}(f) - \widehat{\mu}_A(f)\| &\leq \sup_{i\in I} \left\| \int_{a_i,\Gamma} (z-a_i) f(z)[R_{A_n}(z) - R_A(z)]\, dz \right\| \\
&\leq \sup_{i\in I} \big(\max_{|z-a_i|=|\Gamma|} |(z-a_i)f(z)| \cdot \|R_{A_n}(z) - R_A(z)\|\big) \\
&\leq \frac{\varepsilon}{M} \cdot \sup_{i\in I} \big(\max_{|z-a_i|=|\Gamma|} |(z-a_i)f(z)|\big) \\
&= \varepsilon
\end{aligned}$$

whenever $n \geq N_0$, and therefore, $(A_n)_{n\in\mathbb{N}}$ belongs to $W(c_0(J,\omega,\mathbb{K}))$. Moreover, we have

$$\lim_{n\to\infty} \mu_{A_n}(f) = \widehat{\mu}_A(f).$$

□

The next corollary is an immediate consequence of Theorem 5.10.

Corollary 5.12. *Using the notation and assumptions of Theorem* 5.10 *and assuming that* σ_A *satisfies* (5.2) *and* (5.3), *we have for any* $f \in \widehat{B}_r(\sigma_A) \bowtie \bigcup_{n\in\mathbb{N}} B_r(\sigma_{A_n})$,

$$\begin{aligned}
D(\widehat{\mu}_A(f)) &:= \Big\{ u = (u_i)_{i\in J} \in c_0(J,\omega,\mathbb{K}) : \\
&\qquad \lim_{n\to\infty} \sum_{i\in I} \int_{a_i,\Gamma} (z-a_i) f(z) R_{A_n} u\, dz \quad \textit{exists} \Big\} \\
&= \Big\{ u = (u_i)_{i\in J} \in c_0(J,\omega,\mathbb{K}) : \sum_{i\in I} \int_{a_i,\Gamma} (z-a_i) f(z) R_A u\, dz \quad \textit{exists} \Big\}
\end{aligned}$$

and

$$\begin{aligned}\widehat{\mu}_A(f)u &= \sum_{i\in I}\int_{a_i,\Gamma}(z-a_i)f(z)R_A(z)u\,dz\\ &:= \lim_{n\to\infty}\sum_{i\in I}\int_{a_i,\Gamma}(z-a_i)f(z)R_{A_n}(z)u\,dz\\ &= \sum_{i\in I}\int_{a_i,\Gamma}(z-a_i)f(z)R_A(z)u\,dz\end{aligned}$$

for all $u\in D(\widehat{\mu}_A(f))$.

Remark 5.13. Note that $\widehat{\mu}_A(f)$ is also denoted by $f(A)$. In other words,

$$f(A)u = \sum_{i\in I}\int_{a_i,\Gamma}(z-a_i)f(z)R_A(z)u\,dz$$

for all $u\in D(f(A)) = \{u=(u_i)_{i\in J}\in c_0(J,\omega,\mathbb{K}) : \sum_{i\in I}\int_{a_i,\Gamma}(z-a_i)f(z)R_Au\,dz$ exists$\}$.

Corollary 5.14. *Using the notation and assumptions of Theorem* 5.10 *and assuming that* σ_A *satisfies* (5.2) *and* (5.3), *then for* $f,g\in\widehat{B}_r(\sigma_A)$ *and* $\lambda\in\mathbb{K}$, *the following hold:*

(i) $(\lambda f)(A)=\lambda f(A)$;
(ii) $(f+g)(A)=f(A)+g(A)$;
(iii) $\langle f(A)u,v\rangle=\sum_{i\in I}\int_{a_i,\Gamma}(z-a_i)f(z)\langle R_Au,v\rangle\,dz$ *for* $u\in D(f(A))$ *and* $v\in c_0(J,\omega,\mathbb{K})$; *and*
(iv) $[f(A)]^*=f(A^*)$.

5.3 Examples.

Example 5.15. Consider the linear operators $(A_n)_{n\in\mathbb{N}}$ and A appearing in Example 5.3. We have already seen that $(A_n)_{n\in\mathbb{N}}$ is a sequence of analytic bounded linear operators with compact spectrum that is also r.a.s. for A. In addition, we have seen that $\rho(A)=\mathbb{K}-\{0\}\cup\{\lambda_i\}_{i\geq 1}$ and $\sigma_A=\{0\}\cup\{\lambda_i\}_{i\geq 1}$.

We now compute the values of $f(A)=\widehat{\mu}_A(f)$ for each function $f\in\widehat{B}_r(\sigma_A)$.

Proposition 5.16. *If* $f\in\widehat{B}_r(\sigma_A)$, *then for each* $n\in\mathbb{N}$,

$$\mu_{A_n}(f)e_s=\begin{cases} f(\lambda_s)e_s & \text{if } \lambda_s\in D(\sigma_{A_n},|\Gamma|^-),\\ 0 & \text{otherwise.}\end{cases}$$

PROOF: For each $n \in \mathbb{N}$, we write $\sigma_{A_n} \subset \bigsqcup_{i\in I_n} D(a_i, r^-)$. Then for any $\Gamma \in \mathbb{K}$ with

$$\sup_{b\in\sigma_{A_n}\bigcap D(a_i,r^-)} |b-a_i| < |\Gamma| < r,$$

we have, by using Theorem 2.17,

$$\begin{aligned}\mu_{A_n}(f)e_s &= \sum_{i\in I_n} \int_{a_i,\Gamma} (z-a_i)\frac{f(z)e_s}{z-\lambda_s}\,dz\\ &= \Big(\sum_{i\in I_n} \int_{a_i,\Gamma} (z-a_i)\frac{f(z)}{z-\lambda_s}\,dz\Big)e_s\\ &= \Big(\sum_{\lambda_s\in D(\sigma_{A_n},|\Gamma|^-)} \operatorname{res}_{z=\lambda_s}\frac{f(z)}{z-\lambda_s}\Big)e_s.\end{aligned}$$

Now, $\operatorname{res}_{z=\lambda_s}\frac{f(z)}{z-\lambda_s} = f(\lambda_s)$, and hence

$$\mu_{A_n}(f)e_s = \begin{cases} f(\lambda_s)e_s & \text{if } \ \lambda_s \in D(\sigma_{A_n}, |\Gamma|^-),\\ 0 & \text{otherwise.}\end{cases}$$

□

Corollary 5.17. *Suppose that the function f belongs to $\widehat{B}_r(\sigma_A)$. If $u = (u_s)_{s\in\mathbb{N}} \in c_0(\mathbb{N}, \omega, \mathbb{K})$, then*

$$f(A)u = \mu_{A_n}(f)u = \sum_{\lambda_s\in D(\sigma_{A_n},|\Gamma|^-)} u_s f(\lambda_s)e_s = \sum_{\lambda_s\in\sigma_{A_n}} u_s f(\lambda_s)e_s.$$

Proposition 5.18. *Suppose that the function f belongs to $\widehat{B}_r(\sigma_A)$. If $u \in D(f(A))$, then*

$$f(A)u = \widehat{\mu}_A(f)u = \lim_{n\to\infty} \mu_{A_n}(f|_{D(\sigma_{A_n},r^-)})u.$$

PROOF: First of all, let $\sigma_{A_n} \subset \bigsqcup_{i\in I_n} D(a_i, r^-)$. Now, for each $\Gamma \in \mathbb{K}$ satisfying the condition:

$$\sup_{b\in\sigma\bigcap D(a_i,r^-)} |b-a_i| < |\Gamma| < r,$$

we have that $\widehat{\mu}_A(f)e_s = f(\lambda_s)e_s$ for each $s \in \mathbb{N}$. Consequently, for each $u \in \widehat{\mu}_A(f) = f(A)$, we have that

$$f(A)u = \widehat{\mu}_A(f)u = \sum_{\lambda_s\in D(\sigma_A,|\Gamma|^-)} u_s f(\lambda_s)e_s = \sum_{\lambda_s\in\sigma_A} u_s f(\lambda_s)e_s.$$

Now, using Corollary 5.17 it follows that

$$\begin{aligned}\widehat{\mu}_A(f)u &= \sum_{\lambda_s\in D(\sigma_A,\left|\Gamma\right|^-)} u_s f(\lambda_s)e_s\\ &= \sum_{\lambda_s\in\sigma_A} u_s f(\lambda_s)e_s\\ &= \lim_{n\to\infty}\sum_{\lambda_s\in\sigma_{A_n}} u_s f(\lambda_s)e_s\\ &= \lim_{n\to\infty}\mu_{A_n}(f_{|_{D(\sigma_{A_n},r^-)}})u.\end{aligned}$$

□

Corollary 5.19. *The sequence $(A_n)_{n\in\mathbb{N}}$ is a r.a.s. for the unbounded operator linear operator A and belongs to the class $W(c_0(J,\omega,\mathbb{K}))$.*

The next example is a generalization of the previous one.

Example 5.20. Let A be the linear operator on $c_0(J,\omega,\mathbb{K})$ associated with the block diagonal matrix defined by

$$\begin{bmatrix} A_{i_0} & 0 & \dots & 0 & & & & \\ 0 & Ai_1 & 0 & 0 & \dots & & & \\ 0\dots & 0 & A_{i_2} & 0 & 0 & \dots & & \\ 0 & \dots & 0 & A_{i_3} & 0 & 0 & \dots & \\ & & \dots & 0 & \dots & \dots & \dots & \dots \\ & & & \dots & 0 & A_{i_n} & 0 & \dots \\ & & & & \dots & 0 & \ddots & \end{bmatrix},$$

where A_{i_k} for each $k\in\mathbb{N}$, is an $i_k\times i_k$ square matrix that is chosen so that A is well-defined on $c_0(J,\omega,\mathbb{K})$. Moreover, we suppose that there exists at least an index i_{k_0} such that $0\in\sigma_{A_{i_{k_0}}}$.

Now, solving $(zI-A)u=v$ for $u\in D(A)$, $v\in c_0(\mathbb{N},\omega,\mathbb{K})$, one can easily see that the following lemma holds:

Lemma 5.21. *For the operator A defined above, its resolvent $\rho(A)$ and spectrum σ_A are respectively given by*

$$\rho(A)=\mathbb{K}-\bigcup_{k=0}^{\infty}\sigma_{A_{i_k}} \quad\text{and}\quad \sigma_A=\bigcup_{k=0}^{\infty}\sigma_{A_{i_k}},$$

where $\sigma_{A_{i_k}}$ is the spectrum of the $i_k\times i_k$ square matrix A_{i_k}.

Note that the spectrum $\sigma_{A_{i_k}}$ coincides with the set of eigenvalues of the square matrix A_{i_k}. Moreover, from the algebraic closedness of the field $\mathbb{K}$, we know that $\sigma_{A_{i_k}}$ has exactly i_k elements. From that one can easily see that there exists an orthogonal basis $(\widehat{e}_n)_{n\in\mathbb{N}}$ for $c_0(J,\omega,\mathbb{K})$ in which, A is a diagonal operator, say, A is represented in the orthogonal basis $(\widehat{e}_n)_{n\in\mathbb{N}}$ by the diagonal matrix

$$\begin{bmatrix} \gamma_0 & 0 & \dots & 0 & & & & \\ 0 & \gamma_1 & 0 & 0 & \dots & & & \\ 0\dots & 0 & \gamma_2 & 0 & 0 & \dots & & \\ 0 & \dots & 0 & \gamma_3 & 0 & 0 & \dots & \\ & & \dots & 0 & \dots & \dots & \dots & \dots \\ & & & \dots & 0 & \gamma_n & 0 & \dots \\ & & & & \dots & 0 & \ddots & \end{bmatrix},$$

where $\sigma_A = \bigcup_{k=0}^{\infty} \sigma_{A_{i_k}} = \{\gamma_i\}_{i\in\mathbb{N}}$ with one of these γ_i being 0, by assumption.

In addition to the above, we assume that the eigenvalues γ_i are given so that $\lim_{i\to\infty} |\gamma_i| = \infty$. From the previous assumption, one can easily see that A is an unbounded linear operator on $c_0(J,\omega,\mathbb{K})$.

Let A_n be the linear operator on $c_0(J,\omega,\mathbb{K})$ associated with the block diagonal matrix defined by

$$A_n = \begin{bmatrix} A_{i_0} & 0 & \dots & 0 & & & & \\ 0 & Ai_1 & 0 & 0 & \dots & & & \\ 0\dots & 0 & . & 0 & 0 & \dots & & \\ 0 & \dots & 0 & A_{i_n} & 0 & 0 & \dots & \\ & & \dots & 0 & 0 & \dots & \dots & \dots \\ & & & \dots & 0 & . & 0 & \dots \\ & & & & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the sequence of operators A_n defined above, its spectrum σ_{A_n} is given by

$$\sigma(A_n) = \{0\} \cup \bigcup_{k=0}^{n} \sigma(A_{i_k}),$$

where $\sigma(A_{i_k})$ is the (compact) spectrum of the $i_k \times i_k$ matrix square matrix A_{i_k}.

It is now clear that $\|(zI - A)^{-1} - (zI - A_n)^{-1}\| \to 0$ as $n \to \infty$ for all $z \in \rho(A)$, and hence $(A_n)_{n\in\mathbb{N}}$ is a r.a.s. for A. Moreover, it is also easy to see that the sequence $(A_n)_{n\in\mathbb{N}}$ belongs to the class of $W(c_0(J,\omega,\mathbb{K}))$ with

$$f(A)u = \widehat{\mu}_A(f)u = \sum_{\lambda_s\in D(\sigma_A,|\Gamma|^-)} u_s f(\lambda_s)\widehat{e}_s = \sum_{\lambda_s\in\sigma_A} u_s f(\lambda_s)\widehat{e}_s = \sum_{s=0}^{\infty} u_s f(\gamma_s)\widehat{e}_s$$

for all $u = \sum_{s=0}^{\infty} u_s \widehat{e}_s \in D(f(A))$ and $f \in \widehat{B}_r(\sigma_A) \bowtie \bigcup_{n\in\mathbb{N}} B_r(\sigma_{A_n})$.

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