On r-reflexive Banach spaces

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Abstract. A Banach space X is called r*-reflexive* if for any cover $\mathcal{U}$ of X by weakly open sets there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ covering some ball of radius 1 centered at a point x with $\|x\| \le r$. We prove that an infinite-dimensional separable Banach space X is ∞-reflexive (r-reflexive for some $r \in \mathbb{N}$) if and only if each ε-net for X has an accumulation point (resp., contains a non-trivial convergent sequence) in the weak topology of X. We show that the quasireflexive James space J is r-reflexive for no $r \in \mathbb{N}$. We do not know if each ∞-reflexive Banach space is reflexive, but we prove that each separable ∞-reflexive Banach space X has Asplund dual. As a by-product of the proof we obtain a covering characterization of the Asplund property of Banach spaces.

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1. Introduction

In this paper we address the following problem posed by the third author in 2000 at the Winter School in Křištanovice (Czech Republic):

Question 1. *Is a separable Banach space X reflexive if each net in X has an accumulation point in the weak topology of X?*

By a *net* in a Banach space $(X, \|\cdot\|)$ we understand an ε-net $N \subset X$ for some $\varepsilon > 0$. A subset $N \subset X$ is called an ε*-net* for a subset $B \subset X$ if for every point $x \in B$ there is a point $y \in N$ with $\|x - y\| < \varepsilon$.

It turns out that Question 1 is equivalent to an even more intriguing question concerning ∞-reflexive Banach spaces.

Definition 1. A Banach space $(X, \|\cdot\|)$ is called r*-reflexive* where $r \in [0, +\infty]$ if for every cover $\mathcal{U}$ of X by weakly open sets there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ that covers the open unit ball $x + B_X = \{y \in X : \|x - y\| < 1\}$ centered at some point $x \in X$ with $\|x\| \le r$.

Observe that a Banach space X is reflexive if and only if it is 0-reflexive. We define a Banach space X to be ω*-reflexive* if it is r-reflexive for some $r \in [0, \infty)$.

It turns out that for infinite-dimensional separable Banach spaces the property appearing in Question 1 is equivalent to the ∞-reflexivity.

Theorem 1. *An infinite-dimensional separable Banach space X is ∞-reflexive (resp. ω-reflexive) if and only if every net in X has an accumulation point (resp. contains a non-trivial convergent sequence) in the weak topology of X.*

This theorem is not true for non-separable Banach spaces: for any uncountable set Γ the Banach space $c_0(\Gamma)$ is weakly Lindelöf [Fab, §7.1]. Consequently, each net for $c_0(\Gamma)$, being uncountable, has an accumulation point in the weak topology. On the other hand, the space $c_0(\Gamma)$ is not ∞-reflexive by Proposition 2 below. This example also shows that in the realm of non-separable Banach spaces the answer to Question 1 is negative.

Theorem 1 allows us to reformulate and extend Question 1 as follows:

Question 2. *Is a (separable) Banach space X reflexive if it is ∞-reflexive? ω-reflexive? 1-reflexive?*

In light of the last part of this question, it is interesting to mention that a Banach space X is reflexive if and only if X is r-reflexive for some $r < 1$. This equivalence (observed by the referee) follows from the fact that each cover of a 1-ball $x + B_X$ centered at a point $x \in X$ with $\|x\| \leq r < 1$ covers also the closed ball of radius $\frac{1}{2}(1-r)$ centered at the origin.

Trying to answer Questions 1 and 2, it is natural to look at the quasireflexive James space J (having codimension 1 in its second dual). We recall that a Banach space X is *quasireflexive* if it has finite codimension in its second dual space X^{**}.

Theorem 2. *The quasireflexive James space J is not ω-reflexive.*

However we do not know if the James space is ∞-reflexive.

Question 3. *Is each quasireflexive Banach space ∞-reflexive? Is the James space ∞-reflexive?*

Our principal result on separable ∞-reflexive Banach spaces asserts that any such a space has Asplund dual. We recall that a Banach space X is *Asplund* if each separable subspace Y of X has separable dual Y^*.

Theorem 3. *Each separable ∞-reflexive Banach space X has Asplund dual X^*.*

Since the Banach space l_1 is not Asplund, Theorem 3 implies the result of [Ba] (asserting that the dual space X^* of a separable ∞-reflexive Banach space X contains no copy of l_1). Theorem 3 has also another corollary related to the Fréchet-Urysohn property of the weak topology on bounded subsets of an ∞-reflexive Banach space.

Following [En, §1.6], we say that a topological space X is *Fréchet-Urysohn* if for each accumulation point $x \in X$ of a subset $A \subset X$ some sequence $\{a_n\}_{n=1}^{\infty} \subset A$ converges to x.

Since Eberlein compact spaces are Fréchet-Urysohn, the weak topology of a reflexive space X is Fréchet-Urysohn on bounded subsets of X. A similar property holds for separable ∞-reflexive Banach spaces.

Corollary 1. *If X is a separable ∞-reflexive Banach space, then the unit ball of X endowed with the weak topology is a Fréchet-Urysohn space.*

Proof: First we show that the space X contains no copy of the Banach space l_1. In the opposite case the non-Asplund space $l_\infty = l_1^*$ would be a quotient of the Asplund space X^* which is not possible (because the Asplund property is preserved by quotients). Since X contains no copy of l_1, it is legal to apply the Odell-Rosenthal Theorem [OR] (see also [Dis, p. 215]) to conclude that the second dual unit ball $\bar{B}^{**}$ endowed with the weak* topology is Rosenthal compact; more precisely, $\bar{B}^{**}$ is a compact subspace of the space $B_1(B^*) \subset \mathbb{R}^{\bar{B}^*}$ of functions of the first Baire class on the dual unit ball $\bar{B}^*$. Finally, we apply the Bourgain-Fremlin-Talagrand Theorem [BFT] establishing the Fréchet-Urysohn property of separable Rosenthal compacta to conclude that the unit ball $\bar{B}^{**} \supset \bar{B}$ is Fréchet-Urysohn. □

The proof of Theorem 3 relies on a characterization of the Asplund property of a dual Banach space in terms of so-called weak* covering properties.

Definition 2. A Banach space X is said to satisfy the *τ-covering property*, where τ is a weaker linear topology on X, if for every sequence $(U_i)_{i=1}^\infty$ of τ-open sets in X whose intersection $\bigcap_{i=1}^\infty U_i$ is a norm-neighborhood of the origin in X there are points $x_1, \dots, x_n \in X$ such that the union $\bigcup_{i=1}^n (x_i + U_i)$ contains the open unit ball B_X centered at the origin of X.

If τ is the weak or weak* topology, then we call *the weak or weak* covering properties*, briefly, WCP and W*CP.

Theorem 3 can be derived from the following theorem that can have an independent value.

Theorem 4. (1) *Each separable ∞-reflexive Banach space has the weak covering property.*
(2) *If a Banach space X has the weak covering property, then the second dual space X^{**} has the weak* covering property.*
(3) *A Banach space X is Asplund if and only if the dual space X^* has the weak* covering property.*

The obtained results fit into the following diagram connecting various reflexivity-like properties and holding for any separable Banach space X:

$$\begin{tikzcd}
X \text{ is 0-reflexive} \arrow[r] \arrow[d, leftrightarrow] & X \text{ is } \omega\text{-reflexive} \arrow[r] & X \text{ is } \infty\text{-reflexive} \arrow[r] & X \text{ has WCP} \arrow[d] \\
X \text{ is reflexive} \arrow[r] & X \text{ is quasireflexive} \arrow[r] \arrow[u, "/" marking] & X^* \text{ is Asplund} \arrow[r, leftrightarrow] & X^{**} \text{ has W}^*\text{CP}
\end{tikzcd}$$

Before passing to proofs of Theorems 1–4 we discuss some stability properties of r-reflexive spaces and ask some related questions.

Proposition 1. *Let Z be a Banach subspace of a Banach space X.*

(1) *If X is r-reflexive for some $r \in [0,+\infty]$, then the quotient space X/Z is r-reflexive too.*
(2) *If X is r-reflexive for some $r \in \{0,\omega,\infty\}$, then each Banach space Y isomorphic to X is r-reflexive.*
(3) *If Z is reflexive and X/Z is r-reflexive for some $r \in [0,+\infty)$, then X is r-reflexive too.*

Question 4. *Is a subspace of a (separable) r-reflexive Banach space r-reflexive (at least for $r \in \{\omega,\infty\}$)?*

Question 5. *Is the second dual X^{**} of an r-reflexive Banach space X r-reflexive? Is a Banach space X r-reflexive if its second dual X^{**} if r-reflexive for some $r \in [0,+\infty]$?*

Since the r-reflexivity is an isomorphic property for $r \in \{0,\omega,\infty\}$, we may also ask:

Question 6. *Is the r-reflexivity an isomorphic property for arbitrary $r \in (0,+\infty)$?*

As we already know, Theorem 1 is not true for non-separable Banach spaces. What about Theorem 3?

Question 7. *Has each ∞-reflexive Banach space Asplund dual?*

We can give a partial answer for Banach spaces with $\aleph_0$-monolithic dual space. We recall that a topological space X is *monolithic* (resp. $\aleph_0$-*monolithic*) if each (separable) subspace Y of X has network weight $\mathrm{nw}(Y)$ equal to the density $\mathrm{dens}(Y)$ of Y. It is easy to see that each Banach space is monolithic in norm and weak topologies.

We shall say that a Banach space X has ($\aleph_0$-)*monolithic dual space*, if the dual space X^* is ($\aleph_0$-)monolithic with respect to the weak* topology. It can be shown that a Banach space X has ($\aleph_0$-)monolithic dual space if and only if for any (separable) subset $Y \subset X^*$ the annulator $Y^\top = \{x \in X : \forall y^* \in Y\ y^*(x) = 0\}$ satisfies $\mathrm{dens}(X/Y^\top) = \mathrm{dens}(Y)$ in X. The latter property was introduced in [BPZ] as the property (1). Since Corson compacta are monolithic, each weakly Lindelöf determined Banach space (=Banach space with Corson dual ball) has monolithic dual. In particular, for each set Γ the Banach space $c_0(\Gamma)$ has monolithic dual.

Proposition 2. *Each ∞-reflexive Banach space with $\aleph_0$-monolithic dual has Asplund dual.*

PROOF: Assume that X is an ∞-reflexive Banach space with $\aleph_0$-monolithic dual. To show that X^* is Asplund, take any separable subspace $Y \subset X^*$ and consider its annulator $Y^\top = \{x \in X : \forall y^* \in Y\ y^*(x) = 0\}$ in X. The Hahn-Banach Theorem implies that Y is weak* dense in $(X/Y^\top)^*$ identified with the subspace $(Y^\top)^\perp \subset X^*$ of functionals that annulate $Y^\top$. Since X has $\aleph_0$-monolithic dual, the space $(X/Y^\top)^*$, being separable, has countable network weight in the weak* topology. Consequently, the unit ball of $(X/Y^\top)^*$ in the weak* topology has countable network weight and is metrizable. This is equivalent to the separability of $X/Y^\top$. Being a quotient of the ∞-reflexive space X, the space $X/Y^\top$ is ∞-reflexive. Applying Theorem 3, to the separable ∞-reflexive space $X/Y^\top$, we conclude that the dual space $(X/Y^\top)^*$ is Asplund and consequently, its separable subspace Y has separable dual Y^*. □

Also we do not know if the separability assumption is essential in Corollary 1.

Question 8. *Let X be an ∞-reflexive Banach space (with $\aleph_0$-monolithic dual). Is the unit ball of X endowed with the weak topology a Fréchet-Urysohn space?*

Finally, we ask:

Question 9. *Let X be a separable ∞-reflexive Banach space. Is the dual space X^* separable? Equivalently, is the second dual X^{**} separable?*

Now we present the proofs of the results announced in the introduction.

2. Proof of Theorem 4

The first item of Theorem 4 is established in

Lemma 1. *A separable ∞-reflexive Banach space X has the weak covering property.*

PROOF: To show that X has the weak covering property, take any sequence $(U_n)_{n\in\omega}$ of weakly open sets in X such that $\bigcap_{n\in\omega} U_n$ has non-empty interior in X. Let $\{x_n : n \in \omega\}$ be a countable dense set in X. It follows that $\{x_n + U_n : n \in \omega\}$ is a cover of X by weakly open sets. The ∞-reflexivity of X yields a point $x \in X$ such that the open unit ball $x + B_X$ centered at x lies in the finite union $\bigcup_{n=0}^m x_n + U_n$ for some $m \in \omega$. Then $B_X \subset \bigcup_{n=0}^m (x_n - x + U_n)$ witnessing the weak covering property of X. □

The second item of Theorem 4 is established in

Lemma 2. *If a Banach space has the weak covering property, then the second dual space X^{**} has the weak* covering property.*

PROOF: Suppose that $(V_i)_{i=1}^\infty$ is a sequence of weak* open sets in X^{**} whose intersection $\bigcap_{i=1}^\infty V_i$ contains a closed ε-ball $\varepsilon\bar{B}^{**}$. To show that X^{**} has the weak* covering property, it suffices to find points $x_1, \dots, x_n \in X^{**}$ such that $\bigcup_{i=1}^n (x_i + V_i) \supset \bar{B}^{**}$.

By the compactness of $\varepsilon\bar{B}^{**}$ and the regularity of the weak* topology, for every $i \in \mathbb{N}$, there is a weak* open subset $W_i \subset X^{**}$ such that $\varepsilon\bar{B}^{**} \subset W_i \subset \overline{W_i} \subset V_i$ where the closure is taken in the weak* topology of X^{**}.

Consider the sequence $(U_i)_{i=1}^\infty$, $U_i = W_i \bigcap X$, of weakly open sets in X. Note that $\bigcap_{i=1}^\infty U_i = (\bigcap_{i=1}^\infty W_i) \bigcap X \supset \varepsilon\bar{B}^{**} \bigcap X = \varepsilon\bar{B}$. By definition of the weak covering property of X, there exist points $x_1, \dots, x_n \in X$ such that the union $\bigcup_{i=1}^n (x_i + U_i)$ contains the open unit ball B centered at the origin. According to Goldstine Theorem [HHZ, p. 46], $\overline{B} = \bar{B}^{**}$. Thus we obtain $\bar{B}^{**} = \overline{B} \subset \bigcup_{i=1}^n (x_i + \overline{U_i}) \subset \bigcup_{i=1}^n (x_i + \overline{W_i}) \subset \bigcup_{i=1}^n (x_i + V_i)$, and hence X^{**} has the weak* covering property. □

For the proof of the third item of Theorem 4 we need an auxiliary

Lemma 3. *Let K be a weak* compact subset of a weak* open set U of a dual Banach space X^*. Then there is a weak* open set V in X^* such that $K \subset V \subset U$ and $V = V + L$ for some weak* closed linear subspace L of finite codimension in X^*.*

Proof: By definition, the weak* topology on X^* has a base consisting of sets W such that $W = W + F^\perp$ for some finite subset $F \subset X$. Here, as expected, $F^\perp = \{x^* \in X^* : \forall x \in F\ x^*(x) = 0\}$. Consequently, for every $x \in K$ we may find a weak* open subset $O(x) \subset X^*$ such that $x \in O(x) \subset U$ and $O(x) = O(x) + F_x^\perp$ for some finite subset $F_x \subset X$. Using the weak* compactness of K, choose a finite subcover $\{O(x_1), \dots, O(x_n)\}$ of the cover $\{O(x) : x \in K\}$ of K. Then the weak* open set $V = \bigcup_{i=1}^n O(x_i)$ has the properties $K \subset V \subset U$ and $V = V + F^\perp$, where $F = \bigcup_{i=1}^n F_{x_i}$. □

The following characterization establishes the third item of Theorem 4.

Proposition 3. *For a Banach space X, the following conditions are equivalent:*

(1) *X is Asplund;*
(2) *X^* has W*CP;*
(3) *for each sequence $(U_i)_{i=1}^\infty$ of weak* open subsets of X^* whose intersection $\bigcap_{i=1}^\infty U_i$ is a norm-neighborhood of the origin there is a sequence of points $\{a_i^*\}_{i=1}^\infty \subset X^*$ such that $X^* = \bigcup_{i=1}^\infty (a_i^* + U_i)$.*

Proof: (1) $\Rightarrow$ (3) Fix a sequence $(U_i)_{i=1}^\infty$ of weak* open sets in X^* whose intersection $\bigcap_{i=1}^\infty U_i$ contains the closed ε-ball $\varepsilon\bar{B}^*$ centered at the origin.

By Lemma 3, for every $i \geq 1$ there exists a weak* open set $V_i \subset X^*$ such that $\varepsilon\bar{B}^* \subset V_i \subset U_i$ and $V_i = V_i + F_i^\perp$ for some finite subset $F_i \subset X$. Let Y be the closed linear hull of the set $F = \bigcup_{i=1}^\infty F_i$ in X. As X is Asplund, Y^* is separable. Since Y^* is isomorphic to $X^*/Y^\perp = X^*/F^\perp$, the latter quotient space is separable. Since the quotient map $\pi : X^* \to X^*/F^\perp$ is open, the set $\pi(\varepsilon B^*)$

has non-empty interior in $X^*/F^\perp$. The separability of $X^*/F^\perp$ yields a countable subset $C = \{c_i : i \ge 1\}$ of $X^*/F^\perp$ such that $C + \pi(\varepsilon B^*) = X^*/F^\perp$. For every $i \ge 1$ find a point $a_i^* \in X^*$ with $\pi(a_i^*) = c_i$. Then

$$X^* = \bigcup_{i=1}^{\infty} \big(a_i^* + \pi^{-1}(\pi(\varepsilon B^*))\big)$$
$$= \bigcup_{i=1}^{\infty}(a_i^* + \varepsilon B^* + F^\perp) \subset \bigcup_{i=1}^{\infty}(a_i^* + (V_i + F_i^\perp)) \subset \bigcup_{i=1}^{\infty}(a_i^* + U_i).$$

The implication (3) $\Rightarrow$ (2) trivially follows from the weak* compactness of the unit ball $\bar{B}^* \subset X^*$.

(2) $\Rightarrow$ (1) Assume that X is not Asplund. Then by Theorem 5.2.3 of [Fab], the dual Banach space X^* contains a bounded subset D such that every non-empty relatively weak* open subset U of D has norm diameter $> 8\varepsilon$ for some $\varepsilon > 0$. Without loss of generality, $0 \in D$ and $\|x^*\| < 1$ for every $x^* \in D$.

Let $2 = \{0, 1\}$ and $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ be the set of all finite binary sequences. For each sequence $s = (s_0, \dots, s_{n-1}) \in 2^{<\omega}$, by $|s| = n$ we denote its length and by $s|k = (s_0, \dots, s_{k-1})$ the initial segment of s of length $k \le |s|$. For $i \in \{0, 1\}$ let $s\hat{}i = (s_0, \dots, s_{n-1}, i)$ be the concatenation of s and i.

The set $2^{<\omega}$ is a (binary) tree with respect to the partial order: $s \le t$ if $s = t|n$ for some $n \le |t|$. The empty sequence is the smallest element of $2^{<\omega}$.

Let $x_\emptyset^* = 0$ and $x_\emptyset = 0$. By induction on the tree $2^{<\omega}$, we shall construct sequences $(x_t^*)_{t\in 2^{<\omega}} \subset D$ and $(x_t)_{t\in 2^{<\omega}} \subset X$ such that for every $t \in 2^{<\omega}$ the following conditions are satisfied:

(1) $x_{t\hat{}0}^* = x_t^*$ and $x_{t\hat{}0} = x_t$;
(2) $|x_{t\hat{}1}^*(x_s) - x_t^*(x_s)| < 2^{-|t|}\varepsilon$ for all $s \in 2^{<\omega}$ with $|s| \le |t|$;
(3) $\|x_{t\hat{}1}\| = 1$;
(4) $x_{t\hat{}1}^*(x_{t\hat{}1}) - x_t^*(x_{t\hat{}1}) \ge 4\varepsilon$.

Suppose for some $t \in 2^{<\omega}$ the functionals x_s^* and points x_s have been constructed for all $s \in 2^{<\omega}$ with $|s| < |t|$. If $t = \tau\hat{}0$ for some $\tau \in 2^{<\omega}$, then we put $x_t^* = x_\tau^*$ and $x_t = x_\tau$.

Now consider the other case: $t = \tau\hat{}1$ for some $\tau \in 2^{<\omega}$. Consider the weak* open set

$$W = \{x^* \in D : \forall s \in 2^{<\omega}\ |s| < |t| \ \Rightarrow\ |x^*(x_s) - x_\tau^*(x_s)| < \varepsilon\}$$

in D. Since $W \ne \emptyset$, we have $\operatorname{diam} W > 8\varepsilon$. Consequently there exists a functional $x_t^* \in W$ such that $\|x_t^* - x_\tau^*\| > 4\varepsilon$. Choose a point $x_t \in X$ with $\|x_t\| = 1$ and $(x_t^* - x_\tau^*)(x_t) \ge 4\varepsilon$. This completes the inductive construction.

For every $i \in \mathbb{N}$ let

$$U_i = \{x^* \in X^* : |x^*(x_s)| < \varepsilon \text{ for every } s \in 2^{<\omega} \text{ with } |s| \leq i\}.$$

Evidently, U_i are weak* open sets in X^* and their intersection $\bigcap_{i=1}^{\infty} U_i$ contains the open ε-ball εB^*. To see that W*CP fails for the space X^* it suffices to check that $B^* \not\subset \bigcup_{i=1}^{n} a_i^* + U_i$ for every $n \in \mathbb{N}$ and points $a_1^*, \dots, a_n^* \in X^*$. This will follow as soon as we find $t \in 2^n$ with $x_t^* \notin \bigcup_{j=1}^{n}(a_j^* + U_j)$.

Since $|(x_1^* - x_0^*)(x_1)| \geq 4\varepsilon$, there is $t_0 \in \{0, 1\}$ with $|x_{t_0}^*(x_1) - a_1^*(x_1)| \geq 2\varepsilon$. By the same reason, the inequality $(x_{(t_0,1)} - x_{(t_0,0)})(x_{(t_0,1)}) \geq 4\varepsilon$ yields a number $t_1 \in \{0, 1\}$ such that $|(x^*_{(t_0,t_1)} - a_2^*)(x_{(t_0,1)})| \geq 2\varepsilon$. Proceeding by finite induction and using (4), we may construct a sequence $t = (t_0, t_1, \dots, t_{n-1}) \in 2^n$ such that for every $k \leq n$

(5) $\quad |(x^*_{t|k} - a_k^*)(x_{s_k})| \geq 2\varepsilon$ for some sequence $s_k \in 2^k$.

Let us show that $x_t^* \notin \bigcup_{j=1}^{n}(a_j^* + U_j)$. Assuming the converse, we would find a number $p \leq n$ with $x_t^* - a_p^* \in U_p$ which implies

(6) $\quad |(x_t^* - a_p^*)(x_{s_p})| < \varepsilon.$

It follows from (2) that

$$|(x_t^* - x^*_{t|p})(x_{s_p})| \leq \sum_{k=p}^{n-1} |(x^*_{t|k+1} - x^*_{t|k})(x_{s_p})| \leq \sum_{k=p}^{n-1} 2^{-k}\varepsilon < 2^{-p+1}\varepsilon \leq \varepsilon$$

which together with (6) yields the inequality $|(x^*_{t|p} - a_p^*)(x_{s_p})| < 2\varepsilon$ that contradicts (5). □

3. Proof of Theorem 1

The following two lemmas yield the "∞-reflexive" part of Theorem 1.

Lemma 4. *Each net in an infinite-dimensional ∞-reflexive Banach space has an accumulation point in the weak topology.*

Proof: Assume that some ε-net N in X has no accumulating points in the weak topology. Replacing N by a suitable homothetic copy, we can assume that $\varepsilon = \frac{1}{8}$. Since N has no accumulation points in the weak topology, there is a cover $\mathcal{U}$ of X by weakly open subsets such that each set $U \in \mathcal{U}$ has at most one common point with the net N. Since X is ∞-reflexive, there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ whose union $\bigcup \mathcal{V}$ contains some ball B of radius 1. Then $B \cap N \subset \bigcup_{V \in \mathcal{V}} V \cap N$ is finite. One can easily check that $B \cap N$ is a $\frac{1}{4}$-net for B, which implies that X is finite-dimensional according to the classical Riesz Lemma on an almost orthogonal element, see [HHZ, Lemma 15]. □

Lemma 5. *A separable Banach space X is ∞-reflexive if each net in X has an accumulation point in the weak topology.*

PROOF: Assuming that X is not ∞-reflexive, find a cover $\mathcal{U}$ of X by weakly open sets such that for every finite subfamily $\mathcal{V} \subset \mathcal{U}$ the union $\bigcup \mathcal{V}$ contains no ball of radius 1. Using the separability of X, we can assume that the cover $\mathcal{U}$ is countable and hence can be enumerated as $\mathcal{U} = \{U_n : n \in \omega\}$. Let $\{a_n : n \in \omega\}$ be a countable dense set in X. For every $n \in \omega$ we can find a point $x_n \in X \setminus \bigcup_{i<n} U_i$ with $\|x_n - a_n\| \leq 1$. Such a point x_n exists by the choice of the cover $\mathcal{U}$. Then $\{x_n : n \in \omega\}$ is a 2-net in X having no accumulation point in the weak topology. $\square$

The "ω-reflexive" part of Theorem 1 is established in the following more general characterization of the ω-reflexivity. However we shall need a more general meaning for an ε-net: a subset N of a Banach space $(X, \|\cdot\|)$ is called an *ε-net* for a subset $B \subset X$ if for every $x \in B$ there is $y \in N$ with $\|x - y\| < \varepsilon$.

Lemma 6. *For a separable infinite-dimensional Banach space X the following conditions are equivalent:*

(1) *X is ω-reflexive;*
(2) *each net for X contains a non-trivial sequence convergent in the weak topology of X;*
(3) *there are a bounded set $D \subset X$ and $\varepsilon > 0$ such that each ε-net $N \subset X$ for D has an accumulation point in the weak topology of X.*

PROOF: We shall prove the equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (3). Assume that X is ω-reflexive and find $r \in \mathbb{N}$ such that X is r-reflexive. We claim that each $\frac{1}{4}$-net for the ball $(r+1)B = \{x \in X : \|x\| < r+1\}$ has an accumulating point in the weak topology of X. Assuming that it is not so, find a $\frac{1}{4}$-net $N \subset X$ for $(r+1)B$ having no accumulation point in the weak topology. This allows us to construct a cover $\mathcal{U}$ of X by weakly open sets having at most one-point intersection with the net N. The r-reflexivity of X yields a finite subfamily $\mathcal{V} \subset \mathcal{U}$ covering the 1-ball $x+B = \{y \in X : \|x-y\| < 1\}$ centered at a point $x \in X$ with $\|x\| \leq r$. Then the intersection $(x+B) \cap N \subset \bigcup_{V \in \mathcal{V}} V \cap N$ is finite and thus lies in a finite-dimensional subspace $F \subset X$. The Riesz almost orthogonality Lemma 15 in [HHZ] allows us to find a point $y \in B_x$ such that $\|y - x\| = \frac{1}{2}$ but $\mathrm{dist}(y, F) > \frac{1}{4}$. Using the fact that N is a $\frac{1}{4}$-net for $(r+1)B \supset x + B$, find a point $z \in N$ with $\|z - y\| < \frac{1}{4}$. Then $z \in (x+B) \cap N \setminus F$ which is not possible because $(x+B) \cap N \subset F$.

(3) $\Rightarrow$ (1) Assume that for some bounded set $D \subset X$ and some $\varepsilon > 0$ each ε-net $N \subset X$ for D has an accumulation point in the weak topology. Replacing D by its homothetic copy, we can assume that $\varepsilon = 1$. Let $r = \sup\{\|x\| : x \in D\}$. We claim that the space X is r-reflexive. Otherwise we can find an open cover $\mathcal{U}$ of

X by weakly open subsets such that no finite subfamily of $\mathcal{U}$ covers the open unit ball centered at a point $x \in X$ with $\|x\| \le r$. Use the separability of X to find a countable subcover $\{U_n : n \in \omega\} \subset \mathcal{U}$ of X and let $\{x_n : n \in \omega\}$ be a countable dense set in the r-ball $rB = \{x \in X : \|x\| < r\}$. For every $n \in \omega$ select a point $y_n \in X \setminus \bigcup_{k<n} U_k$ with $\|x_n - y_n\| < 1$ (such a point y_n exists by the choice of the cover $\mathcal{U}$). Then $N = \{y_n : n \in \omega\}$ is a 2-net for D without accumulation points in the weak topology of X. This is a contradiction.

(3) $\Rightarrow$ (2) Assume that for some bounded set $D \subset X$ and some $\varepsilon > 0$ each ε-net $N \subset X$ for D has an accumulation point in the weak topology. Then given any ε-net N in X we can find a bounded subset $A \subset N$ having an accumulation point $a \in X$ in the weak topology of X. The implication (3) $\Rightarrow$ (1) ensures that X is ω-reflexive and hence ∞-reflexive. By Corollary 1, the bounded subset $A \cup \{a\}$, being Fréchet-Urysohn, contains a sequence $\{a_n\}_{n=1}^{\infty} \subset A \setminus \{a\}$ that converges to a.

(2) $\Rightarrow$ (3). Assume conversely that for each bounded set D and every $\varepsilon > 0$ there is an ε-net $N \subset X$ for D having no accumulation point in the weak topology of X. In particular, for every $r \in \omega$, there is an 1-net N_r for the r-ball $B_r = \{x \in X : \|x\| \le r\}$ having no accumulating point in the weak topology of X. Now consider the union $N = \bigcup_{r\in\omega} N_r \setminus B_{r-2}$ and note that it is an 1-net for X. Indeed, given any $x \in X$ find $r \in \omega$ with $r - 1 < \|x\| \le r$ and $y \in N_r$ with $\|x - y\| < 1$. Then $\|y\| > \|x\| - 1 > r - 2$ and hence $y \in N_r \setminus B_{r-2} \subset N$. Assuming that N contains a non-trivial weakly convergent sequence $S \subset N$, find $R \in \omega$ with $S \subset B_{R-2}$ and observe that $S \subset N \cap B_{R-2} \subset \bigcup_{r\le R} N_r$. Then for some $r \le R$ the intersection $S \cap N_r$ is infinite and hence $N_r \supset S \cap N_r$ has an accumulation point in the weak topology, which contradicts the choice of N_r. $\square$

4. Proof of Proposition 1

Let Z be a subspace of a Banach space X and let $\pi : X \to X/Z$ denote the quotient operator.

1. Assuming that X is r-reflexive for some $r \in [0, +\infty]$, we shall prove that the quotient space X/Z is r-reflexive too. Given a cover $\mathcal{U}$ of X/Z by weakly open sets, consider the cover $\pi^{-1}(\mathcal{U}) = \{\pi^{-1}(U) : U \in \mathcal{U}\}$ of X. By the r-reflexivity of X there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ whose preimage $\pi^{-1}(\mathcal{V})$ covers some ball $x + B_X = \{y \in X : \|x - y\| < 1\}$ centered at a point $x \in X$ with $\|x\| \le r$. Then the family $\mathcal{V}$ covers the image $\pi(x + B_X)$ which coincides with the ball $\pi(x) + B_{X/Z} = \{z \in X/Z : \|z - \pi(x)\| < 1\}$ according to the definition of the quotient norm on X/Z. Taking into account that $\|\pi(x)\| \le \|x\| \le r$, we conclude that the space X/Z is r-reflexive.

2. Assume that X is ω-reflexive and a Banach space Y is isomorphic to X. Let $T : X \to Y$ be an isomorphism between X and Y and $M = \max\{\|T\|, \|T^{-1}\|\}$.

Let B_X, B_Y denote the open unit balls centered at the origins of the spaces X, Y, respectively. It follows that $\frac{1}{M}B_Y \subset T(B_X) \subset M \cdot B_Y$.

The space X, being ω-reflexive, is r-reflexive for some r. We claim that Y is M^2r-reflexive. Indeed, given a cover $\mathcal{U}$ of Y by weakly open sets, consider the covers $\mathcal{W} = T^{-1}(\mathcal{U}) = \{T^{-1}(U) : U \in \mathcal{U}\}$ and $\frac{1}{M}\mathcal{W} = \{\frac{1}{M} \cdot W : W \in \mathcal{W}\}$ of X. The r-reflexivity of X implies the existence of a finite subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup_{V \in \mathcal{V}} \frac{1}{M}T^{-1}(V)$ covers the unit ball $x + B_X$ centered at some point $x \in X$ with $\|x\| \le r$. Letting $y = M \cdot T(x)$, observe that $\|y\| = M \cdot T(x) \le M^2 r$ and $y + B_Y \subset T(Mx + MB_X) = M \cdot T(x + B_X) \subset M \cdot T\big(\bigcup_{V \in \mathcal{V}} \frac{1}{M}T^{-1}(V)\big) = \bigcup \mathcal{V}$, witnessing the M^2r-reflexivity of the space Y.

By analogy we can prove that the ∞-reflexivity of X implies the ∞-reflexivity of Y. Finally the 0-reflexivity coincides with the usual reflexivity and also is preserved by isomorphisms.

3. Assume that the space Z is reflexive and X/Z is r-reflexive for some $r \in [0, \infty)$. Since the short sequence $0 \to Z \to X \to X/Z \to 0$ is exact, so is the sequence $0 \to Z^{**} \to X^{**} \to (X/Z)^{**} \to 0$, see [CG, 2.2.d]. Consequently, the second dual $\pi^{**} : X^{**} \to (X/Z)^{**}$ of the quotient operator $\pi : X \to X/Z$ has $Z^{**} = Z$ as the kernel.

We claim that for each bounded weakly closed subset $F \subset X$ the image $\pi(F)$ is weakly closed in X/Z. It follows that the closure $\bar{F}$ of F in the weak*-topology of X^{**} is compact and so is its image $\pi^{**}(\bar{F}) \subset (X/Z)^{**}$. We claim that $\pi(F) = \pi^{**}(\bar{F}) \cap \pi(X)$ which will ensure that $\pi(F)$ is closed in X/Z. Indeed, the inclusion $\pi(F) = \pi^{**}(\bar{F} \cap X) \subset \pi^{**}(\bar{F}) \cap \pi(X)$ is trivial. To prove the reverse inclusion, take any point $y^{**} \in \pi^{**}(\bar{F}) \cap \pi(X)$ and find two points $x^{**} \in \bar{F}$ and $x \in X$ with $\pi^{**}(x^{**}) = \pi(x) = y^{**}$. It follows that $x^{**} - x \in \mathrm{Ker}(\pi^{**}) = Z^{**} = Z$ and hence $x^{**} \in x + Z \subset X$. Now we see that $x^{**} \in \bar{F} \cap X = F$ and hence $y^{**} = \pi^{**}(x^{**}) \in \pi^{**}(F) = \pi(F)$.

Since the quotient homomorphism π maps bounded weakly closed subsets of X to bounded weakly closed sets of X/Z, the image $\pi(\bar{B}_X)$ of the closed unit ball centered at the origin of X coincides with the closed unit ball $\bar{B}_{X/Z}$ centered at the origin of X/Z.

Now we are ready to show that the space X is r-reflexive. Take any weakly open cover $\mathcal{U}$ of X. For every point $y \in (r+1)\bar{B}_{X/Z}$ the set $(r+1)\bar{B}_X \cap \pi^{-1}(y)$ is weakly compact and hence can be covered by a finite subfamily $\mathcal{U}_y \subset \mathcal{U}$. The set $F_y = (r+1)\bar{B}_X \setminus \bigcup \mathcal{U}_y$ is bounded and weakly closed in X. Consequently, its projection $\pi(F_y)$ is weakly closed in X/Z while the complement $V_y = (r+1)\bar{B}_{X/Z} \setminus \pi(F_y)$ is a weakly open neighborhood of y in $(r+1)\bar{B}_{X/Z}$. Since the space X/Z is r-reflexive the cover $\{V_y : y \in (r+1)B_{X/Z}\}$ of the closed ball $(r+1)B_{X/Z}$ contains a finite subcollection $\{V_{y_1}, \dots, V_{y_n}\}$ whose union contains the open 1-ball $y + B_{X/Z}$ centered at some point $y \in X/Z$ with $\|y\| \le r$. Take any point $x \in X$ with $\|x\| = \|y\|$ and $\pi(x) = y$ and observe that $\mathcal{W} = \bigcup_{i=1}^n \mathcal{U}_{y_i}$ is a finite cover of the open 1-ball $x + B_X$ centered at x. This witnesses that the space X is r-reflexive.

5. Proof of Theorem 2

In this section we prove that the James space J fails to be ω-reflexive. We recall that the James space J is the Banach space consisting of all real sequences $(x_n)_{n\in\omega}$ that tend to zero and have norm

$$\|(x_i)\| = \sup_{n_0<\dots<n_k} \sqrt{\sum_{i=1}^{k}(x_{n_i} - x_{n_{i-1}})^2} < \infty.$$

Let J_0 denote the set of all eventually zero sequences.

First we prove

Lemma 7. *For every $M > 0$ there is $\varepsilon > 0$ such that for every $x \in J_0$ with $\|x\| \le M$ there is $y = (y_n) \in J$ such that $\|x - y\| < 1$ and $|y_n - 1| \ge \varepsilon$ for all $n \in \omega$.*

Proof: Given $M > 0$ find an integer $m \ge 2$ with $\frac{20M}{\sqrt{2m+1}} < \frac{1}{2}$ and $4M^2(2m+1) > 1$, and let $\varepsilon = \frac{1}{4m+2}$.

Take any point $x = (x_n) \in J_0$ with $\|x\| \le M$. By induction, construct an increasing finite number sequence $(k_i)_{i=0}^{r}$ such that for $k_{r+1} = \infty$ we get

- $k_0 = 0$;
- $|x_p - x_q| \le \varepsilon$ for all numbers $p, q \in [k_i, k_{i+1})$ and all $0 \le i \le r$;
- for every $0 < i \le r$ there is a number $p_i \in [k_{i-1}, k_i)$ with $|x_{k_i} - x_{p_i}| > \varepsilon$.

It follows that

$$M \ge \|x\| \ge \sqrt{\sum_{0<i\le r} |x_{k_i} - x_{p_i}|^2} > \sqrt{r\varepsilon^2}$$

and hence $r < \frac{M^2}{\varepsilon^2}$. Let $A = 2\varepsilon \cdot \mathbb{Z}$ be the arithmetic progression with step 2ε and let $f : \mathbb{R} \to A$ be a function assigning to each real number $t \in \mathbb{R}$ a number $f(t) \in A$ with $|t - f(t)| \le \varepsilon$. Given a number $a \in A$, let

$$r_a = \big|\{i \le r : f(x_{k_i}) \in \{a - 2\varepsilon, a, a + 2\varepsilon\}\}\big|.$$

Since $|A\cap[\frac{1}{2}, \frac{3}{2}]| = \frac{1}{2\varepsilon} = 2m+1$, there is a point $a \in A\cap[\frac{1}{2}, \frac{3}{2}]$ with $r_a \le \frac{3r}{2m+1} \le \frac{3M^2}{(2m+1)\varepsilon^2} = 12M^2(2m+1)$. Taking into account that $1 < 4M^2(2m+1)$, we get $r_a + 1 \le 16M^2(2m+1)$.

Consider the sequence $z = (z_n)_{n\in\omega}$ such that $z_n = 0$ for $n \in [k_r, \infty)$ and for every $i < r$ and $n \in [k_i, k_{i+1})$ we have

$$z_n = \begin{cases} 0 & \text{if } \ f(x_{k_i}) \notin \{a - 2\varepsilon, a, a + 2\varepsilon\}; \\ -5\varepsilon & \text{if } \ f(x_{k_i}) = a - 2\varepsilon; \\ 5\varepsilon & \text{if } \ f(x_{k_i}) \in \{a, a + 2\varepsilon\}. \end{cases}$$

The definition of the norm on the James space J implies that

$$\begin{aligned} \|z\| \le \sqrt{(r_a + 1)(10\varepsilon)^2} &\le \sqrt{16M^2(2m + 1)100\varepsilon^2} \\ &= \sqrt{\frac{1600M^2}{4(2m + 1)}} = \frac{20M}{\sqrt{2m + 1}} < \frac{1}{2}. \end{aligned}$$

Let $e = (e_n)_{n\in\omega}$ be the element of J such that $e_n = 1$ for all $i < k_r$ and $e_n = 0$ for all $n \ge k_r$. It is clear that $\|e\| = 1$.

Finally, consider the point $y = x + z + (1 - a) \cdot e$. Observe that

$$\|y - x\| = \|z + (1 - a) \cdot e\| \le \|z\| + |1 - a| \cdot \|e\| < \frac{1}{2} + \frac{1}{2} = 1.$$

Now, we show that $|y_n - 1| \ge \varepsilon$ for all $n \in \omega$. Indeed, if $n \ge k_r$, then $y_n = x_n$ and $|y_n - 1| \ge 1 - |x_n| \ge 1 - \varepsilon \ge \varepsilon$.

Next, assume that $n \in [k_i, k_{i+1})$ for some $i < r$. If $f(x_{k_i}) \notin \{a-2\varepsilon, a, a+2\varepsilon\}$, then

$$\begin{aligned} |y_n - 1| &= |x_n + z_n + (1 - a) - 1| = |x_n - a| = |x_n - f(x_{k_i}) + f(x_{k_i}) - a| \\ &\ge |f(x_{k_i}) - a| - |x_{k_i} - f(x_{k_i})| - |x_n - x_{k_i}| \ge 4\varepsilon - \varepsilon - \varepsilon \ge \varepsilon. \end{aligned}$$

If $f(x_{k_i}) = a - 2\varepsilon$, then

$$\begin{aligned} |y_n - 1| &= |x_n + z_n + (1 - a) - 1| = |x_n - x_{k_i} + x_{k_i} + f(x_{k_i}) - f(x_{k_i}) + z_n - a| \\ &\ge |z_n + f(x_{k_i}) - a| - |x_n - x_{k_i}| - |f(x_{k_i}) - x_{k_i}| \ge 3\varepsilon - \varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

The case $f(x_{k_i}) \in \{a, a + 2\varepsilon\}$ can be considered by analogy. □

The following lemma combined with Lemma 6 implies that the James space is not ω-reflexive.

Lemma 8. *For every $R \in \mathbb{N}$ the ball $B_R = \{x \in J : \|x\| \le R\}$ possesses a 2-net in J which is closed and discrete in the weak topology of J.*

Proof: Using Lemma 7, find $\varepsilon > 0$ such that the set

$$A_\varepsilon = \{(y_n) \in J : |y_n - 1| \ge \varepsilon \ \text{ for all } \ n \in \omega\}$$

intersects each open ball of unit radius centered at a point $x \in J_0$ with $\|x\| \le R$. Fix a countable dense set $D = \{x_n : n \in \omega\}$ in $A_\varepsilon \cap B_{R+1}$. It follows that D is a 1-net for the ball B_R. For every $n \in \omega$ consider the sequence $\vec{e}_n = (1, \dots, 1, 0, \dots)$ with first n units. Since $\|e_n\| = 1$ for all $n \in \omega$, the set $D' = \{x_n - e_n : n \in \omega\}$ is a 2-net for the ball B_R in J. We claim that D' is closed and discrete in the weak topology of J. Assuming the converse and using the metrizability of the weak topology of J on bounded subsets, find an increasing number sequence (n_k) such that the sequence $(x_{n_k} - e_{n_k})_{k\in\omega}$ weakly converges to some point $z \in J$. The weak convergence implies the coordinate convergence. Now it is convenient to think of the elements of J as functions defined on ω. It follows that for every $i \in \omega$, $|z(i)| = \lim_{k\to\infty} |x_{n_k}(i) - e_{n_k}(i)| = \lim_{k\to\infty} |x_{n_k}(i) - 1| \ge \varepsilon$, which is not possible because $\lim_{i\to\infty} z(i) = 0$. $\square$

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References

[Ba] Banakh I., *On Banach spaces possessing an ε-net without weak limit points*, Math. Methods and Phys. Mech. Fields **43** (2000), no. 3, 40–43.

[BPZ] Banakh T., Plichko A., Zagorodnyuk A., *Zeros of continuous quadratic functionals on non-separable Banach spaces*, Colloq. Math. **100** (2004), 141–147.

[BFT] Bourgain J., Fremlin D., Talagrand M., *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. **100** (1978), no. 4, 845–886.

[CG] Castillo J., González M., *Three-space problems in Banach space theory*, Lecture Notes in Mathematics, 1667, Springer, Berlin, 1997.

[Dis] Diestel J., *Sequences and Series in Banach Spaces*, Springer, New York, 1984.

[En] Engelking R., *General Topology*, PWN, Warsaw, 1977.

[Fab] Fabian M., *Gâteaux Differentiability of Convex Functions and Topology*, John Wiley & Sons, Inc., New York, 1997.

[HHZ] Habala P., Hájek P., Zizler V., *Introduction to Banach spaces*, Matfyzpress, Praha, 1996.

[OR] Odell E., Rosenthal H.P., *A double-dual characterization of separable Banach spaces containing l^1*, Israel J. Math. **20** (1975), no. 3–4, 375–384.

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