Sequences between d-sequences and sequences of linear type

Hamid Kulosman

Abstract. The notion of a d-sequence in Commutative Algebra was introduced by Craig Huneke, while the notion of a sequence of linear type was introduced by Douglas Costa. Both types of sequences generate ideals of linear type. In this paper we study another type of sequences, that we call c-sequences. They also generate ideals of linear type. We show that c-sequences are in between d-sequences and sequences of linear type and that the initial subsequences of c-sequences are c-sequences. Finally we prove a statement which is useful for computational aspects of the theory of c-sequences.

Keywords: ideal of linear type, c-sequence, d-sequence, sequence of linear type

Classification: Primary 13A30, 13B25; Secondary 13A15, 13C13

1. Introduction

Let R be a Noetherian commutative ring, $\langle \mathbf{a} \rangle = \langle a_1, \dots, a_n \rangle$ a sequence of elements of R, $I = (a_1, \dots, a_n)$ the ideal generated by the a_i's and $I_i = (a_1, \dots, a_i)$, $i = 0, 1, \dots, n$, the ideal generated by the first i elements of the sequence.

Let $S(I) = \bigoplus_{i \geq 0} S^i(I)$ be the symmetric algebra of the ideal I, $R[It] = \bigoplus_{i \geq 0} I^i t^i$ its Rees algebra and $\alpha : S(I) \to R[It]$ the canonical map, which maps $a_i \in S^1(I)$ to $a_i t$. The ideal I is said to be of *of linear type* if α is an isomorphism. There are also the canonical maps $\rho : R[T_1, \dots, T_n] \to R[It]$, mapping T_i to $a_i t$, and $\sigma : R[T_1, \dots, T_n] \to S(I)$, mapping T_i to $a_i \in S^1(I)$. Let $Q_\infty = \ker(\rho)$ and $Q = \ker(\sigma)$. Then $Q \subset Q_\infty$ and $\mathcal{A} := \ker(\alpha)$ can be identified with Q_∞ / Q.

Let us observe a simple property of ideals of linear type, used later.

Lemma 1.1 ([1, Theorem 4(i)]). *If $I = (a_1, \dots, a_n)$ is an ideal of linear type, then*

$$I_{n-1} I^{k-1} : a_n^k = I_{n-1} : a_n$$

for every $k \geq 1$.

Now we list various types of sequences related to the notion of ideals of linear type.

We say that $\langle \mathbf{a} \rangle$ is a *relative regular* or *d-sequence* ([6]) if

$$[I_{i-1} : a_i] : a_j = I_{i-1} : a_j$$

for every $i, j \in \{1, 2, \ldots, n\}$ with $j \geq i$. Equivalently

$$[I_{i-1} : a_i] \cap I = I_{i-1}$$

for every $i \in \{1, 2, \ldots, n\}$.

We say that $\langle \mathbf{a} \rangle$ is a *weakly relative regular* sequence ([2]) if

$$[I_{i-1} I : a_i] \cap I = I_{i-1}$$

for every $i \in \{1, 2, \ldots, n\}$.

We say that $\langle \mathbf{a} \rangle$ is a *proper* sequence ([3]) if

$$a_i \cdot H_j(a_1, \ldots, a_{i-1}) = 0,$$

for $i = 1, \ldots, n$, $j \geq 1$, where $H_j(a_1, \ldots, a_{i-1})$ denotes the j-th homology module of the Koszul complex on $a_1, \ldots, a_{i-1}$. (Actually it is enough to have this property for $j = 1$, it is then true for all $j \geq 1$ by [7].)

We say that $\langle \mathbf{a} \rangle$ is a *sequence of linear type* ([1]) if each of the ideals $I_i = (a_1, \ldots, a_i)$, $i = 1, \ldots, n$, is of linear type.

It is well-known that the ideals generated by d-sequences are of linear type ([4], [8]), in fact that the d-sequences are sequences of linear type. Every d-sequence is weakly relative regular and every weakly relative regular sequence is proper ([3]).

2. c-sequences and their initial subsequences

It was proved in [1] that d-sequences satisfy the following property:

$$[I_{i-1} I^k : a_i] \cap I^k = I_{i-1} I^{k-1}$$

for every $i \in \{1, \ldots, n\}$ and every $k \geq 1$. It was also proved ([1, Theorem 3]) that, if a sequence satisfies this property, it generates an ideal of linear type. We call the sequences that satisfy this property *c-sequences*.

Definition 2.1. We say that $\langle \mathbf{a} \rangle$ is a *c-sequence* if

$$[I_{i-1} I^k : a_i] \cap I^k = I_{i-1} I^{k-1}$$

for every $i \in \{1, \ldots, n\}$ and every $k \geq 1$.

Now we show that the notion of a c-sequence is strictly weaker than the notion of a d-sequence, i.e., that there are sequences which are c-sequences but not d-sequences.

Example 2.2. Let $R = k[X,Y,Z,U]/(XU - Y^2Z) = k[x,y,z,u]$, where k is a field. Consider the sequence $\langle x, y\rangle$ and the ideal $I = (x,y)$. This sequence is not a d-sequence since $z \in (x) : y^2$ and $z \notin (x) : y$, although I is an ideal of linear type, which was shown in [8, Example 3.16].

Let us show that $\langle x, y\rangle$ is a c-sequence. We should show two relations:

$$[0 : x] \cap I^k = 0, \ k \geq 1$$
$$[xI^k : y] \cap I^k = xI^{k-1}, \ k \geq 1,$$

the first of which is trivial since R is an integral domain. For the second one, note that $[xI^k : y] \cap I^k$=$[xI^k : y] \cap (xI^{k-1} + (y)^k) = [xI^k : y] \cap (y)^k + xI^{k-1}$. So it is enough to prove that $[xI^k : y] \cap (y)^k \subset xI^{k-1}$. Let $\alpha = ay^k$, $a \in R$, be an element of $[xI^k : y] \cap (y)^k$. Then $ay^{k+1} \in xI^k$, i.e., $a \in xI^k : y^{k+1} = (x) : y$ by Lemma 1.1. Hence $ay \in (x)$ and so $\alpha = ay^k = ay \cdot y^{k-1} \in xI^{k-1}$.

If $\langle \mathbf{a}\rangle$ is a d-sequence, it is obvious that then for each $i = 1,\ldots,n$, $\langle a_1,\ldots,a_i\rangle$ is also a d-sequence. The analogous property for c-sequences is far from being obvious. We establish it in the following main theorem of the paper.

Theorem 2.3. *If $\langle a_1,\ldots,a_n\rangle$ is a c-sequence, then for each $i = 1,\ldots,n$, $\langle a_1,\ldots,a_i\rangle$ is a c-sequence.*

Proof: It is enough to prove that $\langle a_1,\ldots,a_{n-1}\rangle$ is a c-sequence, but it is about the same to prove it for $\langle a_1,\ldots,a_j\rangle$ for any $j \in \{1,\ldots,n\}$. Fix $j \in \{1,2,\ldots,n\}$. Then $\langle a_1,\ldots,a_j\rangle$ is a c-sequence if and only if

$$[I_{i-1}I_j^k : a_i] \cap I_j^k = I_{i-1}I_j^{k-1}, \tag{1}$$

for $i = 1,\ldots,j$, $k \geq 1$.

Claim 1. *If $i = j$, the equality* (1) *holds. In other words,*

$$[I_{j-1}I_j^k : a_j] \cap I_j^k = I_{j-1}I_j^{k-1}, \tag{2}$$

for $k \geq 1$.

Note that, if we divide both sides of the equality

$$[I_{j-1}I^k : a_j] \cap I^k = I_{j-1}I^{k-1}$$

by a_j^k, we get

$$I_{j-1}I^k : a_j^{k+1} = I_{j-1}I^{k-1} : a_j^k,$$

and hence, by induction on k,

$$I_{j-1}I^{k-1} : a_j^k = I_{j-1} : a_j, \ \ k \geq 1. \tag{3}$$

It then follows

$$I_{j-1}I_j^{k-1} : a_j^k = I_{j-1} : a_j, \quad k \geq 1. \tag{4}$$

Now

$$\begin{aligned}
&[I_{j-1}I_j^k : a_j] \cap I_j^k \\
&= [I_{j-1}I_j^k : a_j] \cap [I_{j-1}I_j^{k-1} + (a_j)^k] \\
&= I_{j-1}I_j^{k-1} + [I_{j-1}I_j^k : a_j] \cap (a_j)^k.
\end{aligned} \tag{5}$$

If $ra_j^k \in I_{j-1}I_j^k : a_j$, then by (4), $r \in I_{j-1}I_j^k : a_j^{k+1} = I_{j-1} : a_j$, hence $ra_j \in I_{j-1}$ and so $ra_j^k = ra_ja_j^{k-1} \in I_{j-1}(a_j)^{k-1} \subset I_{j-1}I_j^{k-1}$. This together with (5) gives (2). Claim 1 is proved.

Claim 2. *In order to prove* (1), *it is enough to prove that*

$$I_{i-1}I^{k-1} \cap (a_i, \ldots, a_j)^k \subset I_{i-1}(a_i, \ldots, a_j)^{k-1} \tag{6}$$

for $i = 1, \ldots, j$, $k \geq 1$.

Indeed, since $\langle \mathbf{a} \rangle$ is a c-sequence we would then have

$$([I_{i-1}I^k : a_i] \cap I^k) \cap (a_i, \ldots, a_j)^k \subset I_{i-1}(a_i, \ldots, a_j)^{k-1},$$

hence

$$[I_{i-1}I_j^k : a_i] \cap (a_i, \ldots, a_j)^k \subset I_{i-1}(a_i, \ldots, a_j)^{k-1}. \tag{7}$$

Now

$$\begin{aligned}
[I_{i-1}I_j^k : a_i] \cap I_j^k &= [I_{i-1}I_j^k : a_i] \cap [I_{i-1}I_j^{k-1} + (a_i, \ldots, a_j)^k] \\
&= I_{i-1}I_j^{k-1} + [I_{i-1}I_j^k : a_i] \cap (a_i, \ldots, a_j)^k \\
&= I_{i-1}I_j^{k-1} + I_{i-1}(a_i, \ldots, a_j)^{k-1} \qquad \text{(by (7))} \\
&= I_{i-1}I_j^{k-1}.
\end{aligned}$$

Claim 2 is proved.

Denote

$$\Lambda_t = (a_i)^{k-t}(a_i, \ldots, a_j)^t \tag{8}$$

for $t = 0, 1, \ldots, k$.

Claim 3. *In order to prove* (6)*, it is enough to prove that*

$$I_{i-1}I^{k-1} \cap \Lambda_t \ \subset \ I_{i-1}I^{k-1} \cap \Lambda_{t-1} + I_{i-1}(a_i)^{k-t}(a_{i+1}, \ldots, a_j)^{t-1}, \tag{9}$$

for $i = 1, \ldots, j$, $k \geq 1$, $t = 1, \ldots, k$.

Indeed, if (9) holds we would have:

$$\begin{aligned} I_{i-1}I^{k-1} \cap (a_i, \ldots, a_j)^k &= I_{i-1}I^{k-1} \cap \Lambda_k \\ &\subset I_{i-1}I^{k-1} \cap \Lambda_{k-1} + I_{i-1}(a_i)^0(a_{i+1}, \ldots, a_j)^{k-1} \\ &\subset \ldots \\ &\subset I_{i-1}I^{k-1} \cap \Lambda_0 + I_{i-1}\sum_{t=1}^{k}(a_i)^{k-t}(a_{i+1}, \ldots, a_j)^{t-1} \qquad (10) \\ &= I_{i-1}I^{k-1} \cap \Lambda_0 + I_{i-1}(a_i, \ldots, a_j)^{k-1}. \end{aligned}$$

Now let $\alpha \in I_{i-1}I^{k-1} \cap \Lambda_0 = I_{i-1}I^{k-1} \cap (a_i)^k$. Then $\alpha = ra_i^k$ and $r \in I_{i-1}I^{k-1} : a_i^k = I_{i-1} : a_i$ by (3). Hence $ra_i \in I_{i-1}$ and so $\alpha = ra_ia_i^{k-1} \in I_{i-1}(a_i)^{k-1}$. Thus

$$I_{i-1}I^{k-1} \cap \Lambda_0 \subset I_{i-1}(a_i)^{k-1}. \tag{11}$$

Now from (10) and (11) we have

$$\begin{aligned} I_{i-1}I^{k-1} \cap (a_i, \ldots, a_j)^k &\subset I_{i-1}(a_i)^{k-1} + I_{i-1}(a_i, \ldots, a_j)^{k-1} \\ &= I_{i-1}(a_i, \ldots, a_j)^{k-1}. \end{aligned}$$

Claim 3 is proved.

We will now prove (6) by induction on i. Let us first treat the case $i = j$. We need to show that

$$I_{j-1}I^{k-1} \cap (a_j)^k \subset I_{j-1}(a_j)^{k-1}$$

for $k \geq 1$. Since $\langle \mathbf{a} \rangle$ is a c-sequence, this is equivalent to

$$[I_{j-1}I^k : a_j] \cap I^k \cap (a_j)^k \subset I_{j-1}(a_j)^{k-1},$$

i.e., with

$$[I_{j-1}I^k : a_j] \cap (a_j)^k \subset I_{j-1}(a_j)^{k-1}.$$

If $ra_j^k \in I_{j-1}I^k : a_j$, then by (3), $r \in I_{j-1}I^k : a_j^{k+1} = I_{j-1} : a_j$, hence $ra_j \in I_{j-1}$ and so $ra_j^k = ra_ja_j^{k-1} \in I_{j-1}(a_j)^{k-1}$.

Now let $i < j$ and suppose that (6) holds for $i+1$ and any $k \geq 1$. By Claim 3, to prove that then (6) holds for i and any $k \geq 1$, it is enough to prove that (9) holds for i and any $k \geq 1$, $t = 1, \ldots, k$. For that purpose let

$$\begin{aligned} \alpha &\in I_{i-1}I^{k-1} \cap \Lambda_t \\ &= I_{i-1}I^{k-1} \cap [\Lambda_{t-1} + (a_i)^{k-t}(a_{i+1}, \ldots, a_j)^t]. \end{aligned}$$

We can write $\alpha = \delta + \varepsilon \in I_{i-1}I^{k-1}$, where

$$\begin{aligned} &\delta \in \Lambda_{t-1}, \\ &\varepsilon \in (a_i)^{k-t}(a_{i+1}, \ldots, a_j)^t. \end{aligned}$$

More concretely, let

$$\begin{aligned} \delta &= a_i^{k-t+1}d, \\ \varepsilon &= a_i^{k-t}e, \end{aligned}$$

where

$$\begin{aligned} &d \in (a_i, \ldots, a_j)^{t-1}, \\ &e \in (a_{i+1}, \ldots, a_j)^t. \end{aligned} \tag{12}$$

Then

$$\alpha = \delta + \varepsilon = a_i^{k-t}(e + a_i d) \in I_{i-1}I^{k-1}. \tag{13}$$

Now if we divide both sides of the equality

$$[I_{i-1}I^k : a_i] \cap I^k = I_{i-1}I^{k-1}$$

by $a_i^{k-t}(e + a_i d)$, we get

$$I_{i-1}I^k : a_i^{k-t+1}(e + a_i d) = I_{i-1}I^{k-1} : a_i^{k-t}(e + a_i d).$$

By induction on k we get

$$I_{i-1}I^k : a_i^{k-t+1}(e + a_i d) = I_{i-1}I^{t-1} : (e + a_i d).$$

Since by (13), $\alpha = \delta + \epsilon = a_i^{k-t+1}(e + a_i d) \in I_{i-1}I^k$, we have

$$e + a_i d \in I_{i-1}I^{t-1}. \tag{14}$$

Note that

$$a_i d \in I_i I^{t-1}. \tag{15}$$

From (14) and (15) we get

$$e \in I_i I^{t-1}. \tag{16}$$

Now from (12) and (16), using the inductive hypothesis (6) for $i+1$ and any $k \geq 1$, we get

$$e \in I_i (a_{i+1}, \dots, a_j)^{t-1}. \tag{17}$$

Hence

$$\begin{aligned}
\varepsilon &= a_i^{k-t} e \\
&\in I_i a_i^{k-t} (a_{i+1}, \dots, a_j)^{t-1} \qquad \text{(by (17))} \\
&= I_{i-1} a_i^{k-t} (a_{i+1}, \dots, a_j)^{t-1} + a_i^{k-t+1} (a_{i+1}, \dots, a_j)^{t-1} \\
&\in \Lambda_{t-1} + I_{i-1} a_i^{k-t} (a_{i+1}, \dots, a_j)^{t-1} \qquad \text{(by (8))}
\end{aligned}$$

and consequently

$$\begin{aligned}
\alpha = \delta + \varepsilon &\in [\Lambda_{t-1} + I_{i-1} a_i^{k-t} (a_{i+1}, \dots, a_j)^{t-1}] \cap I_{i-1} I^{k-1} \\
&= I_{i-1} I^{k-1} \cap \Lambda_{t-1} + I_{i-1} a_i^{k-t} (a_{i+1}, \dots, a_j)^{t-1}.
\end{aligned}$$

The theorem is proved. □

3. Relations between c-sequences and other types of sequences

Theorem 3.1. *Every c-sequence is a sequence of linear type.*

Proof: Every c-sequence generates an ideal of linear type ([1, Theorem 3]). Now the statement follows from Theorem 2.3. □

Thus, by Introduction and Theorem 3.1, we have

$$\{\text{d-sequences}\} \subset \{\text{c-sequences}\} \subset \{\text{sequences of linear type}\}.$$

For one-element sequences all three notions coincide. But, in general, for sequences of at least two elements, both of the above inclusions are strict, as Example 2.2 and the example that follows illustrate.

Example 3.2. Let $R = k[X, Y, U, V]/(UX, VX, UY, U^2, V^2, UV) = k[x, y, u, v]$, where k is a field. Then $\langle x, y\rangle$ is a sequence of linear type which is not a c-sequence.

Indeed, let us first show that $I = (x, y)$ is an ideal of linear type. We can write

$$R = A[X, Y]/(uX, vX, uY),$$

where $A = k[u, v]$ with $u^2 = v^2 = uv = 0$. Hence R is a symmetric algebra of some A-module (namely $A^2/(A(u, 0) + A(v, 0) + A(0, u))$) and so (by [3, p. 87]) its augmentation ideal $I = (x, y)$ is an ideal of linear type.

Also it is easy to verify that $(0 : x) = (0 : x^2) = (u, v)$. Thus $\langle x, y\rangle$ is a sequence of linear type.

But $(0 : x) \cap (x, y)$ contains a nonzero element vy and thus the first condition for $\langle x, y\rangle$ to be a c-sequence is not satisfied.

Thus it can happen that an ideal I of linear type can be generated by a sequence $\langle \mathbf{a}\rangle$ of linear type which is not a c-sequence. In the remaining part of the paper we will show that this cannot happen if $\langle \mathbf{a}\rangle$ is a weakly relative regular or a proper sequence. We will first establish an analogue for c-sequences of the following two statements:

(i) $\langle \mathbf{a}\rangle$ is a *proper sequence* if and only if the corresponding sequence of 1-forms $\langle \overline{\mathbf{a}}\rangle$ in $S_R(I)$ is a d-sequence ([7, Theorem 2.2]);

(ii) $\langle \mathbf{a}\rangle$ is a *d-sequence* if and only if the corresponding sequence of 1-forms $\langle \mathbf{a}^*\rangle$ in $gr_I(R)$ is a d-sequence ([5, Theorem 1.2] ($\Rightarrow$) and [3, Theorem 12.10] ($\Leftarrow$)).

Proposition 3.3. *Let $a_1, \dots, a_n \in R$ and let $a_1t, \dots, a_nt$ be the corresponding 1-forms in $R[It]$. Then $\langle a_1, \dots, a_n\rangle$ is a c-sequence in R if and only if $\langle a_1t, \dots, a_nt\rangle$ is a d-sequence in $R[It]$.*

Proof: Denote $\mathcal{I} = (a_1t, \dots, a_nt) = R[It]_+$, the ideal in $R[It]$ generated by $a_1t, \dots, a_nt$. Also $\mathcal{I}_{i-1} = (a_1t, \dots, a_{i-1}t)$ and $I_{i-1} = (a_1, \dots, a_{i-1})$, $i = 1, 2, \dots, n$. Then $\langle \mathbf{a}t\rangle$ is a d-sequence in $R[It]$ if and only if

$$[\mathcal{I}_{i-1} : a_it] \cap \mathcal{I} = \mathcal{I}_{i-1},\ i = 1, 2, \dots, n.$$

This is equivalent to

$$(c_1t + c_2t^2 + \dots)a_it \in \mathcal{I}_{i-1} \Rightarrow c_1t + c_2t^2 + \dots \in \mathcal{I}_{i-1},\ i = 1, 2, \dots, n,$$

where $c_j \in I^j$, $j = 1, 2, \dots$ are arbitrary elements. This in turn is equivalent to

$$c_k \in I_{i-1}I^k : a_i \Rightarrow c_k \in I_{i-1}I^{k-1},\ i = 1, 2, \dots, n,\ k \geq 1,$$

where each $c_k \in I^k$. This is the same as

$$[I_{i-1}I^k : a_i] \cap I^k \subset I_{i-1}I^{k-1},\ i = 1, 2, \dots, n,\ k \geq 1,$$

which is the condition for $\langle \mathbf{a}\rangle$ to be a c-sequence. □

Corollary 3.4. *Let $\langle a_1, \ldots, a_n \rangle$ be a sequence in R and let $I = (a_1, \ldots, a_n)$. Then the following are equivalent:*

(i) $\langle \mathbf{a} \rangle$ *is a c-sequence;*
(ii) $\langle \mathbf{a} \rangle$ *is a weakly relative regular sequence and I is of linear type;*
(iii) $\langle \mathbf{a} \rangle$ *is a proper sequence and I is of linear type.*

Proof: (i)⇒(ii): follows from the definition of a weakly relatively regular sequence and [1, Theorem 3].
(ii)⇒(iii): follows from [3, p. 113].
(iii)⇒(i): By [7, Theorem 2.2], if $\langle \mathbf{a} \rangle$ is a proper sequence, then the corresponding sequence of 1-forms $\langle \overline{\mathbf{a}} \rangle$ is a d-sequence in $S_R(I)$. Since I is assumed to be of linear type, $S_R(I)$ is canonically isomorphic to $R[It]$. Hence $\langle a_1 t, \ldots, a_n t \rangle$ is a d-sequence in $R[It]$. Now by Proposition 3.3, $\langle a_1, \ldots, a_n \rangle$ is a c-sequence in R. □

Thus if $I = (\mathbf{a})$ is an ideal of linear type, where $\langle \mathbf{a} \rangle$ is a proper or weakly relative regular sequence, then $\langle \mathbf{a} \rangle$ is necessarily a c-sequence. Note that neither proper nor weakly relative regular sequences are sequences of linear type.

Remark 3.5. Corollary 3.4 is useful for *computational purposes*. Namely, in order to test by computer programs whether some sequence is a c-sequence, we would have to test infinitely many conditions. Using Corollary 3.4(ii), it is enough to test only two things: that the ideal generated by the sequence is of linear type (which is a known procedure) and that the sequence is weakly relative regular (which is easy).

References

[1] Costa D., *Sequences of linear type*, J. Algebra **94** (1985), 256–263.
[2] Fiorentini M., *On relative regular sequences*, J. Algebra **18** (1971), 384–389.
[3] Herzog J., Simis A., Vasconcelos W., *Koszul homology and blowing-up rings*, Commutative Algebra (Trento, 1981), Lecture Notes in Pure and Appl. Math. 84, Dekker, New York, 1983, pp. 79–169.
[4] Huneke C., *On the symmetric and Rees algebra of an ideal generated by a d-sequence*, J. Algebra **62** (1980), 268–275.
[5] Huneke C., *Symbolic powers of prime ideals and special graded algebras*, Comm. Algebra **9** (1981), 339–366.
[6] Huneke C., *The theory of d-sequences and powers of ideals*, Adv. in Math. **46** (1982), 249–279.
[7] Kühl M., *On the symmmetric algebra of an ideal*, Manuscripta Math. **37** (1982), 49–60.
[8] Valla G., *On the symmetric and Rees algebras of an ideal*, Manuscripta Math. **30** (1980), 239–255.

Department of Mathematics, University of Louisville, USA

(*Received* June 22, 2008, *revised* October 12, 2008)