Linear identities in graph algebras

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Abstract. We find the basis of all linear identities which are true in the variety of entropic graph algebras. We apply it to describe the lattice of all subvarieties of power entropic graph algebras.

Keywords: graph algebra, linear identity, entropic algebra, equational basis, lattice of subvarieties, power algebra of subsets

Classification: Primary 08B15, 03C05, 03C13, 08C10; Secondary 17D99, 08A05, 08A40, 08A62, 08A99

1. Introduction

In 1979 Caroline Shallon introduced in her dissertation [9] algebras associated with graphs. Let $G = (V, E)$ be a (undirected) graph with a set V of vertices and a set $E \subseteq V \times V$ of edges. Its *graph algebra* $A(G) = (V \cup \{0\}, \cdot)$ is a groupoid with the multiplication defined as follows:

$$x \cdot y := \begin{cases} x, & \text{if } (x, y) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

C. Shallon proved that many finite graph algebras are nonfinitely based. Moreover it was shown in [4] that a graph algebra $A(G)$ is finitely based if and only if it is entropic.

In this paper we find all linear identities which are true in the variety of entropic graph algebras. Some new linear identity (3.1) true in this variety is crucial for the final result. Linear identities play an important rôle in the theory of power algebras.

The power (complex or global) *algebra* $\mathbf{CmA} = (\mathcal{P}(A), F)$ of an algebra (A, F) is the family $\mathcal{P}(A)$ of all non-empty subsets of A with complex operations given by

$$f(A_1, \ldots, A_n) := \{f(a_1, \ldots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \ldots, A_n \subseteq A$ and $f \in F$ is an n-ary operation.

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Power algebras were studied by several authors, for instance by G. Grätzer and H. Lakser [5], S. Whitney [6], A. Shafaat [8], C. Brink [3], I. Bošnjak and R. Madarász [2].

Complex operations may preserve some of properties of (A, F), but not all identities true in (A, F) must be satisfied in **CmA**. For example, the power algebra of a group is not again a group [5]. For an arbitrary variety $\mathcal{V}$, G. Grätzer and H. Lakser determined the identities satisfied by the variety generated by **CmA**, for $\mathbf{A} \in \mathcal{V}$. They applied their result to describe all subvarieties of power algebras of lattices and groups [5]. They showed that there is exactly one non-trivial variety of power algebras of lattices and there are exactly three non-trivial varieties of power algebras of groups.

In this paper we show that the lattice of all subvarieties of the variety generated by power algebras of entropic graph algebras has eight elements.

In Section 2 we recall main known results concerning graph algebras. Section 3 is devoted to linear identities satisfied in entropic graph algebras. In the last Section 4 we present some theorems concerning identities satisfied by power algebras of sets and describe all subvarieties of power entropic graph algebras.

We say that an algebra (A, F) is *idempotent* if each element $a \in A$ forms a one-element subalgebra of (A, F). An algebra (A, F) is *entropic*, if each operation $f \in F$ as a mapping from a direct power of the algebra into the algebra is actually a homomorphism.

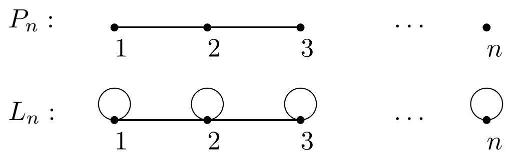
We call a term t *linear*, if every variable occurs in t at most once. An identity $t \approx u$ is called *linear*, if both terms t and u are linear. An identity $t \approx u$ is called *regular*, if t and u contain the same variables. The notation $t(x_1, \ldots, x_n)$ means that the term t contains no other variables than $x_1, \ldots, x_n$ (but not necessarily all of them).

In the case of groupoid terms we will use non-brackets notation, as follows:

$$xx_1x_2\ldots x_n := \begin{cases} (\ldots((xx_1)x_2)\ldots)x_n, & n \geq 1, \\ x, & n = 0. \end{cases}$$

2. Graph algebras

For each natural number $n \in \mathbb{N}$, let P_n denote n-vertex graphs in the form of a path without loops, and L_n denote n-vertex graphs in the form of a path with loops, as in the diagrams:



For graphs G and H, $G + H$ will denote the disjoint union of G and H, with no edges between G and H.

It is not difficult to see that the direct product of two graph algebras is not necessarily a graph algebra. So the class of all graph algebras does not form a variety. Let us denote by $\mathcal{V}_{A(G)}$ the variety generated by a graph algebra $A(G)$.

As was shown in [1], the variety of all entropic graph algebras is generated by the algebra $A(P_2 + L_2)$. The following identities

$$xy \approx xyy \tag{2.1}$$
$$x(yz) \approx xy(yz) \tag{2.2}$$
$$xyz \approx xzy \tag{2.3}$$
$$xy \approx x(yx) \tag{2.4}$$
$$x(yzu) \approx x(yz)(yu) \tag{2.5}$$
$$x(y(zu)) \approx x(yz)(uz) \tag{2.6}$$
$$x(yz)(uv) \approx x(yv)(uz) \tag{2.7}$$
$$x(xy) \approx x(yy) \tag{2.8}$$
$$xx(yz) \approx x(yy)(zz) \tag{2.9}$$

form a basis for $\mathcal{V}_{A(P_2+L_2)}$. It is easy to see that the entropic law

$$xy(zw) \approx xz(yw). \tag{E}$$

follows by the above identities:

$$xy(zw) \overset{(2.4)}{\approx} x(yx)(zw) \overset{(2.7)}{\approx} x(yw)(zx) \overset{(2.3)}{\approx} x(zx)(yw) \overset{(2.4)}{\approx} xz(yw).$$

The lattice of all subvarieties of the variety of entropic graph algebras was discussed in [1] and it is given in Figure 1.

$\mathcal{T}$ denotes the trivial variety, $\mathcal{SL}$ is the variety of semilattices, $\mathcal{LZ}$ is the variety of left zero bands (groupoids determined by identity $xy \approx x$), $\mathcal{LN}$ is the variety of left normal bands — idempotent semigroups satisfying the additional left-normal law (2.3), $\mathcal{U}_1$ is the variety defined by two identities: (2.1) and

$$xx \approx xy \tag{2.10}$$

and $\mathcal{U}_2$ is the variety defined by all identities (2.1)–(2.9) and additional one

$$xx \approx x(yy). \tag{2.11}$$

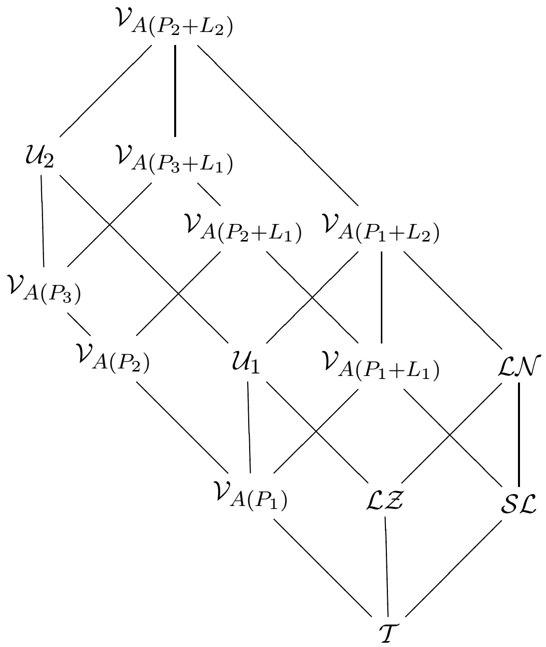


Figure 1

The remaining subvarieties of entropic graph algebras have the following basis:

$$
\begin{array}{lll}
\mathcal{V}_{A(P_1)} & : & xy \approx uz; \\
\mathcal{V}_{A(P_1+L_1)} & : & x(yz) \approx (xy)z,\ xy \approx yx,\ xy \approx x(yy); \\
\mathcal{V}_{A(P_1+L_2)} & : & (2.1)\text{–}(2.9),\ \ x(yz) \approx (xy)z; \\
\mathcal{V}_{A(P_2)} & : & (2.1)\text{–}(2.9),\ \ xx \approx yy,\ x(yz) \approx z(yx); \\
\mathcal{V}_{A(P_2+L_1)} & : & (2.1)\text{–}(2.9),\ \ x(yz) \approx z(yx); \\
\mathcal{V}_{A(P_3)} & : & (2.1)\text{–}(2.9),\ \ xx \approx yy; \\
\mathcal{V}_{A(P_3+L_1)} & : & (2.1)\text{–}(2.9),\ \ x(yy) \approx y(xx).
\end{array}
$$

As it was proved in [1] using the regular identities (2.1)–(2.9), every term in the variety $\mathcal{V}_{A(P_2+L_2)}$ may be expressed in one of the standard forms:

(i) x,
(ii) $x_1(x_1x_1)(x_2x_2)\dots(x_nx_n)$, for $n \geq 1$,
(iii) $x_1(y_1x_1)(y_2x_2)\dots(y_nx_n) \approx (x_1y_1)(y_2x_2)\dots(y_nx_n)$,

for $n \geq 1$, where $\{x_1,\dots,x_n\} \cap \{y_1,\dots,y_n\} = \emptyset$. The variables $x_1,\dots,x_n$ will be referred to as bottom and $y_1,\dots,y_n$ will be referred to as top variables.

Theorem 2.1 ([1]). *An identity $p \approx q$ is derivable from identities* (2.1)–(2.9) *if and only if both sides of the identity $p \approx q$ have the same standard form, the same leftmost variables and* (*in the case of type* (iii)) *the same top and bottom variables.*

3. Linear identities in entropic graph algebras

In this section we will find all linear identities which are true in the variety $\mathcal{V}_{A(P_2+L_2)}$. First we prove that all entropic graph algebras satisfy some linear identity which plays an important rôle in this paper.

Theorem 3.1. *The following linear identity*

$$x(y(zt)) \approx xy(tz) \tag{3.1}$$

is satisfied in the variety $\mathcal{V}_{A(P_2+L_2)}$.

PROOF:

$$\begin{aligned} x(y(zt)) &\overset{(2.6)}{\approx} x(yz)(tz) \overset{(2.2)}{\approx} xy(yz)(tz) \\ &\overset{(E)}{\approx} xyt(yzz) \overset{(2.1)}{\approx} xyt(yz) \overset{(E)}{\approx} xyy(tz) \overset{(2.1)}{\approx} xy(tz). \end{aligned}$$

□

Let Σ be the following set of identities:

$$\Sigma : \begin{cases} (E) & xy(zt) \approx xz(yt), \\ (2.3) & xyz \approx xzy, \\ (3.1) & x(y(zt)) \approx xy(tz). \end{cases}$$

These identities are all linear and regular and they hold in any variety of entropic graph algebras. Moreover, the linear identity (3.1) also follows from Σ:

$$x(yz)(tw) \overset{(E)}{\approx} xt(yzw) \overset{(2.3)}{\approx} xt(ywz) \overset{(E)}{\approx} x(yw)(tz).$$

We will show that any linear identity true in $\mathcal{V}_{A(P_2+L_2)}$ is a consequence of Σ. Before proving this fact we present a sequence of technical lemmas.

Let $n \geq 1$ and π be any permutation of the set $\{1, 2, \ldots, n\}$. By using identities from Σ repeatedly we obtain:

$$(xy_1y_2\ldots y_n)(zt_1t_2\ldots t_n) \approx xz(y_1t_1)\ldots(y_nt_n) \text{ (by (E))}, \tag{3.2}$$

$$xy_1y_2\ldots y_n \approx xy_{\pi(1)}y_{\pi(2)}\cdots y_{\pi(n)} \text{ (by (2.3))}, \tag{3.3}$$

$$x(y(z_1t_1)(z_2t_2)\ldots(z_nt_n)) \approx xy(t_1z_1)(t_2z_2)\ldots(t_nz_n) \text{ (by (3.1))}. \tag{3.4}$$

Moreover, as it was proved in [1], the following identity follows by entropicity and identities (2.3) and (2.7):

$$\begin{aligned} & xy_1 \dots y_k(y_{k+1}z_1)\dots(y_{k+n}z_n) \approx \\ & xy_{\pi(1)} \dots y_{\pi(k)}(y_{\pi(k+1)}z_{\delta(1)})\dots(y_{\pi(k+n)}z_{\delta(n)}), \end{aligned} \tag{3.5}$$

where π is a permutation on the set $\{1, 2, \dots, k+n\}$ and δ is a permutation on the set $\{1, 2, \dots, n\}$.

Lemma 3.2. *The following identities*

$$xy_1 \dots y_n \approx x(y_1x)(y_2x)\dots(y_nx), \tag{3.6}$$

$$x(yz_1z_2 \dots z_n) \approx x(yx)(yz_1)(yz_2)\dots(yz_n) \tag{3.7}$$

hold in the variety $\mathcal{V}_{A(P_2+L_2)}$ for any $n \geq 1$.

Proof: The proof of (3.6) goes by induction. For $n = 1$, the identity (3.6) follows by (2.4). Now let us assume that (3.6) is true for some $n > 1$. Then using (2.3) and (2.4) we obtain

$$\begin{aligned} xy_1 \dots y_{n+1} &\approx x(y_1x)(y_2x)\dots(y_nx)y_{n+1} \overset{(2.3)}{\approx} xy_{n+1}(y_1x)(y_2x)\dots(y_nx) \\ &\overset{(2.4)}{\approx} x(y_{n+1}x)(y_1x)(y_2x)\dots(y_nx) \overset{(2.3)}{\approx} x(y_1x)(y_2x)\dots(y_{n+1}x). \end{aligned}$$

The identity (3.7) follows directly by (3.6) and (3.4):

$$\begin{aligned} x(yz_1z_2 \dots z_n) &\overset{(3.6)}{\approx} x(y(z_1y)(z_2y)\dots(z_ny)) \\ &\overset{(3.4)}{\approx} xy(yz_1)\dots(yz_n) \overset{(2.4)}{\approx} x(yx)(yz_1)\dots(yz_n). \end{aligned}$$

□

Lemma 3.3. *Let $n, k \geq 0$. The following identities follow from the set Σ:*

$$x(y(t_1t_2z_1 \dots z_n)) \approx xyz_1 \dots z_n(t_2t_1), \tag{3.8}$$

$$(x(y_1y_2 \dots y_n))(t_1t_2z_1 \dots z_k) \approx (x(y_1y_2 \dots y_nz_1 \dots z_k))(t_1t_2), \tag{3.9}$$

$$(x(ty_1 \dots y_nw))z \approx (x(ty_1 \dots y_n))(zw), \tag{3.10}$$

$$(x(z_1y_1 \dots y_n))(z_2w) \approx (x(z_2y_1 \dots y_n))(z_1w), \tag{3.11}$$

$$\text{(3.12)} \qquad (yy_1y_2\dots y_n)(zz_1z_2\dots z_k) \approx$$

$$\begin{cases} (y(zz_1\dots z_{k-n}))(y_1z_{k-n+1})\dots(y_nz_k), & \textit{if } k > n \geq 0 \\ yy_1\dots y_{n-k}z(y_{n-k+1}z_1)\dots(y_nz_k), & \textit{if } n \geq k \geq 0 \end{cases}$$

$$\text{(3.13)} \qquad (x(z_1y_1\dots y_k))(z_2y_{k+1})\dots(z_{n+1}y_{k+n}) \approx$$
$$(x(z_{\pi(1)}y_{\delta(1)}\dots y_{\delta(k)}))(z_{\pi(2)}y_{\delta(k+1)})\dots(z_{\pi(n+1)}y_{\delta(k+n)}),$$

for any permutation δ on the set $\{1,2,\dots,k+n\}$ and any permutation π on the set $\{1,2,\dots,n+1\}$.

Proof: 1. For $n=0$, the identity (3.8) follows by (3.1). If $n>0$, then

$$x(y(t_1t_2z_1\dots z_{n-1}z_n)) \overset{(3.1)}{\approx} (xy)(z_n(t_1t_2z_1\dots z_{n-1})).$$

By induction on n and (3.3) we obtain

$$(xy)(z_n(t_1t_2z_1\dots z_{n-1})) \approx xyz_nz_1\dots z_{n-1}(t_2t_1) \overset{(3.3)}{\approx} xyz_1\dots z_n(t_2t_1).$$

2. For $n=0$ and $k=0$, the identity (3.9) is obvious. If $n>0$ and $k>0$, then

$$(x(y_1y_2\dots y_n))(t_1t_2z_1\dots z_{k-1}z_k) \overset{(3.1)}{\approx} x(y_1y_2\dots y_n(z_k(t_1t_2z_1\dots z_{k-1})))$$
$$\overset{(3.3)}{\approx} x(y_1(z_k(t_1t_2z_1\dots z_{k-1}))y_2\dots y_n).$$

Using (3.8) we conclude that $y_1(z_k(t_1t_2z_1\dots z_{k-1})) \approx y_1z_kz_1\dots z_{k-1}(t_2t_1)$. Thus

$$x(y_1(z_k(t_1t_2z_1\dots z_{k-1}))y_2\dots y_n) \approx x(y_1z_kz_1\dots z_{k-1}(t_2t_1)y_2\dots y_n)$$
$$\overset{(3.3)}{\approx} x(y_1y_2\dots y_nz_1\dots z_k(t_2t_1)) \overset{(3.1)}{\approx} (x(y_1y_2\dots y_nz_1\dots z_k))(t_1t_2).$$

3. For $n=0$, the identity (3.10) follows by (2.3) and entropicity. If $n>0$, then

$$(x(ty_1\dots y_nw))z \overset{(2.3)}{\approx} (xz)(ty_1\dots y_nw) \overset{(E)}{\approx} (x(ty_1\dots y_n))(zw).$$

4. For $n=0$, the identity (3.11) follows by entropicity. If $n>0$, then

$$(x(z_1y_1\dots y_n))(z_2w) \overset{(2.3)}{\approx} (x(z_2w))(z_1y_1\dots y_n) \overset{(E)}{\approx} (x(z_1y_1\dots y_{n-1}))(z_2wy_n).$$

By induction on n, (2.3) and (2.7) we obtain

$$(x(z_1y_1\dots y_{n-1}))(z_2wy_n) \approx (x(z_2wy_1\dots y_{n-1}))(z_1y_n)$$
$$\overset{(2.3)}{\approx} (x(z_2y_1\dots y_{n-1}w))(z_1y_n) \overset{(2.7)}{\approx} (x(z_2y_1\dots y_n))(z_1w).$$

5. The identity (3.12) follows directly by (3.2).

6. The identity (3.13) follows by (2.3), (2.7) and (3.11). □

The next theorem describes all linear terms in the variety $\mathcal{V}_{A(P_2+L_2)}$.

Theorem 3.4. *Every term in $\mathcal{V}_{A(P_2+L_2)}$ may be expressed in one of the following standard forms:*

$$\text{(I1)} \qquad xy_1\dots y_k(t_1w_1)\dots(t_nw_n), \qquad k\geq 0,\ n\geq 0;$$
$$\text{(I2)} \qquad (x(y_1y_2\dots y_k))(t_1w_1)\dots(t_nw_n), \qquad k\geq 2,\ n\geq 0.$$

Proof: We show (using Σ) that the set of terms of the form (I1) and (I2) is closed under multiplication.

(a) Let $\gamma\approx xy_1\dots y_k(t_1w_1)\dots(t_nw_n)$ and $\delta\approx ab_1\dots b_l(c_1d_1)\dots(c_md_m)$. By (3.3) and (3.4) we have

$$\begin{aligned}\gamma\delta &\approx [xy_1\dots y_k(t_1w_1)\dots(t_nw_n)][ab_1\dots b_l(c_1d_1)\dots(c_md_m)]\\ &\overset{(3.3)}{\approx} xy_1\dots y_k(ab_1\dots b_l(c_1d_1)\dots(c_md_m))(t_1w_1)\dots(t_nw_n)\\ &\overset{(3.4)}{\approx} ((xy_1\dots y_k)(ab_1\dots b_l))(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n).\end{aligned}$$

If $l>k\geq 0$, then

$$\begin{aligned}&((xy_1\dots y_k)(ab_1\dots b_l))(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n)\\ &\overset{(3.12)}{\approx} (x(ab_1\dots b_{l-k}))(y_1b_{l-k+1})\dots(y_kb_l)(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n).\end{aligned}$$

Hence $\gamma\delta$ can be written in the form (I2).
For $k\geq l\geq 0$,

$$\begin{aligned}&((xy_1\dots y_k)(ab_1\dots b_l))(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n)\\ &\overset{(3.12)}{\approx} xy_1\dots y_{k-l}a(y_{k-l+1}b_1)\dots(y_kb_l)(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n),\end{aligned}$$

and $\gamma\delta$ has the form (I1).

(b) Let $\gamma\approx xy_1\dots y_k(t_1w_1)\dots(t_nw_n)$ and $\delta\approx(a(b_1b_2\dots b_l))(c_1d_1)\dots(c_md_m)$ for $l\geq 2$. Then

$$\begin{aligned}\gamma\delta &\approx [xy_1\dots y_k(t_1w_1)\dots(t_nw_n)][(a(b_1b_2\dots b_l))(c_1d_1)\dots(c_md_m)]\\ &\overset{(3.3)}{\approx} x((a(b_1b_2\dots b_l))(c_1d_1)\dots(c_md_m))y_1\dots y_k(t_1w_1)\dots(t_nw_n)\\ &\overset{(3.4),(3.3)}{\approx} x(a(b_1b_2\dots b_l))y_1\dots y_k(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n).\end{aligned}$$

By (3.8) we have that $x(a(b_1b_2\dots b_l))\approx xab_3\dots b_l(b_2b_1)$. Consequently

$$\begin{aligned}&x(a(b_1b_2\dots b_l))y_1\dots y_k(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n)\approx\\ &xab_3\dots b_ly_1\dots y_k(b_2b_1)(d_1c_1)\dots(d_mc_m)(t_1w_1)\dots(t_nw_n),\end{aligned}$$

and $\gamma\delta$ is reduced to the form (I1).

(c) Let $\gamma \approx (a(b_1b_2\dots b_l))(c_1d_1)\dots(c_md_m)$ and $\delta \approx xy_1\dots y_k(t_1w_1)\dots(t_nw_n)$, for $l \geq 2$. Then

$$\begin{aligned}\gamma\delta &\approx [(a(b_1b_2\dots b_l))(c_1d_1)\dots(c_md_m)][xy_1\dots y_k(t_1w_1)\dots(t_nw_n)]\\ &\overset{(3.3)}{\approx} (a(b_1b_2\dots b_l))(xy_1\dots y_k(t_1w_1)\dots(t_nw_n))(c_1d_1)\dots(c_md_m)\\ &\overset{(3.4)}{\approx} (a(b_1b_2\dots b_l))(xy_1\dots y_k)(w_1t_1)\dots(w_nt_n)(c_1d_1)\dots(c_md_m).\end{aligned}$$

If $k=0$, then by (3.10)

$$\gamma\delta \approx (a(b_1b_2\dots b_{l-1}))(xb_l)(w_1t_1)\dots(w_nt_n)(c_1d_1)\dots(c_md_m).$$

If $k \geq 1$, then by (3.9) we have

$$(a(b_1b_2\dots b_l))(xy_1\dots y_k) \overset{(3.9)}{\approx} (a(b_1b_2\dots b_ly_2\dots y_k))(xy_1).$$

It follows that

$$\begin{aligned}&(a(b_1b_2\dots b_l))(xy_1\dots y_k)(w_1t_1)\dots(w_nt_n)(c_1d_1)\dots(c_md_m) \approx\\ &(a(b_1b_2\dots b_ly_2\dots y_k))(xy_1)(w_1t_1)\dots(w_nt_n)(c_1d_1)\dots(c_md_m).\end{aligned}$$

Hence, for any $k \geq 0$, $\gamma\delta$ may be reduced to the form (I2).

(d) Let $\gamma \approx (x(y_1y_2\dots y_k))(t_1w_1)\dots(t_nw_n)$ and $\delta \approx (a(b_1b_2\dots b_l))(c_1d_1)\dots(c_md_m)$, for $k,l \geq 2$. By (3.4) and (E), we have

$$\begin{aligned}\gamma\delta &\approx [(x(y_1y_2\dots y_k))(t_1w_1)\dots(t_nw_n)][(a(b_1b_2\dots b_l))(c_1d_1)\dots(c_md_m)]\\ &\overset{(3.4)}{\approx} (x((y_1\dots y_k)(w_1t_1)\dots(w_nt_n)))(a((b_1b_2\dots b_l)(d_1c_1)\dots(d_mc_m)))\\ &\overset{(E)}{\approx} xa(((y_1\dots y_k)(w_1t_1)\dots(w_nt_n))((b_1b_2\dots b_l)(d_1c_1)\dots(d_mc_m))).\end{aligned}$$

Thus by (a) and (b), also in this case, $\gamma\delta$ reduces to the required form. □

Proposition 3.5. *The set Σ is a basis for all linear identities true in the variety of entropic graph algebras.*

Proof: According to results of [1], each identity which is satisfied in $\mathcal{V}_{A(P_2+L_2)}$ is regular. Let $p \approx q$ be linear and regular identity true in $\mathcal{V}_{A(P_2+L_2)}$. By Theorem 3.4, the terms p and q are in one of the two standard forms.

Case 1. Let p and q be of type (I1). There are distinct variables $\{x, y_1, \dots, y_k, y_{k+1}\dots, y_{k+n}, z_1, \dots, z_n\} = \{a, b_1, \dots, b_l, b_{l+1}, \dots, b_{l+m}, c_1, \dots, c_m\}$, such that

$$p \approx xy_1\dots y_k(y_{k+1}z_1)\dots(y_{k+n}z_n)$$

and

$$q \approx ab_1 \dots b_l(b_{l+1}c_1)\dots(b_{l+m}c_m).$$

By (3.6),

$$p \approx x(y_1x)\dots(y_kx)(y_{k+1}z_1)\dots(y_{k+n}z_n)$$

and

$$q \approx a(b_1a)\dots(b_la)(b_{l+1}c_1)\dots(b_{l+m}c_m).$$

By Theorem 2.1, the identity $p \approx q$ is satisfied in $\mathcal{V}_{A(P_2+L_2)}$ if and only if $x = a$, $\{x, z_1, \dots, z_n\} = \{a, c_1, \dots, c_m\}$ and $\{y_1, \dots, y_{k+n}\} = \{b_1, \dots, b_{l+m}\}$. This implies that $n = m$, $k = l$ and there are permutations π on $\{1, 2, \dots, k+n\}$ and δ on $\{1, 2, \dots, n\}$ such that

$$q \approx xy_{\pi(1)}\cdots y_{\pi(k)}(y_{\pi(k+1)}z_{\delta(1)})\cdots(y_{\pi(k+n)}z_{\delta(n)}).$$

Consequently, by identity (3.5), the identity $p \approx q$ follows from Σ.

Case 2. Let p and q be of type (I2). There are distinct variables $\{x, z_1, \dots, z_{n+1}, y_1, \dots, y_{k+n}\} = \{a, b_1, \dots, b_l, b_{m+1}, c_1, \dots, c_{l+m}\}$, such that

$$p \approx (x(z_1y_1\dots y_k))(z_2y_{k+1})\dots(z_{n+1}y_{k+n}),$$
$$q \approx (a(b_1c_1\dots c_l))(b_2c_{l+1})\dots(b_{m+1}c_{l+m}),$$

and $k, l \geq 2$.
By the identity (3.7),

$$p \approx x(z_1x)(z_1y_1)\dots(z_1y_k)(z_2y_{k+1})\dots(z_{n+1}y_{k+n})$$

and

$$q \approx a(b_1a)(b_1c_1)\dots(b_1c_l)(b_2c_{l+1})\dots(b_{m+1}c_{l+m}).$$

By Theorem 2.1, the identity $p \approx q$ is satisfied in $\mathcal{V}_{A(P_2+L_2)}$ if and only if $x = a$, $\{x, y_1, \dots, y_{k+n}\} = \{a, c_1, \dots, c_{l+m}\}$ and $\{z_1, \dots, z_{n+1}\} = \{b_1, \dots, b_{m+1}\}$. This implies that $n = m$, $k = l$ and there are permutations δ on $\{1, 2, \dots, k+n\}$ and π on $\{1, 2, \dots, n+1\}$ such that

$$q \approx (x(z_{\pi(1)}y_{\delta(1)}\cdots y_{\delta(k)}))(z_{\pi(2)}y_{\delta(k+1)})\cdots(z_{\pi(n+1)}y_{\delta(k+n)}).$$

Hence, by identity (3.13), the identity $p \approx q$ follows from Σ.

Case 3. Let p be of type (I1) and q be of type (I2). As before,

$$p \approx xy_1\dots y_k(y_{k+1}z_1)\dots(y_{k+n}z_n)$$

and

$$q \approx (a(b_1c_1\ldots c_l))(b_2c_{l+1})\ldots(b_{m+1}c_{l+m}),$$

where $\{x, z_1, \ldots, z_n, y_1, \ldots, y_{k+n}\} = \{a, b_1, \ldots, b_{m+1}, c_1, \ldots, c_{l+m}\}$ and $l \geq 2$. Similarly as in Case 1 and Case 2, the identities (3.6), (3.7) and Theorem 2.1 imply that the identity $p \approx q$ is satisfied in $\mathcal{V}_{A(P_2+L_2)}$ if and only if $x = a$, $\{x, z_1, \ldots, z_n\} = \{a, c_1, \ldots, c_{l+m}\}$ and $\{y_1, \ldots, y_{k+n}\} = \{b_1, \ldots, b_{m+1}\}$. It follows that $n = m+l$ and $m+1 = k+n$. Hence $k+l = 1$. But we assumed that $l \geq 2$, so this gives a contradiction.

This shows that any linear identity holding in $\mathcal{V}_{A(P_2+L_2)}$ is a consequence of Σ. $\square$

4. The variety of power entropic graph algebras

Using the result of the previous section we will describe the lattice of all subvarieties of power entropic graph algebras.

Let $\mathcal{V}$ be a variety. We will denote by $\mathbf{Cm}\mathcal{V}$ the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathbf{Cm}\mathcal{V} := \mathrm{HSP}(\{\mathbf{CmA} \mid \mathbf{A} \in \mathcal{V}\}).$$

Evidently, $\mathcal{V} \subseteq \mathbf{Cm}\mathcal{V}$, because every algebra $\mathbf{A}$ in $\mathcal{V}$ embeds into $\mathbf{CmA}$ by $x \mapsto \{x\}$. G. Grätzer and H. Lakser proved the following theorem.

Theorem 4.1 ([5])**.** *Let $\mathcal{V}$ be a variety. The variety $\mathbf{Cm}\mathcal{V}$ satisfies precisely those identities resulting through identification of variables from the linear identities true in $\mathcal{V}$.*

For example, an idempotent law is satisfied in the variety $\mathbf{Cm}\mathcal{V}$ if and only if it is a consequence of linear identities true in $\mathcal{V}$.

As an immediate corollary of the Theorem 4.1 we have the following result.

Corollary 4.2 ([5])**.** *For a variety $\mathcal{V}$, $\mathbf{Cm}\mathcal{V} = \mathcal{V}$ if and only if $\mathcal{V}$ is defined by a set of linear identities.*

As it was shown in Section 3, any linear identity true in $\mathcal{V}_{A(P_2+L_2)}$ is derivable from the set Σ. Hence, by Theorem 4.1, we obtain the following result.

Proposition 4.3. $\mathbf{Cm}\mathcal{V}_{A(P_2+L_2)} = \mathrm{Mod}(\Sigma)$.

Similarly as in the case of the variety $\mathcal{V}_{A(P_2+L_2)}$ we can show that varieties $\mathbf{Cm}\mathcal{V}_{A(P_3)}$, $\mathbf{Cm}\mathcal{U}_2$ and $\mathbf{Cm}\mathcal{V}_{A(P_3+L_1)}$ are also defined by the set Σ.

Now we will give a description of varieties generated by power algebras from remaining subvarieties of $\mathcal{V}_{A(P_2+L_2)}$. The varieties $\mathcal{T}$, $\mathcal{LZ}$ and $\mathcal{V}_{A(P_1)}$ are defined only by linear identities. Hence by Corollary 4.2, $\mathbf{Cm}\mathcal{T} = \mathcal{T}$, $\mathbf{Cm}\mathcal{LZ} = \mathcal{LZ}$ and $\mathbf{Cm}\mathcal{V}_{A(P_1)} = \mathcal{V}_{A(P_1)}$.

The varieties $\mathcal{SL}$ and $\mathcal{V}_{A(P_1+L_1)}$ satisfy associativity and commutativity. Hence, $\mathbf{Cm}\mathcal{SL} = \mathbf{Cm}\mathcal{V}_{A(P_1+L_1)}$ is the variety $\mathcal{CS}$ of commutative semigroups.

It is easy to see that by left-normal law (2.3) and associativity, a linear regular identity $p \approx q$ is true in the variety $\mathcal{LN}$ or $\mathcal{V}_{A(P_1+L_2)}$ if and only if terms p and q have the same leftmost variable. This implies that $\mathbf{Cm}\mathcal{LN} = \mathbf{Cm}\mathcal{V}_{A(P_1+L_2)}$ is the variety $\mathcal{LS}$ of left-normal semigroups.

Moreover, the variety $\mathcal{U}_1$ satisfies the associative law. This implies that each linear identity in $\mathcal{U}_1$ is a consequence of the identity $xy \approx xz$ and associativity. Hence $\mathbf{Cm}\mathcal{U}_1$ coincides with the variety $\mathcal{US}$ defined by $xy \approx xz$ and associativity.

Now let us consider the following linear identity:

$$x(yz) \approx z(yx). \tag{4.1}$$

It is not difficult to see that the identity (3.1) is a consequence of entropicity, left-normality and the identity (4.1):

$$x(y(zt)) \overset{(4.1)}{\approx} zt(yx) \overset{(2.3)}{\approx} z(yx)t \overset{(4.1)}{\approx} x(yz)t \overset{(2.3)}{\approx} xt(yz) \overset{(E)}{\approx} xy(tz).$$

Let Γ be the following set of identities:

$$\Gamma : \begin{cases} (E) & xy(zt) \approx xz(yt), \\ (2.3) & xyz \approx xzy, \\ (4.1) & x(yz) \approx z(yx). \end{cases}$$

Lemma 4.4. *For $n \geq 1$, the identity*

$$x_1y_1(y_2x_2)\dots(y_nx_n) \approx x_{\pi(1)}y_{\delta(1)}(y_{\delta(2)}x_{\pi(2)})\dots(y_{\delta(n)}x_{\pi(n)})$$

follows from the set Γ for any permutations π and δ of the set $\{1, 2, \dots, n\}$.

PROOF: This follows directly by (3.5) and (4.1). □

Theorem 4.5. *Any linear identity holding in the variety $\mathcal{V}_{A(P_2+L_1)}$ is a consequence of the set Γ.*

PROOF: Using the same methods as in proofs of Theorem 3.4 and Proposition 3.5 we obtain that each linear regular identity true in $\mathcal{V}_{A(P_2+L_1)}$ is one of the two following types:

$$x_1y_1\dots y_k(y_{k+1}x_2)\dots(y_{n+k}x_{n+1}) \approx$$
$$x_{\delta(1)}y_{\pi(1)}\dots y_{\pi(k)}(y_{\pi(k+1)}x_{\delta(2)})\dots(y_{\pi(n+k)}x_{\delta(n+1)}),$$

for any permutation π on the set $\{1,\ldots,n+k\}$ and any permutation δ on the set $\{1,\ldots,n+1\}$, or

$$(x_1(y_1x_2\ldots x_k))(y_2x_{k+1})\ldots(y_{n+1}x_{k+n})\approx$$
$$(x_{\delta(1)}(y_{\pi(1)}x_{\delta(2)}\ldots x_{\delta(k)}))(y_{\pi(2)}x_{\delta(k+1)})\ldots(y_{\pi(n+1)}x_{\delta(k+n)}),$$

for any permutation π on the set $\{1,\ldots,n+1\}$ and any permutation δ on the set $\{1,\ldots,n+k\}$.

By identities (3.5), (3.13), (4.1) and (4.2), both follow from Γ. $\square$

Similarly, we can show that any linear identity true in $\mathcal{V}_{A(P_2)}$ is also a consequence of Γ. All these observations prove the following proposition.

Proposition 4.6. *There are eight varieties of power entropic graph algebras:*

- $\mathbf{Cm}\mathcal{V}_{A(P_2+L_2)}=\mathbf{Cm}\mathcal{V}_{A(P_3)}=\mathbf{Cm}\mathcal{U}_2=\mathbf{Cm}\mathcal{V}_{A(P_3+L_1)}=\mathrm{Mod}(\Sigma)$
- $\mathbf{Cm}\mathcal{V}_{A(P_2+L_1)}=\mathbf{Cm}\mathcal{V}_{A(P_2)}=\mathrm{Mod}(\Gamma)$
- $\mathbf{Cm}\mathcal{LN}=\mathbf{Cm}\mathcal{V}_{A(P_1+L_2)}=\mathcal{LS}=\mathrm{Mod}(xyz\approx xzy, xyz\approx x(yz))$
- $\mathbf{Cm}\mathcal{SL}=\mathbf{Cm}\mathcal{V}_{A(P_1+L_1)}=\mathcal{CS}=\mathrm{Mod}(xy\approx yx, xyz\approx x(yz))$
- $\mathbf{Cm}\mathcal{U}_1=\mathcal{US}=\mathrm{Mod}(xy\approx xz, xyz\approx x(yz))$
- $\mathbf{Cm}\mathcal{V}_{A(P_1)}=\mathcal{V}_{A(P_1)}=\mathrm{Mod}(xy\approx uz)$
- $\mathbf{Cm}\mathcal{LZ}=\mathcal{LZ}=\mathrm{Mod}(xy\approx x)$
- $\mathbf{Cm}\mathcal{T}=\mathcal{T}=\mathrm{Mod}(x\approx y)$

The lattice of all subvarieties of $\mathbf{Cm}\mathcal{V}_{A(P_2+L_2)}$ *is given in Figure 2:*

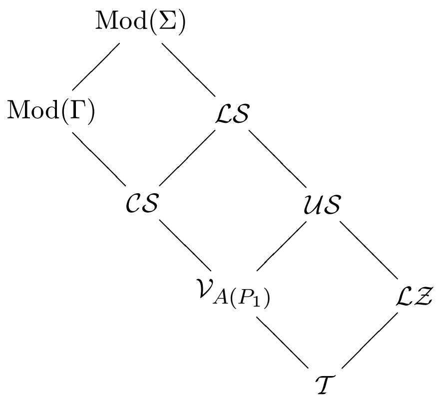


Figure 2

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