Local monotonicity of Hausdorff measures restricted to curves in $\mathbb{R}^n$

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Abstract. We give a sufficient condition for a curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ to ensure that the 1-dimensional Hausdorff measure restricted to γ is locally monotone.

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1. Introduction

Study of monotone measures is motivated by open problems on existence and regularity of minimal surfaces. For known results, compactness argument is used to achieve existence of a generalized minimal surface (e.g. a stationary varifold) and then the monotonicity formula is used to obtain the tangential regularity and the regularity of the surface, see for example [6].

Definition 1.1. Let μ be a Radon measure on $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that μ is *k-monotone* if the function $r \mapsto \frac{\mu B(z,r)}{r^k}$ is nondecreasing on $(0, \infty)$ for every $z \in \mathbb{R}^n$. Instead of 1-monotone, we simply write *monotone*.

In 1999, Huovinen, Kirchheim, Kolář and De Pauw studied the question concerning the generalization of the famous Allard Theorem, see [1]. It was natural to ask whether there exists a monotone measure with non unique tangential behaviour. Such a measure was given by Kolář in [4]. However, this measure is not minimal surface-like enough to be applied to the question concerning the Allard Theorem.

As another candidate for a suitable measure, there was further considered the 1-dimensional Hausdorff measure restricted to a symmetrical pair of logarithmic spirals. For such measures, it is difficult to check the monotonicity directly from the definition, because of long technical computation even for very small radii. These measures were found not to be monotone, but locally monotone. For future trials to construct a measure denying the generalization of the Allard Theorem, it would be useful to have a simple method to check the local monotonicity.

In recent paper [2], it is shown that if $\gamma : \mathbb{R} \mapsto \mathbb{R}^2$ is a C^2-curve and its curvature is bounded, bounded away from zero and uniformly continuous, then the 1-dimensional Hausdorff measure restricted to γ is locally monotone. In this paper, we want to obtain a similar result for curves in $\mathbb{R}^n$. We prove

Theorem 1.2. *Let $0 < \alpha \le K < \infty$, $t_0 > 0$ and $\omega : [0,\infty) \to [0,\infty)$ be a function satisfying $\omega(t) \le \frac{\alpha}{60}$ on $(0,t_0)$. Then there is $\sigma > 0$ with the following property: If $-\infty < a < b < \infty$ and $\gamma : [a,b] \to \mathbb{R}^n$ is a C^2-curve such that $|\dot\gamma| \equiv 1$, $|\ddot\gamma(s) - \ddot\gamma(t)| \le \omega(|s-t|)$, $s,t \in (a,b)$, with the curvature k_γ satisfying $k_\gamma((a,b)) \subset [\alpha, K]$, then $r \mapsto \frac{\mu_\gamma B(z,r)}{r}$ is nondecreasing on $(0, \min(\sigma, |z-\gamma(a)|, |z-\gamma(b)|))$ for every centre $z \in \mathbb{R}^n$.*

Hence we observe that if the assumptions of Theorem 1.2 are satisfied, then the torsion does not disturb the local monotonicity. This result is used in the proof of the main theorem in [3], where the local monotonicity in the case of real analytic curves is considered.

In the last section, we show some interesting facts about the local k-monotonicity. Our main goal is to prove that the $(n-1)$-dimensional Hausdorff measure restricted to a sphere in $\mathbb{R}^n$ is locally $(n-1)$-monotone for $n=2$ and $n=3$ only.

We refer to [5] and [6] for other information about the geometry of measures and the Monotonicity Formula.

2. Preliminaries

The *scalar product* of $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$ and the Euclidean *norm* of x is denoted by $|x|$. Further

$$B(z,r) = \{x \in \mathbb{R}^n : |x-z| \le r\} \qquad \text{and} \qquad S(z,r) = \{x \in \mathbb{R}^n : |x-z| = r\}.$$

For $h \in \mathbb{R}$, $g \in \mathbb{R}^{n-2}$, $x > 0$ and functions $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}^{n-2}$, we define

$$z_{h,g} = (0,h,g) \in \mathbb{R}^n \qquad \text{and} \qquad r_{h,g}(x) = |(x, f(x), u(x)) - z_{h,g}|.$$

We write $u'(x)$ instead of $(u_1'(x), \dots, u_{n-2}'(x))$, similarly for $u''(x)$.

We say that a Radon measure μ is *monotone at* (z,r) if $\underline{\mathrm{D}}_{\mathrm{r}} \frac{\mu B(z,r)}{r} \ge 0$, where $\underline{\mathrm{D}}_{\mathrm{r}} f(r) = \liminf\limits_{\delta \to 0} \frac{f(r+\delta) - f(r)}{\delta}$.

Some notes on curves in $\mathbb{R}^n$. Let $I \subset \mathbb{R}$ be an open interval and let $\gamma : I \to \mathbb{R}^n$ be a regular C^2-curve. We denote $\dot\gamma(t) = \left(\frac{\partial \gamma_1(t)}{\partial t}, \dots, \frac{\partial \gamma_n(t)}{\partial t}\right)$ and $\ddot\gamma(t) = \left(\frac{\partial^2 \gamma_1(t)}{\partial t^2}, \dots, \frac{\partial^2 \gamma_n(t)}{\partial t^2}\right)$, $t \in I$. Further, we suppose $|\dot\gamma| \equiv 1$ on I, which is obtained, for a regular curve, after a change of parameterization. In this case, the *curvature* is defined by $k_\gamma(t) = |\ddot\gamma(t)|$.

For a C^1-curve $\gamma : [a,b] \to \mathbb{R}^n$, $-\infty < a < b < \infty$ and for a Borel set A, we define

$$\mu_\gamma(A) = \int_{\{t \in (a,b) : \gamma(t) \in A\}} |\dot\gamma(t)| \, dt.$$

3. Local monotonicity

We prove Theorem 1.2 showing that μ_γ is monotone at (z,r) for every centre $z \in B(\sigma)$ and for every radius $r \in (0, \min(\sigma, |z-\gamma(a)|, |z-\gamma(b)|))$. As every curve in $\mathbb{R}^2$ can be considered as a curve in $\mathbb{R}^3$ with the third coordinate equal to zero, let us suppose $n \geq 3$ in the sequel. The proof is based on the following proposition, where we are interested in the curves of the type $\gamma(x) = (x, f(x), u(x))$.

Proposition 3.1. *Assume $\varepsilon, \delta \in (0, \frac{1}{20}]$ and let the functions $f \in C^2((-\delta,\delta),\mathbb{R})$ and $u \in C^2((-\delta,\delta),\mathbb{R}^{n-2})$ satisfy*

$$f(0) = f'(0) = |u(0)| = |u'(0)| = 0,$$

further

$$|f''(x) - 1| \leq \varepsilon \quad \text{and} \quad |u''(x)| \leq \varepsilon \quad \text{on} \quad (-\delta,\delta)$$

and $\gamma(x) = (x, f(x), u(x))$. Then $r \mapsto \frac{\mu_\gamma B(z_{h,g},r)}{r}$ is nondecreasing on $(0,\delta)$ for every $h \in \mathbb{R}$ and every $g \in \mathbb{R}^{n-2}$.

In the following, let f, u and $\varepsilon, \delta \in (0, \frac{1}{20}]$ satisfy the assumptions of Proposition 3.1. Since

$$\mu_{(\cdot,f(\cdot),u(\cdot))} B(z_{h,g},r) = \tfrac{1}{2}\Big(\mu_{(\cdot,f(|\cdot|),u(|\cdot|))} B(z_{h,g},r) + \mu_{(\cdot,f(-|\cdot|),u(-|\cdot|))} B(z_{h,g},r)\Big)$$

and trivially

$$\mu_\gamma B(z_{h,g},r) = \mu_{\gamma|_{\{t\in\mathbb{R}:\gamma(t)\in B(z_{h,g},r)\}}} B(z_{h,g},r)$$

provided $h \in \mathbb{R}$, $g \in \mathbb{R}^{n-2}$, $r > 0$, we suppose, without loss of generality, that f and each component of u are even functions satisfying $1-\varepsilon \leq f''(x) \leq 1+\varepsilon$, $|u''(x)| \leq \varepsilon$ on $\mathbb{R}$. Hence

$$\begin{aligned} &f'(x) \in [(1-\varepsilon)x, (1+\varepsilon)x], \qquad f(x) \in \Big[\frac{(1-\varepsilon)}{2}x^2, \frac{(1+\varepsilon)}{2}x^2\Big], \\ &|u'(x)| \leq \varepsilon x, \qquad |u(x)| \leq \frac{\varepsilon}{2}x^2. \end{aligned} \tag{1}$$

To simplify our notation, we define auxiliary vectors

$$\begin{aligned} a_x &= (1, f'(x), u'(x)), \\ b_x &= (x, f(x) - h, u(x) - g), \\ c_x &= (x, 0, \dots, 0). \end{aligned}$$

Further, let $x \in (0,\delta)$ and P_x be the orthogonal projection to the two-dimensional subspace of $\mathbb{R}^n$ generated by a_x and c_x. Now, let $\varphi(x)$ be the angle between c_x

and $P_x b_x$, and let $\eta(x)$ be the angle between c_x and a_x. We observe that if $u, v, w \in \mathbb{R}^n \setminus \{(0,\dots,0)\}$ and Pw is the orthogonal projection of w to the two-dimensional subspace of $\mathbb{R}^n$ generated by u and v, then $u \cdot Pw = u \cdot w$. (to see this, it is enough to rotate the coordinates so that $u = (a, 0, \dots, 0)), a \neq 0$ and $v = (b, c, 0, \dots, 0)), (b, c) \neq (0, 0)$.) Therefore we have

$$\begin{aligned} \cos(\varphi(x)) &= \frac{c_x \cdot P_x b_x}{|c_x||P_x b_x|} = \frac{c_x \cdot b_x}{|c_x||P_x b_x|} \geq \frac{c_x \cdot b_x}{|c_x||b_x|}\,, \\ \cos(\eta(x) - \varphi(x)) &= \frac{a_x \cdot P_x b_x}{|a_x||P_x b_x|} = \frac{a_x \cdot b_x}{|a_x||P_x b_x|} \geq \frac{a_x \cdot b_x}{|a_x||b_x|}\,. \end{aligned} \tag{2}$$

Lemma 3.2. *Assume $x \in [0, \delta)$, $|h| \leq \frac{1}{2}$ and $|g| \leq \frac{1}{2}$. Then*

$$S(z_{h,g}, r_{h,g}(x)) \cap \operatorname{spt} \mu_\gamma = \{(x, f(x), u(x)), (-x, f(x), u(x))\}.$$

Moreover, if $x \in (0, \delta)$, then $\frac{\partial r_{h,g}(t)}{\partial t}|_{t=x} \in (0, \infty)$ and

$$\mu_\gamma B(z_{h,g}, r_{h,g}(x)) = 2 \int_0^x \sqrt{1 + f'^2(t) + |u'|^2(t)}\, dt, \tag{3}$$

$$\frac{\partial \mu_\gamma B(z_{h,g}, r_{h,g}(x))}{\partial x} = 2\sqrt{1 + f'^2(x) + |u'|^2(x)} = 2|a_x|, \tag{4}$$

$$\frac{\partial \mu_\gamma B(z_{h,g}, r)}{\partial r} r \Big|_{r = r_{h,g}(x)} \geq \frac{4x}{1 + \cos(\eta(x))}\,. \tag{5}$$

Proof:

$$S(z_{h,g}, r_{h,g}(x)) \cap \operatorname{spt} \mu_\gamma \supset \{(x, f(x), u(x)), (-x, f(x), u(x))\}$$

is trivially satisfied. Conversely, as

$$r_{h,g}(t) = \sqrt{t^2 + (f(t) - h)^2 + |u(t) - g|^2},$$

using (1) we get

$$\begin{aligned} \frac{\partial r_{h,g}(t)}{\partial t} &= \frac{t + (f(t) - h) f'(t) + (u(t) - g) \cdot u'(t)}{r_{h,g}(t)} \\ &\geq \frac{t + \frac{(1-\varepsilon)}{2} t^2 (1 - \varepsilon) t - \frac{1}{2}(1 + \varepsilon) t - (\frac{\varepsilon}{2} t^2 + \frac{1}{2}) \varepsilon t}{r_{h,g}(t)} > 0 \end{aligned}$$

on $(0, \infty)$. Hence, the continuous function $t \mapsto r_{h,g}(t)$ is increasing on $[0, \infty)$, thus

$$S(z_{h,g}, r_{h,g}(x)) \cap \operatorname{spt} \mu_\gamma = \{(x, f(x), u(x)), (-x, f(x), u(x))\}.$$

From which we obtain (3). Further (4) follows from (3).

Finally, let

$$F(x,r) = x^2 + (f(x) - h)^2 + |u(x) - g|^2 - r^2.$$

Then $\frac{\partial F}{\partial x} = 2a_x \cdot b_x$ and $\frac{\partial F}{\partial r} = -2r$. The Implicit Function Theorem, the identities $\frac{x|c_x|}{b_x \cdot c_x} = \frac{x^2}{x^2} = 1$, $r_{h,g}(x) = |b_x|$, $2\cos\alpha\,\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$, (4) and (2) imply

$$\begin{aligned}
&\frac{\partial \mu_\gamma B(z_{h,g}, r)}{\partial r} r\Big|_{r=r_{h,g}(x)} \\
&\quad= \frac{\partial \mu_\gamma B(z_{h,g}, r_{h,g}(x))}{\partial x}\frac{\partial x}{\partial r} r\Big|_{r=r_{h,g}(x)} = 2|a_x|(-1)\frac{\frac{\partial F}{\partial r}(x,r)}{\frac{\partial F}{\partial x}(x,r)} r\Big|_{r=r_{h,g}(x)} \\
&\quad= 2|a_x|\frac{2r_{h,g}(x)}{2a_x \cdot b_x} r_{h,g}(x) = 2|a_x|\frac{|b_x|}{a_x \cdot b_x}|b_x|\frac{x|c_x|}{b_x \cdot c_x} = 2\frac{|a_x||b_x|}{a_x \cdot b_x}\frac{|b_x||c_x|}{b_x \cdot c_x} x \\
&\quad\geq \frac{2x}{\cos(\eta(x) - \varphi(x))\cos(\varphi(x))} = \frac{4x}{\cos(\eta(x) - 2\varphi(x)) + \cos(\eta(x))} \\
&\quad\geq \frac{4x}{1 + \cos(\eta(x))}.
\end{aligned}$$

$\square$

Proof of Proposition 3.1: Suppose $\varepsilon, \delta \in (0, \frac{1}{20}]$, $h \in \mathbb{R}$, $g \in \mathbb{R}^{n-2}$ and $r \in (0, \delta)$. If $|h| \geq \frac{1}{2}$ or $|g| \geq \frac{1}{2}$, then for every $x \in (0, \delta)$, we have by (1)

$$\begin{aligned}
r_{h,g}(x) &= \sqrt{x^2 + (h - f(x))^2 + |g - u(x)|^2} \geq \max(|h - f(x)|, |g - u(x)|) \\
&\geq \tfrac{1}{2} - \max(|f(x)|, |u(x)|) \geq \tfrac{1}{2} - \tfrac{1+\varepsilon}{2}\delta^2 > \tfrac{1}{2} - \tfrac{1+\frac{1}{20}}{2}\tfrac{1}{20^2} > \tfrac{1}{20} \geq \delta > r.
\end{aligned}$$

Hence $B(z_{h,g}, r) \cap \operatorname{spt}\mu_\gamma = \emptyset$ and therefore for any $r_1 \in (0, r)$ and $r_2 > r$ we have

$$\frac{\mu_\gamma B(z_{h,g}, r_1)}{r_1} \leq \frac{\mu_\gamma B(z_{h,g}, r)}{r} \leq \frac{\mu_\gamma B(z_{h,g}, r_2)}{r_2}$$

and thus μ_γ is monotone at $(z_{h,g}, r)$. Similarly, if $|h| \leq \frac{1}{2}$, $|g| \leq \frac{1}{2}$ and $S(z_{h,g}, r) \cap \operatorname{spt}\mu_\gamma \subset \{(0, \ldots, 0)\}$, then $\mu_\gamma B(z_{h,g}, r) = 0$ and we are done again.

In the remaining case, we have $|h| \leq \frac{1}{2}$, $|g| \leq \frac{1}{2}$ and there is $x \in (0, r] \subset (0, \delta)$ such that $r = r_{h,g}(x)$. Hence we can use Lemma 3.2. From (1) and $x, \varepsilon \in (0, \frac{1}{20}]$ we obtain

$$(6)\quad (1-\varepsilon)^2 x^3 - \frac{1}{4}((1+\varepsilon)^2 + \varepsilon^2)^2 x^5 \geq \tfrac{(1+\varepsilon)^2+\varepsilon^2}{3} x^3\Big(\sqrt{1 + f'^2(x) + |u'|^2(x)} + 1\Big).$$

Now, using the estimate $1+\frac{t}{2}-\frac{t^2}{8}\le\sqrt{1+t}$ for $t\ge 0$ and estimates (1) we obtain

$$\begin{aligned}
&4x\sqrt{1+f'^2(x)+|u'|^2(x)}\\
&\quad\ge 2x\sqrt{1+f'^2(x)+|u'|^2(x)}+2x\Big(1+\frac{1}{2}f'^2(x)-\frac{1}{8}\Big(f'^2(x)+|u'|^2(x)\Big)^2\Big)\\
&\quad\ge 2x\sqrt{1+f'^2(x)+|u'|^2(x)}+2x+(1-\varepsilon)^2x^3-\frac{1}{4}((1+\varepsilon)^2+\varepsilon^2)^2x^5.
\end{aligned}$$

Further by (6), (1) and $\sqrt{1+t}\le 1+\frac{t}{2}$ for $t\ge 0$ we have

$$\begin{aligned}
&4x\sqrt{1+f'^2(x)+|u'|^2(x)}\\
&\quad\ge 2x\sqrt{1+f'^2(x)+|u'|^2(x)}+2x+\tfrac{(1+\varepsilon)^2+\varepsilon^2}{3}x^3\Big(\sqrt{1+f'^2(x)+|u'|^2(x)}+1\Big)\\
&\quad= 2(x+\tfrac{(1+\varepsilon)^2}{6}x^3+\tfrac{\varepsilon^2}{6}x^3)\Big(\sqrt{1+f'^2(x)+|u'|^2(x)}+1\Big)\\
&\quad\ge 2\int_0^x 1+\frac{1}{2}f'^2(t)+\frac{1}{2}|u'|^2(t)\,dt\ \Big(\sqrt{1+f'^2(x)+|u'|^2(x)}+1\Big)\\
&\quad\ge 2\int_0^x \sqrt{1+f'^2(t)+|u'|^2(t)}\,dt\ \Big(\sqrt{1+f'^2(x)+|u'|^2(x)}+1\Big).
\end{aligned}$$

And thus, (3), (5) and $\cos(\eta(x))=\frac{c_x\cdot a_x}{|c_x||a_x|}=\frac{1}{\sqrt{1+f'^2(x)+|u'|^2(x)}}$ imply

$$\begin{aligned}
&\frac{\partial}{\partial r}\frac{\mu_\gamma B(z_{h,g},r)}{r}\Big|_{r=r_{h,g}(x)}\\
&\quad=\frac{1}{r^2}\Big(\frac{\partial\mu_\gamma B(z_{h,g},r)}{\partial r}r-\mu_\gamma B(z_{h,g},r)\Big)\Big|_{r=r_{h,g}(x)}\\
&\quad\ge\frac{1}{r^2}\Big(\frac{4x}{1+\cos(\eta(x))}-\mu_\gamma B(z_{h,g},r)\Big)\Big|_{r=r_{h,g}(x)}\\
&\quad=\frac{1}{r_{h,g}(x)^2}\left(\frac{4x\sqrt{1+f'^2(x)+|u'|^2(x)}}{\sqrt{1+f'^2(x)+|u'|^2(x)}+1}-2\int_0^x\sqrt{1+f'^2(t)+|u'|^2(t)}\,dt\right)\ge 0.
\end{aligned}$$

Therefore, μ_γ is monotone at $(z_{h,g},r_{h,g}(x))$. □

As every regular C^2-curve in $\mathbb{R}^n$ is locally a graph of a C^2-function from $\mathbb{R}$ to $\mathbb{R}^n$, up to a rotation, Theorem 1.2 follows from Proposition 3.1 after suitable rescaling of the coordinates. The rest of this section is devoted to its detailed proof. In the following lemma, our goal is to show that the assumptions concerning $\ddot\gamma$ and k_γ in Theorem 1.2 imply assumptions concerning f,f',f'',u,u',u'' in Proposition 3.1.

Lemma 3.3. *Assume $\varepsilon \in (0, \frac{1}{20}]$, $K \in (0,\infty)$ and $\tau \in (0, \frac{\varepsilon}{20}]$. Let $I \subset (-\frac{\tau}{K}, \frac{\tau}{K})$ be a closed interval such that $0 \in I$ and let $\gamma : I \to \mathbb{R}^n$ be a C^2-curve satisfying*

$$\gamma(0) = (0,\dots,0), \quad \dot\gamma(0) = (1,0,\dots,0), \quad \gamma_1''(0) = 0, \quad \gamma_3''(0) = \dots = \gamma_n''(0) = 0,$$
$$|\dot\gamma| \equiv 1, \quad k_\gamma(I) \subset [0,K] \quad \text{and} \quad |\ddot\gamma(t) - \ddot\gamma(0)| \le \frac{\varepsilon}{3} k_\gamma(0) \quad \text{on } I.$$

Then

$$\tilde f(x) = \gamma_2(\gamma_1^{-1}(x)) \qquad \text{and} \qquad \tilde u(x) = (\gamma_3(\gamma_1^{-1}(x)),\dots,\gamma_n(\gamma_1^{-1}(x)))$$

are C^2-functions defined on $\gamma_1(I)$. These functions satisfy

$$\tilde f(0) = \tilde f'(0) = |\tilde u(0)| = |\tilde u'(0)| = 0$$

and

$$|\tilde f''(x) - \tilde f''(0)| \le \varepsilon k_\gamma(0) = \varepsilon \tilde f''(0), \quad |\tilde u''(x)| \le \varepsilon k_\gamma(0) \quad \text{on } \gamma_1(I).$$

Proof: First, since $I \subset (-\frac{\tau}{K}, \frac{\tau}{K})$, for $t \in I$, we have

$$\begin{aligned}\gamma_1'(t) &\ge \gamma_1'(0) - \Big|\int_0^t |\gamma_1''(s)|\,ds\Big| \ge 1 - \Big|\int_0^t |\ddot\gamma(s)|\,ds\Big| = 1 - \Big|\int_0^t |k_\gamma(s)|\,ds\Big| \\ &\ge 1 - K|t| > 1-\tau.\end{aligned} \tag{7}$$

As $\tau < 1$, $\tilde f(x) = \gamma_2(\gamma_1^{-1}(x))$, $\tilde u(x) = (\gamma_3(\gamma_1^{-1}(x)),\dots,\gamma_n(\gamma_1^{-1}(x)))$ are well defined on $\gamma_1(I)$. Further, it can be shown that $\tilde f$ and $\tilde u$ are C^2-functions, by the Implicit Function Theorem. Moreover we obviously have

$$\tilde f(0) = \tilde f'(0) = |\tilde u(0)| = |\tilde u'(0)| = 0.$$

Using (7), $\frac{1}{(1-\tau)^2} - 1 \le 3\tau$ and the assumptions on $\dot\gamma$ and $\ddot\gamma$ we obtain for $x \in \gamma_1(I)$

$$\begin{aligned}&\Big|\frac{\partial^2}{\partial s^2}\gamma(\gamma_1^{-1}(s))\big|_{s=x} - \frac{\partial^2}{\partial s^2}\gamma(\gamma_1^{-1}(s))\big|_{s=0}\Big| \\ &\qquad = \Big|\ddot\gamma(\gamma_1^{-1}(x))\frac{1}{\gamma_1'^{\,2}(\gamma_1^{-1}(x))} - \dot\gamma(\gamma_1^{-1}(x))\frac{\gamma_1''(\gamma_1^{-1}(x))}{\gamma_1'^{\,3}(\gamma_1^{-1}(x))} - \ddot\gamma(\gamma_1^{-1}(0))\Big| \\ &\qquad \le |\ddot\gamma(\gamma_1^{-1}(x)) - \ddot\gamma(\gamma_1^{-1}(0))| + |\ddot\gamma(\gamma_1^{-1}(x))|\Big|1 - \frac{1}{\gamma_1'^{\,2}(\gamma_1^{-1}(x))}\Big| \\ &\qquad\qquad + \frac{|\dot\gamma(\gamma_1^{-1}(x))||\gamma_1''(\gamma_1^{-1}(x))|}{|\gamma_1'^{\,3}(\gamma_1^{-1}(x))|} \\ &\qquad \le \tfrac{\varepsilon}{3}k_\gamma(0) + (1+\tfrac{\varepsilon}{3})k_\gamma(0)\left(\frac{1}{(1-\tau)^2} - 1\right) + \frac{\frac{\varepsilon}{3}k_\gamma(0)}{(1-\tau)^3} \\ &\qquad \le \tfrac{\varepsilon}{3}k_\gamma(0) + (1+\tfrac{1}{60})k_\gamma(0)3\tau + \tfrac{101}{100}\tfrac{\varepsilon}{3}k_\gamma(0) \le \varepsilon k_\gamma(0).\end{aligned}$$

Since $\tilde{f}(x)$ and $\tilde{u}(x)$ are defined as components of $\gamma(\gamma_1^{-1}(x))$ the estimates concerning $\tilde{f}''$ and $\tilde{u}''$ follow from the above estimate. $\square$

Proof of Theorem 1.2: Let us set $\varepsilon = \frac{1}{20}$ and $\tau = \min(\frac{\varepsilon}{20}, Kt_0)$. Hence the assumption $\omega(t) \le \frac{\alpha}{60}$ on $(0, t_0)$ implies $\omega(t) \le \frac{\varepsilon\alpha}{3}$ on $(0, \frac{\tau}{K})$. We set $\tilde{\sigma} = \frac{\tau}{K}$. By the geometrical meaning of the curvature, there is $\varrho_0 = \varrho_0(K) > 0$ such that for any C^2-curve $\gamma\colon [a,b] \mapsto \mathbb{R}^n$ satisfying $k_\gamma(x) \le K$ on $[a,b]$ and any $z \in \mathbb{R}^n$, there are

$$a \le a_1 \le b_1 < a_2 \le b_2 < \cdots < a_m \le b_m \le b$$

such that $|b_i - a_i| < \frac{\tau}{K}$,

$$\{t \in [a,b]\colon \gamma(t) \in B(z, \varrho_0)\} = \bigcup_{i=1}^{m} [a_i, b_i],$$

and for any $\varrho \in (0, \varrho_0]$

$$\{t \in [a,b]\colon \gamma(t) \in B(z, \varrho)\} = \bigcup_{i=1}^{m} I_i,$$

where either $I_i = [\tilde{a}_i, \tilde{b}_i] \subset [a_i, b_i]$ or $I_i = \emptyset$ is satisfied for every $i = 1, \dots, m$. We set $\sigma = \min(\tilde{\sigma}, \varrho_0)$. Let us prove that this is the estimate of the radius we demand.

Let γ be a curve satisfying the assumptions of Theorem 1.2 and $z \in \mathbb{R}^n$. We set $\varrho = \min(\sigma, |z - \gamma(a)|, |z - \gamma(b)|)$.

If $\{\gamma(t) : t \in [a,b]\} \cap B(z, \varrho) = \emptyset$, then $r \mapsto \frac{1}{r}\mu_\gamma B(z, r)$ is trivially nondecreasing on $(0, \varrho]$.

Otherwise

$$\{t \in [a,b] : \gamma(t) \in B(z, \varrho)\} = \bigcup_{i=1}^{m} I_i.$$

For fixed $i \in \{1, \dots, m\}$ such that $I_i = [\tilde{a}_i, \tilde{b}_i] \ne \emptyset$, let us find $t_i \in [\tilde{a}_i, \tilde{b}_i]$ satisfying

$$|\gamma(t_i) - z| = \operatorname{dist}\big(z, \{\gamma(t) : t \in [\tilde{a}_i, \tilde{b}_i]\}\big).$$

As $|\dot{\gamma}(t)| \equiv 1$ implies $0 = \frac{d}{dt}(\dot{\gamma}(t) \cdot \dot{\gamma}(t)) = 2\dot{\gamma}(t) \cdot \ddot{\gamma}(t)$ and thus $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are orthogonal, we can shift and rotate the coordinates so that

$$t_i = 0, \gamma(t_i) = (0, \dots, 0), \dot{\gamma}(t_i) = (1, 0, \dots, 0), \gamma_1''(t_i) = \gamma_3''(t_i) = \cdots = \gamma_n''(t_i) = 0. \tag{8}$$

Otherwise we rotate and shift the coordinates. There are $h \in \mathbb{R}$ and $g \in \mathbb{R}^{n-2}$ such that $z = z_{h,g}$. Since we have (8),

$$[\tilde{a}_i, \tilde{b}_i] \subset \left(t_i - \frac{\tau}{K}, t_i + \frac{\tau}{K}\right) = \left(-\frac{\tau}{K}, \frac{\tau}{K}\right)$$

and $\omega(t) \le \frac{\varepsilon\alpha}{3} \le \frac{\varepsilon}{3} k_\gamma(t_i)$ for $t \in [\tilde{a}_i, \tilde{b}_i]$, we can use Lemma 3.3 and we obtain C^2-functions $\tilde{f}$ and $\tilde{u}$ satisfying

$$\tilde{f}(0) = \tilde{f}'(0) = |\tilde{u}(0)| = |\tilde{u}'(0)| = 0,$$
$$|\tilde{f}''(x) - \tilde{f}''(0)| \le \varepsilon \tilde{f}''(0) \qquad \text{and} \qquad |\tilde{u}''(x)| \le \varepsilon \tilde{f}''(0)$$

such that $(x, \tilde{f}(x), \tilde{u}(x))$ parameterizes the set $\{\gamma(t) : t \in [\tilde{a}_i, \tilde{b}_i]\}$.

Let $\beta = \frac{1}{\tilde{f}''(0)}$, $f(x) = \frac{1}{\beta}\tilde{f}(\beta x)$ and $u(x) = \frac{1}{\beta}\tilde{u}(\beta x)$. This is a suitable rescaling of coordinates, because we have $f''(0) = 1$ and thus we can use Proposition 3.1. Moreover, observing that the rescaling of coordinates does not change the ratio $\frac{\mu_{\gamma|_{[\tilde{a}_i,\tilde{b}_i]}} B(z,r)}{r}$, we obtain $r \mapsto \frac{1}{r}\mu_{\gamma|_{[\tilde{a}_i,\tilde{b}_i]}} B(z,r)$ is nondecreasing on $(0, \varrho)$. Therefore the function $r \mapsto \frac{1}{r}\mu_\gamma B(z,r) = \sum_{i=1}^m \frac{1}{r}\mu_{\gamma|_{[\tilde{a}_i,\tilde{b}_i]}} B(z,r)$ is also nondecreasing on $(0, \varrho)$. □

Remark 3.4. The estimate of the maximal radius θ given by Proposition 3.1 was influenced by assumptions $|f''(x) - 1| \le \varepsilon$, $|u''(x)| \le \varepsilon$, with $\varepsilon \in [0, \frac{1}{20}]$, and the fact that we always considered the worst possible case when dealing with $f(x)$, $f'(x)$, $f''(x)$, $u(x)$, $u'(x)$ and $u''(x)$. Therefore the computations for a particular function usually give better estimate of the maximal radius.

4. Local k-monotonicity

Throughout this section the local k-monotonicity has the following meaning.

Definition 4.1. Let μ be a Radon measure on $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that μ is *locally k-monotone* if there is $r_0 > 0$ such that the function $r \mapsto \frac{\mu B(z,r)}{r^k}$ is nondecreasing on $(0, r_0)$ for every $z \in \mathbb{R}^n$.

Positive results in this paper and in [C] are motivated by a well known fact that the 1-dimensional Hausdorff measure restricted to a circle in $\mathbb{R}^2$ is locally 1-monotone. Let us prove

Proposition 4.2. *The $(n-1)$-dimensional Hausdorff measure restricted to a sphere in $\mathbb{R}^n$ is locally $(n-1)$-monotone for $n = 2$ and $n = 3$, but is not locally $(n-1)$-monotone for any $n > 3$.*

Proof: Because of the symmetry of a sphere and the fact that for the restricted $(n-1)$-dimensional Hausdorff measure the ratio $\frac{\mu B(z,r)}{r^{n-1}}$ is rescaling invariant it is enough to consider the unit sphere centered at $(0, \dots, 0, 1)$ and test balls centred at $z_h = (0, \dots, 0, h)$, for $h \in \mathbb{R}$ sufficiently close to the origin, with very small radii. Suppose $\mu B(z_h, r_0) \ne 0$, otherwise we trivially have $\underline{\mathrm{D}}_\mathrm{r} \frac{\mu B(z,r)}{r}\big|_{r=r_0} \ge 0$. In our case there is $x > 0$ such that $(x, 0, \dots, 0, f(x))$, where $f(x) = 1 - \sqrt{1 - x^2}$, is a point of intersection and $r_0 = r_h(x) = |(x, 0, \dots, 0, f(x)) - z_h|$. Hence our measure μ satisfies

$$\begin{aligned}\mu B(z_h, r_h(x)) &= \int_0^x \alpha_{n-2} t^{n-2} \sqrt{1 + f'^2(t)}\, dt = \int_0^x \alpha_{n-2} t^{n-2} \sqrt{1 + \tfrac{t^2}{1-t^2}}\, dt \\ &= \int_0^x \alpha_{n-2} t^{n-2} \sqrt{\tfrac{1}{1-t^2}}\, dt,\end{aligned} \tag{9}$$

where α_{n-2} is the surface of the unit sphere in $\mathbb{R}^{n-1}$. Therefore we have

$$\frac{\partial \mu B(z_h, r_h(x))}{\partial x} = \alpha_{n-2} x^{n-2} \sqrt{1 + f'^2(x)}. \tag{10}$$

Let $\eta(x)$ be the angle between the vectors $(x, 0, \dots, 0, 0)$ and $(1, 0, \dots, 0, f'(x))$ and $\varphi(x)$ be the angle between $(x, 0, \dots, 0, 0)$ and $(x, 0, \dots, 0, f(x) - h)$. Similarly as in the proof of Lemma 3.2 we obtain from (10)

$$\begin{aligned}&\frac{\partial \mu B(z_h, r)}{\partial r} r\Big|_{r=r_h(x)} = \frac{\partial \mu B(z_h, r_h(x))}{\partial x} \frac{\partial x}{\partial r} r\Big|_{r=r_h(x)} \\ &= \alpha_{n-2} x^{n-2} \frac{2x}{\cos(\eta(x) - 2\varphi(x)) + \cos(\eta(x))} \\ &\geq \alpha_{n-2} x^{n-2} \frac{2x}{1 + \cos(\eta(x))} = 2\alpha_{n-2} x^{n-1} \frac{1}{1 + \frac{1}{\sqrt{1+f'^2(x)}}} \\ &= 2\alpha_{n-2} x^{n-1} \frac{1}{1 + \frac{1}{\sqrt{1+\frac{x^2}{1-x^2}}}} = 2\alpha_{n-2} x^{n-1} \frac{1}{1 + \sqrt{1 - x^2}}.\end{aligned} \tag{11}$$

If $n = 2$, then from (9) and (11) we obtain for $x > 0$ small enough

$$\begin{aligned}\frac{\partial}{\partial r} \frac{\mu B(z_h, r)}{r}\Big|_{r=r_h(x)} &= \frac{1}{r^2}\Big(\frac{\partial \mu B(z_h, r)}{\partial r} r - \mu B(z_h, r)\Big)\Big|_{r=r_h(x)} \\ &\geq \frac{\alpha_0}{r_h^2(x)}\Big(\frac{2x}{1 + \sqrt{1 - x^2}} - \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt\Big) \\ &= \frac{\alpha_0}{r_h^2(x)}\Big(x \frac{1}{1 + \frac{\sqrt{1-x^2}-1}{2}} - \arcsin(x)\Big) \\ &= \frac{2\alpha_1}{r_h^2(x)}\Big(x\Big(1 + \frac{x^2}{4} + \mathrm{O}(x^4)\Big) - \Big(x + \frac{x^3}{6} + \mathrm{O}(x^5)\Big)\Big) \geq 0.\end{aligned}$$

Therefore $r \mapsto \frac{\mu B(z_h, r)}{r^2}$ is nondecreasing on $(0, r_1)$ for some $r_1 > 0$.

In case $n = 3$, from (9) and (11) we obtain

$$\begin{aligned}
\frac{\partial}{\partial r}\frac{\mu B(z_h, r)}{r^2}\Big|_{r=r_h(x)} &= \frac{1}{r^3}\Big(\frac{\partial \mu B(z_h, r)}{\partial r} r - 2\mu B(z_h, r)\Big)\Big|_{r=r_h(x)} \\
&\geq \frac{2\alpha_1}{r_h^3(x)}\Big(\frac{x^2}{1+\sqrt{1-x^2}} - \int_0^x \frac{t}{\sqrt{1-t^2}}\,dt\Big) \\
&= \frac{2\alpha_1}{r_h^3(x)}\Big(\frac{x^2}{1+\sqrt{1-x^2}} - (-\sqrt{1-x^2}+\sqrt{1})\Big) \\
&= \frac{2\alpha_1}{r_h^3(x)}\Big(\frac{x^2}{1+\sqrt{1-x^2}} - \frac{1-(1-x^2)}{1+\sqrt{1-x^2}}\Big) = 0.
\end{aligned}$$

And thus $r \mapsto \frac{\mu B(z_h, r)}{r^2}$ is nondecreasing on $(0, r_1)$ for some $r_1 > 0$.

In case $n > 3$, let us find $h = h(x)$ such that $\eta(x) = 2\varphi(x)$. For this h, the only inequality in (11) turns to equality. Further we observe

$$2x^{n-1}\frac{1}{1+\sqrt{1-x^2}} = x^{n-1}\frac{1}{1+\frac{\sqrt{1-x^2}-1}{2}} = x^{n-1}\Big(1+\frac{x^2}{4}+\mathrm{O}(x^4)\Big), \tag{12}$$

$$\begin{aligned}
(n-1)\int_0^x t^{n-2}\sqrt{\frac{1}{1-t^2}}\,dt &= (n-1)\int_0^x t^{n-2}\sqrt{1+t^2+\mathrm{O}(t^4)}\,dt \\
&= (n-1)\int_0^x t^{n-2}\Big(1+\frac{t^2}{2}+\mathrm{O}(t^4)\Big)\,dt \\
&= x^{n-1} + \frac{n-1}{2(n+1)}x^{n+1} + \mathrm{O}(x^{n+3}).
\end{aligned} \tag{13}$$

Since $\frac{1}{2} < \frac{n-1}{2(n+1)}$ for $n > 3$, if $x > 0$ is small enough, then from (9), (11), (12) and (13) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial r}\frac{\mu B(z_h, r)}{r^{n-1}}\Big|_{r=r_h(x)} \\
&\quad= \frac{1}{r^n}\Big(\frac{\partial \mu B(z_h, r)}{\partial r} r - (n-1)\mu B(z_h, r)\Big)\Big|_{r=r_h(x)} \\
&\quad= \frac{\alpha_{n-2}}{r_h^n(x)}\Big(\frac{2x^{n-1}}{1+\sqrt{1-x^2}} - (n-1)\int_0^x t^{n-2}\sqrt{\frac{1}{1-t^2}}\,dt\Big) < 0.
\end{aligned}$$

Therefore the measure μ cannot be locally $(n-1)$-monotone. $\square$

Remark 4.3. The 2-dimensional Hausdorff measure restricted to the graph of the function $|x|^2$, $x = (x_1, x_2)$, in $\mathbb{R}^3$ is not locally 2-monotone (even though the graph of $|x|^2$ and a sphere in $\mathbb{R}^3$ are very similar in the sense of curvature).

Proof: We use similar computation to the one from the proof of Proposition 4.2 and the result follows from

$$\frac{x^2\sqrt{1+4x^2}}{1+\sqrt{1+4x^2}} < \frac{1}{12}\Big((1+4x^2)^{\frac{3}{2}} - 1\Big) = \int_0^x t\sqrt{1+4t^2}\,dt,$$

which is satisfied for $x > 0$ small enough. □

More generally, we can show the same way that the $(n-1)$-dimensional Hausdorff measure restricted to the graph of the function $|x|^p$, $p \geq 1$, in $\mathbb{R}^n$ is not locally $(n-1)$-monotone for $p \leq \frac{n+1}{2}$. In [3], it is shown for $n = 2$ that if $p > \frac{n+1}{2}$, then the restricted measure is locally 1-monotone. Even in such a small dimension, the computations become very complicated when we consider the test ball centres with a non-zero first coordinate. For $n \geq 3$, the question concerning the sufficient condition on p is open. We guess that $p > \frac{n+1}{2}$ is still a sufficient condition for the local $(n-1)$-monotonicity in the higher dimension, because for the test ball centres with the first $(n-1)$ coordinates equal to zero similar computation as above gives $r \mapsto \frac{\mu B(z_h, r)}{r^{n-1}}$ is nondecreasing on $(0, r_1)$, for some $r_1 > 0$, and moreover in case $n = 2$ the test balls with centres with the first coordinate equal to zero were crucial for the local 1-monotonicity.

If we want to construct a locally k-monotone measure which is not just a k-dimensional Hausdorff measure restricted to a k-dimensional subspace of $\mathbb{R}^n$, it may be more convenient to use the following proposition instead of studying graphs of $|x|^p$.

Proposition 4.4. *Let $M \subset \mathbb{R}^n$ be a Borel set, μ be a k-dimensional Hausdorff measure restricted to M and $\tilde{\mu}$ be a $(k+1)$-dimensional Hausdorff measure restricted to $M \times \mathbb{R} = \{(x, y) : x \in M, y \in \mathbb{R}\}$.*

If $r \mapsto \frac{\mu B(z,r)}{r^k}$ is nondecreasing on $[0, r_0]$ for every $z \in \mathbb{R}^n$, then $r \mapsto \frac{\tilde{\mu} B(\tilde{z},r)}{r^{k+1}}$ is nondecreasing on $[0, r_0]$ for every $\tilde{z} \in \mathbb{R}^{n+1}$.

Hence, if μ is locally k-monotone, then $\tilde{\mu}$ is locally $(k+1)$-monotone. If μ is k-monotone, then $\tilde{\mu}$ is $(k+1)$-monotone.

Proof: Without loss of generality suppose that the last component of $\tilde{z}$ is 0. Hence we can write $\tilde{z} = (z, 0)$, where $z \in \mathbb{R}^n$. Let $0 < r_1 < r_2 \leq r_0$. Using the monotonicity of $r \mapsto \frac{\mu B(z,r)}{r^k}$ we obtain

$$\begin{aligned}\frac{\tilde{\mu} B(\tilde{z}, r_2)}{r_2^{k+1}} &= \int_{-r_2}^{r_2} \frac{\mu B\big(z, \sqrt{r_2^2 - t^2}\big)}{r_2^k} \frac{dt}{r_2} = \int_{-r_1}^{r_1} \frac{\mu B\big(z, \sqrt{r_2^2 - (\frac{r_2}{r_1} s)^2}\big)}{r_2^k} \frac{ds}{r_1} \\ &= \int_{-r_1}^{r_1} \frac{\mu B\big(z, \frac{r_2}{r_1}\sqrt{r_1^2 - s^2}\big)}{r_2^k} \frac{ds}{r_1} \geq \int_{-r_1}^{r_1} \frac{\mu B\big(z, \sqrt{r_1^2 - s^2}\big)}{r_2^k (\frac{r_1}{r_2})^k} \frac{ds}{r_1} \\ &= \frac{\tilde{\mu} B(\tilde{z}, r_1)}{r_1^{k+1}}.\end{aligned}$$

□

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