A continuous operator extending ultrametrics

I. Stasyuk, E.D. Tymchatyn

Abstract. The problem of continuous simultaneous extension of all continuous partial ultrametrics defined on closed subsets of a compact zero-dimensional metric space was recently solved by E.D. Tymchatyn and M. Zarichnyi and improvements to their result were made by I. Stasyuk. In the current paper we extend these results to complete, bounded, zero-dimensional metric spaces and to both continuous and uniformly continuous partial ultrametrics.

Keywords: ultrametric, space of partial ultrametrics, continuous extension operator

Classification: 54E35, 54C20, 54E40

1. Introduction

The theory of extension of metrics develops in parallel with the theory of extension of functions. A counterpart for Tietze's extension theorem of continuous real-valued functions is Hausdorff's theorem which states that a metric which generates the topology of a closed subspace of a metrizable topological space can be extended to a continuous metric on the whole space. Dugundji [9] showed that if A is a closed subset of a metric space then there is a continuous linear extension operator $\varphi: C^*(A) \to C^*(X)$ where $C^*(A)$ is the family of bounded continuous real-valued functions on A. The analogous problem of linear extensions for (pseudo)metrics was formulated and solved for some special cases by C. Bessaga [5]. The complete solution was obtained by T. Banakh [1], see also [2]. Further generalizations of known results in the theory of extensions are related to existence of operators which simultaneously extend functions (pseudometrics) defined on variable domains. H. P. Kunzi and L. Shapiro [11] solved the problem of simultaneous linear continuous extension of uniformly continuous real-valued functions defined on the family of compact subsets of a metric space. Recently E. D. Tymchatyn and M. Zarichnyi [16] proved that there exists a continuous linear operator which simultaneously extends continuous pseudometrics defined on closed subsets of a compact metric space. A similar problem was also considered by the same authors [17] for continuous ultrametrics defined on the family of closed subsets of a compact zero-dimensional metric space (see also [14]). In this note we generalize the results from [14] and [17] for partial ultrametrics to the class of complete metric spaces.

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2. Preliminaries

Recall that a metric r on a set Y is called an ultrametric if it satisfies the strong triangle inequality

$$r(x,y) \le \max\{r(x,z), r(y,z)\}$$

for all $x,y,z \in Y$. It is known [8] that a metric space (Y,r) admits an ultrametric compatible with its topology if and only if $\dim Y = 0$. By $\exp Y$ we denote the set of all bounded closed subsets of (Y,r) equipped with the Hausdorff metric H_r.

Let (X,d) be a bounded complete ultrametric space. It is known that the Hausdorff metric generated by an ultrametric is an ultrametric as well (see for instance [10]). Therefore, the space $(\exp X, H_d)$ is an ultrametric space. For every $A \in \exp X$ with $|A| \ge 2$ denote by $\mathcal{UM}(A)$ (respectively $\mathcal{UM}_{\mathrm{u}}(A)$) the family of all bounded continuous (respectively bounded uniformly continuous) ultrametrics defined on A. These sets are closed under the operations of multiplying by a positive constant and taking pointwise maximum of two ultrametrics. Let

$$\mathcal{UM} = \bigcup\{\mathcal{UM}(A) : A \in \exp X,\ |A| \ge 2\}$$

and

$$\mathcal{UM}_{\mathrm{u}} = \bigcup\{\mathcal{UM}_{\mathrm{u}}(A) : A \in \exp X,\ |A| \ge 2\}.$$

We will write $\operatorname{dom}\rho = A$ if $\rho \in \mathcal{UM}(A)$. Suppose that each ultrametric $\rho \in \mathcal{UM}$ is identified with its graph

$$\Gamma_\rho = \{(x,y,\rho(x,y)) : x,y \in \operatorname{dom}\rho\}$$

which is a closed and bounded subset of the space $X \times X \times \mathbb{R}$. We assume that the set $\exp(X \times X \times \mathbb{R})$ is equipped with the Hausdorff metric $H_{d'}$ generated by the metric d' on $X \times X \times \mathbb{R}$ where

$$d'[(x,y,z),(x_1,y_1,z_1)] = \max\{d(x,x_1), d(y,y_1), |z-z_1|\}$$

for $x,y,x_1,y_1 \in X$ and $z,z_1 \in \mathbb{R}$. Thus $\mathcal{UM}$ can be viewed as a subspace of the space $(\exp(X \times X \times \mathbb{R}), H_{d'})$ and $\mathcal{UM}_{\mathrm{u}}$ is a subspace of $\mathcal{UM}$. For every $\rho \in \mathcal{UM}$ the norm of ρ is defined by $\|\rho\| = \max\{\rho(x,y) : x,y \in \operatorname{dom}\rho\}$.

Let $K = \{(x,A) \in X \times \exp X : x \in A\}$ and let σ be the ultrametric on the set $X \times \exp X$ defined as $\sigma((x,A),(y,B)) = \max\{d(x,y), H_d(A,B)\}$ for all $(x,A),(y,B) \in X \times \exp X$. Thus K is a closed subset of $X \times \exp X$. Denote by $S((x,A),t)$ the open ball of radius $t > 0$ centered at (x,A) in the metric space $(X \times \exp X, \sigma)$. For $j \in \{1,2\}$ let pr_j denote the jth coordinate projection in $X \times \exp X$. Let $\mathbb{N}_+$ stand for the set of all positive integers and let $\mathbb{R}^{X\times X}$ denote the set of all real-valued functions on $X \times X$.

3. Main results

In the following lemma we establish an auxiliary result which is a "uniformly continuous" analogue of a selection theorem due to M. Choban [6] (see also R. Engelking, R. Heath and E. Michael [7]).

Lemma. *For a complete ultrametric space X there exists a uniformly continuous function $f\colon X \times \exp X \to X$ such that $f(x,A) \in A$ for every $x \in X$, $A \in \exp X$ and $f(x,A) = x$ whenever $x \in A$.*

Proof: For every $i \in \mathbb{N}_+$ let

$$\mathcal{V}_i = \{S((x,A),1/i) : (x,A) \in X \times \exp X\}.$$

Since in every ultrametric space two balls of the same radius either coincide or have empty intersection, the members of $\mathcal{V}_i$ are pairwise disjoint. For each $V \in \mathcal{V}_1$ choose $A_V \in \exp X$ such that $(X \times \{A_V\}) \cap V \neq \emptyset$ and such that $V \cap K \neq \emptyset$ implies $(X \times \{A_V\}) \cap V \cap K \neq \emptyset$. Choose $z_V \in A_V$ such that if $V \cap K \neq \emptyset$ then $(z_V, A_V) \in V$. Note that if $A \in \exp X$ so $(X \times \{A\}) \cap V \neq \emptyset$ there exists $z \in A$ such that (z,A) and (z_V, A_V) are in the same element of $\mathcal{V}_1$. Define a map $s_1\colon X \times \exp X \to K$ by setting $s_1(V) = \{(z_V, A_V)\}$ for each $V \in \mathcal{V}_1$. Then s_1 is a well-defined uniformly continuous function since elements of $\mathcal{V}_1$ are pairwise disjoint and if $(x,A),(y,B) \in X \times \exp X$ are such that $\sigma((x,A),(y,B)) < 1$ then (x,A) and (y,B) lie in the same element V of $\mathcal{V}_1$, so $s_1(x,A) = s_1(y,B) = (z_V, A_V)$. Note also that for $(x,A) \in K$ we have $\sigma(s_1(x,A),(x,A)) < 1$ and for $(x,A) \in X \times \exp X$ we have $H_d(\mathrm{pr}_2(s_1(x,A)),A) < 1$.

Suppose $1 < i \in \mathbb{N}_+$ and for each $j \in \{1,\dots,i-1\}$, $s_j\colon X \times \exp X \to K$ is a uniformly continuous function such that if $1 \leq j < k \leq i-1$ then $\sigma(s_j(x,A), s_k(x,A)) < 1/j$ and $H_d(\mathrm{pr}_2(s_k(x,A)),A) < 1/k$ for each $(x,A) \in X \times \exp X$. Also, for each $V \in \mathcal{V}_{i-1}$ suppose we have $s_{i-1}(V) = \{(z_V, A_V)\} \subset K$ with $V \cap (X \times \{A_V\}) \neq \emptyset$ and if $V \cap K \neq \emptyset$ then $(z_V, A_V) \in V \cap K$.

Let $U \in \mathcal{V}_i$. Then $U \subset V$ for some unique $V \in \mathcal{V}_{i-1}$. Choose $A_U \in \exp X$ such that $(X \times \{A_U\}) \cap U \neq \emptyset$ and such that $(X \times \{A_U\}) \cap K \neq \emptyset$ if $U \cap K \neq \emptyset$. Then $H_d(A_U, A_V) < 1/(i-1)$. Let $z_U \in A_U$ so $d(z_U, z_V) < 1/(i-1)$ and $(z_U, A_U) \in U$ if $U \cap K \neq \emptyset$. Then $(z_U, A_U) \in S((z_V, A_V), 1/(i-1))$. Define $s_i\colon X \times \exp X \to K$ by setting $s_i(U) = \{(z_U, A_U)\}$ for each $U \in \mathcal{V}_i$. Then, as above, s_i is a uniformly continuous function and for $1 \leq j < i$ we obtain $\sigma(s_j(x,A), s_i(x,A)) < 1/j$ and $H_d(\mathrm{pr}_2(s_i(x,A)),A) < 1/i$ for each $(x,A) \in X \times \exp X$. Also, $\sigma(s_i(x,A),(x,A)) < 1/i$ for every $(x,A) \in K$.

By induction $s_i\colon X \times \exp X \to K$ is defined for each $i \in \mathbb{N}_+$ as above. Let $s\colon X \times \exp X \to K$ be the uniform limit of the Cauchy sequence $\{s_i\}$. Then s is a uniformly continuous retraction of $X \times \exp X$ onto K such that $\mathrm{pr}_2(s(x,A)) = A$ for each $(x,A) \in X \times \exp X$ since K is closed in the complete space $X \times \exp X$.

Let $f\colon X \times \exp X \to X$ be defined as $f(x, A) = \mathrm{pr}_1(s(x, A))$ for every $(x, A) \in X \times \exp X$. Since the map s is uniformly continuous, so is f. Hence, f is a uniformly continuous map such that $f(x, A) \in A$ for all $(x, A) \in X \times \exp X$ and $f(x, A) = x$ whenever $x \in A$. □

The completeness of the space X is essential in the above lemma. Note that a metrizable space which admits a continuous selection on the space of its nonempty closed bounded subsets must be completely metrizable (see [13]). More generally the problem of existence of uniformly continuous selections for multivalued maps defined on complete ultrametric spaces was considered by the authors in [15].

Theorem 1. *Let (X, d) be a bounded complete ultrametric space. There exists an operator $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ that satisfies the following conditions for every $\rho, \rho_1 \in \mathcal{UM}$ and $c > 0$.*

(1) *$u(\rho)$ is an extension of ρ over X.*
(2) *u is positive-homogeneous i.e. $u(c\rho) = cu(\rho)$.*
(3) *$u(\max\{\rho, \rho_1\}) = \max\{u(\rho), u(\rho_1)\}$ if $\mathrm{dom}\,\rho = \mathrm{dom}\,\rho_1$.*
(4) *$\|u(\rho)\| = \|\rho\|$.*
(5) *If $\{\rho_n\}$ is a sequence in $\mathcal{UM}$ such that $\{\Gamma_{\rho_n}\}$ converges to Γ_ρ for some $\rho \in \mathcal{UM}$ then $\{u(\rho_n)\}$ converges to $u(\rho)$ pointwise on $X \times X$.*

Proof: We will follow the notation from the previous lemma. For every $i \in \mathbb{N}_+$ let $V_i = \bigcup\{U \in \mathcal{V}_i : U \cap K \neq \emptyset\}$. Then V_i is both open and closed in $X \times \exp X$ and $K \subset V_{i+1} \subset V_i$ for every $i \in \mathbb{N}_+$. Then

$$\mathcal{W}_i = \{V_i\} \cup \{V \in \mathcal{V}_i : V \cap K = \emptyset\}$$

is a pairwise disjoint clopen cover of $X \times \exp X$ and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$. Let

$$w_i(\rho)(x, y) = \begin{cases} 1 & \text{if } (x, \mathrm{dom}\,\rho) \text{ and } (y, \mathrm{dom}\,\rho) \text{ lie in distinct elements of } \mathcal{W}_i; \\ 0 & \text{if } (x, \mathrm{dom}\,\rho) \text{ and } (y, \mathrm{dom}\,\rho) \text{ lie in the same element of } \mathcal{W}_i \end{cases}$$

for every $\rho \in \mathcal{UM}$ and $x, y \in X$.

Define a map $u\colon \mathcal{UM} \to \mathbb{R}^{X \times X}$ by the formula

$$u(\rho)(x, y) = \max\left\{\rho(f(x, \mathrm{dom}\,\rho), f(y, \mathrm{dom}\,\rho)), \|\rho\| \max_{i \in \mathbb{N}_+}\left\{\frac{1}{2^i} w_i(\rho)(x, y)\right\}\right\}$$

for $\rho \in \mathcal{UM}$ and $x, y \in X$. We are going to prove that the operator u satisfies conditions (1)–(5). Clearly the function $\rho'\colon X \times X \to \mathbb{R}$ defined by

$$\rho'(x, y) = \rho(f(x, \mathrm{dom}\,\rho), f(y, \mathrm{dom}\,\rho))$$

is a continuous ultrapseudometric on X. Note that ρ' is uniformly continuous if ρ is. Let $w(\rho)\colon X \times X \to \mathbb{R}$ be defined by

$$w(\rho)(x,y) = \max_{i \in \mathbb{N}_+} \left\{ \frac{1}{2^i} w_i(\rho)(x,y) \right\}.$$

Each map $2^{-i} w_i(\rho)$ is an ultrapseudometric on X because if $(x, \operatorname{dom}\rho)$ and $(y, \operatorname{dom}\rho)$ belong to different elements of the cover $\mathcal{W}_i$, that is $w_i(\rho)(x,y) = 1$, then for arbitrary $z \in X$ we have either $w_i(\rho)(x,z) = 1$ or $w_i(\rho)(y,z) = 1$. Note that $w_i(\rho)$ is uniformly continuous because if $d(x,x') < 1/i$ and $d(y,y') < 1/i$ then $(x, \operatorname{dom}\rho), (x', \operatorname{dom}\rho) \in W_1$ and $(y, \operatorname{dom}\rho), (y', \operatorname{dom}\rho) \in W_2$ for some $W_1, W_2 \in \mathcal{W}_i$ and $w_i(\rho)(x,y) = w_i(\rho)(x',y')$. Hence, the map $\|\rho\| w(\rho)$ is a uniformly continuous ultrapseudometric on X.

Now let us show that $u(\rho) \in \mathcal{UM}(X)$ for every $\rho \in \mathcal{UM}$. It is obvious that $u(\rho)$ is a bounded continuous ultrapseudometric on X because it is the maximum of ρ' and $\|\rho\| w(\rho)$, two bounded continuous ultrapseudometrics. Note for later use that if $\rho \in \mathcal{UM}_{\mathrm{u}}$ then $u(\rho)$ is uniformly continuous. Take arbitrary points $x, y \in X$ such that $x \neq y$. If $x, y \in \operatorname{dom}\rho$ then $f(x, \operatorname{dom}\rho) = x$, $f(y, \operatorname{dom}\rho) = y$ and $(x, \operatorname{dom}\rho), (y, \operatorname{dom}\rho) \in V_i$ for all $i \in \mathbb{N}_+$. Therefore, $w_i(\rho)(x,y) = 0$ for all $i \in \mathbb{N}_+$ and we obtain

$$u(\rho)(x,y) = \rho(f(x, \operatorname{dom}\rho), f(y, \operatorname{dom}\rho)) = \rho(x,y) > 0.$$

If $y \notin \operatorname{dom}\rho$ and $x \in X$ then there is a number $i \in \mathbb{N}_+$ such that $d(x,y) > 1/i$ and $\sigma((y, \operatorname{dom}\rho), K) > 1/i$. Hence, $(y, \operatorname{dom}\rho)$ and $(x, \operatorname{dom}\rho)$ lie in disjoint elements of $\mathcal{W}_i$, so $w_i(\rho)(x,y) = 1$. Thus,

$$u(\rho)(x,y) \geq \|\rho\| \frac{1}{2^i} w_i(\rho)(x,y) > 0.$$

Since $u(\rho)(x,y) = \rho(x,y)$ for $x, y \in \operatorname{dom}\rho$, condition (1) of Theorem 1 is satisfied.

Since $\operatorname{dom}(c\rho) = \operatorname{dom}\rho$ and $\|c\rho\| = c\|\rho\|$ for every $c > 0$, the operator u satisfies condition (2).

As in [17] one easily shows that the operator u preserves maxima of ultrametrics with a common domain. So (3) is satisfied.

Since $u(\rho)$ is an extension of ρ, we have $\|u(\rho)\| \geq \|\rho\|$. But from the definition of the extension we see that $\|u(\rho)\| \leq \|\rho\|$. Therefore, (4) is satisfied.

Now let $\{\rho_n\}$ be a sequence of ultrametrics in the space $\mathcal{UM}$ such that $\{\Gamma_{\rho_n}\}$ converges to Γ_ρ in the Hausdorff distance for some $\rho \in \mathcal{UM}$. This implies $\operatorname{dom}\rho_n \to \operatorname{dom}\rho$ in $\exp X$ and $\|\rho_n\| \to \|\rho\|$. First let us prove that $w(\rho_n)$ converges to $w(\rho)$ uniformly on $X \times X$. Choose any $i_0 \in \mathbb{N}_+$. Since $\operatorname{dom}\rho_n$ converges to $\operatorname{dom}\rho$ there is $n_0 \in \mathbb{N}_+$ such that $H_d(\operatorname{dom}\rho, \operatorname{dom}\rho_n) < 1/i_0$ for

$n > n_0$. Let us show that $|w(\rho)(x,y) - w(\rho_n)(x,y)| < 1/2^{i_0}$ for all $x, y \in X$ provided that $n > n_0$. It is clear that $\sigma((x, \operatorname{dom}\rho_n), (x, \operatorname{dom}\rho)) < 1/i_0$ and $\sigma((y, \operatorname{dom}\rho_n), (y, \operatorname{dom}\rho)) < 1/i_0$ for $n > n_0$. This means that

$$(x, \operatorname{dom}\rho), (x, \operatorname{dom}\rho_n) \in W_{i_0} \subset W_{i_0-1} \subset \cdots \subset W_1$$

and

$$(y, \operatorname{dom}\rho), (y, \operatorname{dom}\rho_n) \in W'_{i_0} \subset W'_{i_0-1} \subset \cdots \subset W'_1,$$

where $W_i, W'_i \in \mathcal{W}_i$, $i \in \{1, \ldots, i_0\}$. Therefore, $w_i(\rho_n)(x,y) = w_i(\rho)(x,y)$ for $i \in \{1, \ldots, i_0\}$ for all $x, y \in X$ and $n > n_0$. Thus, we obtain $|w(\rho)(x,y) - w(\rho_n)(x,y)| < 1/2^{i_0}$.

Fix arbitrary $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}_+$ such that for all $n > N_1$ and $x, y \in X$ we have

$$\text{(i)} \qquad \big|\|\rho\| - \|\rho_n\|\big| < \frac{\varepsilon}{2}\,;$$

$$\text{(ii)} \qquad |w(\rho)(x,y) - w(\rho_n)(x,y)| < \frac{\varepsilon}{2\|\rho\|}\,.$$

We obtain from (i), (ii) and the triangle inequality

$$(*) \qquad \big|\|\rho\| w(\rho)(x,y) - \|\rho_n\| w(\rho_n)(x,y)\big| < \varepsilon$$

for every $x, y \in X$ and $n > N_1$. Thus, the sequence $\{\|\rho_n\| w(\rho_n)\}$ converges uniformly to $\|\rho\| w(\rho)$ on $X \times X$.

Now let $(x_0, y_0) \in X \times X$ be fixed. Since ρ is continuous at the point $(f(x_0, \operatorname{dom}\rho), f(y_0, \operatorname{dom}\rho))$ there is $\delta > 0$ such that

$$\text{(iii)} \qquad |\rho(a,b) - \rho(f(x_0, \operatorname{dom}\rho), f(y_0, \operatorname{dom}\rho))| < \frac{\varepsilon}{2}$$

for every $a, b \in \operatorname{dom}\rho$ with $d(a, f(x_0, \operatorname{dom}\rho)) < \delta$ and $d(b, f(y_0, \operatorname{dom}\rho)) < \delta$. Since $\operatorname{dom}\rho_n \to \operatorname{dom}\rho$ and f is (uniformly) continuous, there is $N_2 \in \mathbb{N}_+$ such that

$$\text{(iv)} \qquad d(f(x_0, \operatorname{dom}\rho), f(x_0, \operatorname{dom}\rho_n)) < \delta,\ d(f(y_0, \operatorname{dom}\rho), f(y_0, \operatorname{dom}\rho_n)) < \delta$$

for all $n > N_2$.

Since $\Gamma_{\rho_n} \to \Gamma_\rho$, there is $N_3 \in \mathbb{N}_+$ such that for every $n > N_3$ there exist $a_n, b_n \in \operatorname{dom}\rho$ with

$$\text{(v)} \qquad d(a_n, f(x_0, \operatorname{dom}\rho_n)) < \delta,\ d(b_n, f(y_0, \operatorname{dom}\rho_n)) < \delta$$

and

$$|\rho(a_n, b_n) - \rho_n(f(x_0, \operatorname{dom}\rho_n), f(y_0, \operatorname{dom}\rho_n))| < \frac{\varepsilon}{2}\,.$$

Using (iv), (v) and the strong triangle inequality we obtain

$$d(f(x_0, \operatorname{dom}\rho), a_n) < \delta \text{ and } d(f(y_0, \operatorname{dom}\rho), b_n) < \delta.$$

Therefore,

$$\begin{aligned}
&|\rho(f(x_0, \operatorname{dom}\rho), f(y_0, \operatorname{dom}\rho)) - \rho_n(f(x_0, \operatorname{dom}\rho_n), f(y_0, \operatorname{dom}\rho_n))| \\
&\leq |\rho(f(x_0, \operatorname{dom}\rho), f(y_0, \operatorname{dom}\rho)) - \rho(a_n, b_n)| \\
&\quad + |\rho(a_n, b_n) - \rho_n(f(x_0, \operatorname{dom}\rho_n), f(y_0, \operatorname{dom}\rho_n))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This together with $(*)$ implies

$$|u(\rho)(x_0, y_0) - u(\rho_n)(x_0, y_0)| < \varepsilon$$

for all $n > \max\{N_1, N_2, N_3\}$. Thus, condition (5) is also satisfied. □

The following theorem establishes some properties of the operator u applied to extension of uniformly continuous ultrametrics.

Theorem 2. *There exists an operator $v{:}\,\mathcal{UM}_{\mathrm{u}} \to \mathcal{UM}_{\mathrm{u}}(X)$ which has properties* (1), (2), (3) *and* (4) *from Theorem* 1. *Moreover, if $\{\rho_n\}$ is a sequence in $\mathcal{UM}_{\mathrm{u}}$ such that $\{\Gamma_{\rho_n}\}$ converges to Γ_ρ for some $\rho \in \mathcal{UM}_{\mathrm{u}}$ then $\{v(\rho_n)\}$ converges to $v(\rho)$ uniformly on $X \times X$.*

Proof: Let v be the restriction of the operator u from Theorem 1 onto the set $\mathcal{UM}_{\mathrm{u}}$. Then $v(\mathcal{UM}_{\mathrm{u}}) \subset \mathcal{UM}(X)$. Since ρ is uniformly continuous, so is $v(\rho)$ for every $\rho \in \mathcal{UM}_{\mathrm{u}}$ as we remarked in the proof of Theorem 1. Since u satisfies conditions (1)–(4) of Theorem 1, so does its restriction v.

Consider a sequence $\{\rho_n\}$ in the space $\mathcal{UM}_{\mathrm{u}}$ which converges to $\rho \in \mathcal{UM}_{\mathrm{u}}$, that is $\{\Gamma_{\rho_n}\} \to \Gamma_\rho$. Then $\operatorname{dom}\rho_n \to \operatorname{dom}\rho$ in $\exp X$ and $\|\rho_n\| \to \|\rho\|$. Let $\tilde{\rho}_n = v(\rho)|_{\operatorname{dom}\rho_n \times \operatorname{dom}\rho_n}$. Then $H_{d'}(\Gamma_{\rho_n}, \Gamma_{\tilde{\rho}_n}) \to 0$ since $v(\rho)$ is uniformly continuous and $\operatorname{dom}\tilde{\rho}_n = \operatorname{dom}\rho_n \to \operatorname{dom}\rho$.

Fix arbitrary $\varepsilon > 0$. There exists $N \in \mathbb{N}_+$ such that for all $n > N$ and $x, y \in X$ we have

(i) $H_{d'}(\Gamma_{\rho_n}, \Gamma_{\tilde{\rho}_n}) < \frac{\varepsilon}{2}$;
(ii) $|\rho(f(x, \operatorname{dom}\rho), f(y, \operatorname{dom}\rho)) - v(\rho)(f(x, \operatorname{dom}\rho_n), f(y, \operatorname{dom}\rho_n))| < \frac{\varepsilon}{2}$ because $v(\rho)$ and f are uniformly continuous and $v(\rho) = \rho$ on $\operatorname{dom}\rho \times \operatorname{dom}\rho$;
(iii) $|\|\rho\| w(\rho)(x, y) - \|\rho_n\| w(\rho_n)(x, y)| < \frac{\varepsilon}{2}$ (this condition follows from the proof of Theorem 1).

Now, for every $x, y \in X$ and $n > N$ we obtain

$$\begin{aligned}
&|\rho(f(x, \operatorname{dom}\rho), f(y, \operatorname{dom}\rho)) - \rho_n(f(x, \operatorname{dom}\rho_n), f(y, \operatorname{dom}\rho_n))| \\
&\le |\rho(f(x, \operatorname{dom}\rho), f(y, \operatorname{dom}\rho)) - \tilde{\rho}_n(f(x, \operatorname{dom}\rho_n), f(y, \operatorname{dom}\rho_n))| \\
&\quad + |\tilde{\rho}_n(f(x, \operatorname{dom}\rho_n), f(y, \operatorname{dom}\rho_n)) - \rho_n(f(x, \operatorname{dom}\rho_n), f(y, \operatorname{dom}\rho_n))| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Together with condition (iii) this implies $|v(\rho)(x, y) - v(\rho_n)(x, y)| < \varepsilon$ for all $n > N$ and $x, y \in X$. □

The next two corollaries are analogues for ultrametrics of Dugundji's theorems on extension of continuous bounded real-valued functions [9]. We remark that on $\mathcal{UM}(A)$ the topology of pointwise convergence is weaker than the topology of convergence of graphs in the Hausdorff metric and this is in turn weaker than the topology of uniform convergence. On $\mathcal{UM}_{\mathrm{u}}(A)$ the latter two topologies coincide.

Corollary 1. *The restriction $u|_{\mathcal{UM}(A)}: \mathcal{UM}(A) \to \mathcal{UM}(X)$ of the operator u from Theorem 1 is continuous with respect to the uniform convergence topology on $\mathcal{UM}(A)$ and on $\mathcal{UM}(X)$ for every $A \in \exp X$ with $|A| \ge 2$.*

Proof: Suppose that $A \in \exp X$ and $\{\rho_n\}$ is a sequence in $\mathcal{UM}(A)$ which converges uniformly to some $\rho \in \mathcal{UM}(A)$ on $A \times A$. Since $\operatorname{dom}\rho_n = \operatorname{dom}\rho = A$ for all n, it follows from the definition of the map w that $w(\rho)(x, y) = w(\rho_n)(x, y)$ for every $x, y \in X$ and $n \in \mathbb{N}_+$. Also for every $\varepsilon > 0$ there is $N \in \mathbb{N}_+$ such that $|\rho(f(x, A), f(y, A)) - \rho_n(f(x, A), f(y, A))| < \varepsilon$ for all $n > N$. This implies $|u(\rho)(x, y) - u(\rho_n)(x, y)| < \varepsilon$ for all $n > N$, that is, $\{u(\rho_n)\}$ converges uniformly to $u(\rho)$ on $X \times X$. □

Corollary 2. *The restriction $u|_{\mathcal{UM}(A)}: \mathcal{UM}(A) \to \mathcal{UM}(X)$ of the operator u from Theorem 1 is continuous with respect to the pointwise convergence topology on $\mathcal{UM}(A)$ and on $\mathcal{UM}(X)$ for every $A \in \exp X$ with $|A| \ge 2$.*

The proof is similar to that of Corollary 1.

4. Operators preserving Assouad dimension

In this section we construct an extension operator for partial ultrametrics in a separable zero-dimensional metric space which possesses in addition to properties (1)–(5) stated in Theorem 1 the additional property of preserving Assouad dimension. This result is already known for compact zero-dimensional spaces [17]. We recall the notion of the Assouad dimension of a metric space.

Definition. Let $\alpha, \beta \ge 0$. A metric space (Y, r) is called (α, β)-homogeneous if for every $a, b > 0$ and $B \subset Y$ such that $a \le r(x, y) \le b$ whenever $x, y \in B$ and $x \ne y$, we have $|B| \le \alpha(b/a)^\beta$. The space (Y, r) is called β-homogeneous if it is

(α,β)-homogeneous for some $\alpha \geq 0$. We define the Assouad dimension $\dim_A(Y,r)$ of the space (Y,r) as

$$\dim_A(Y,r) = \inf\{\beta \geq 0 : (Y,r) \text{ is } \beta \text{ homogeneous }\}.$$

The Assouad dimension of a non-separable ultrametric space is infinite (see [12]). The proof of our last theorem is some adaptation of proofs from [14] and [17].

Theorem 3. *Let (X,d) be a separable complete ultrametric space. Then there exists a map $v\colon \mathcal{UM}_{\mathrm{u}} \to \mathcal{UM}_{\mathrm{u}}(X)$ that satisfies the following conditions for every $\rho, \rho_1 \in \mathcal{UM}$ and $c > 0$.*

(1) *$v(\rho)$ is an extension of ρ over X.*
(2) *$v(c\rho) = cv(\rho)$.*
(3) *$v(\max\{\rho,\rho_1\}) = \max\{v(\rho), v(\rho_1)\}$ if $\operatorname{dom}\rho = \operatorname{dom}\rho_1$.*
(4) *$\|v(\rho)\| = \|\rho\|$.*
(5) *v is a continuous map with respect to the Hausdorff metric topology on $\mathcal{UM}_{\mathrm{u}}$ and on $\mathcal{UM}_{\mathrm{u}}(X)$.*
(6) *$\dim_A(X, v(\rho)) = \dim_A(\operatorname{dom}\rho, \rho)$.*

Proof: There exists an ultrametric r on the standard Cantor set C such that $\dim_A(C,r) = 0$ (see [17]). Let e be a uniformly continuous embedding of (X,d) into (C,r). For every $A \in \exp X$ let $\tilde{e}(A) = \overline{e(A)} \in \exp C$. Let $K = \{(x,A) \in X \times \exp X : x \in A\}$ and $\tilde{K} = \{(y,B) \in C \times \exp C : y \in B\}$. Then K and $\tilde{K}$ are closed subsets of $X \times \exp X$ and $C \times \exp C$, respectively. If $(x,A) \notin K$ that is $x \notin A$ then $e(x) \notin \overline{e(A)}$ and $(e(x), \tilde{e}(A)) \notin \tilde{K}$. We also have $\tilde{e}(A) \neq \tilde{e}(B)$ for distinct A and B from $\exp X$. Consider the quotient space $(C \times \exp C)/\tilde{K}$. Since $C \times \exp C$ is metrizable, compact and zero-dimensional, so is the quotient space $(C \times \exp C)/\tilde{K}$. Denote by q the quotient map acting from $C \times \exp C$ to $(C \times \exp C)/\tilde{K}$. Since $C \times \exp C$ is compact, q is uniformly continuous (we assume that there is a fixed metric which is compatible with the topology of $(C \times \exp C)/\tilde{K}$). Since the Cantor set is topologically homogeneous, there exists a uniformly continuous one to one function $s\colon (C \times \exp C)/\tilde{K} \to C$ with $s(q(\tilde{K})) = 0$. Let $g\colon X \times \exp X \to C$ be defined as $g = s \circ q \circ (e \times \tilde{e})$. We are going to establish some properties of the map g. First note that since the maps $e, \tilde{e}, q$ and s are uniformly continuous, so is g. If $(x,A) \in K$ then $(e(x), \tilde{e}(A)) \in \tilde{K}$ and consequently $g(x,A) = 0$. For distinct points $(x,A), (y,B) \in X \times \exp X$ which are not both in K we have $g(x,A) \neq g(x,B)$ because $(e \times \tilde{e})$ and s are one to one and q is one to one outside $\tilde{K}$ (we also use the fact that $(e(z), \tilde{e}(D)) \notin \tilde{K}$ whenever $(z,D) \notin K$ for every $(z,D) \in X \times \exp X$).

Define a map $v\colon \mathcal{UM}_{\mathrm{u}} \to \mathbb{R}^{X \times X}$ by the formula

$$v(\rho)(x,y) = \max\{\rho(f(x,\operatorname{dom}\rho), f(y,\operatorname{dom}\rho)), \|\rho\| r(g(x,\operatorname{dom}\rho), g(y,\operatorname{dom}\rho))\}$$

where $f\colon X \times \exp X \to K$ is the retraction defined in Theorem 1, $\rho \in \mathcal{UM}_{\mathrm{u}}$, $x, y \in X$. Since $v(\rho)$ is a composition of uniformly continuous maps, it is uniformly continuous. This construction is an analogue of a construction used in [17] for the compact case and the proof of properties (1)–(6) is similar to that of the main result from [17] (see also [14]). $\square$

5. Remarks

Let (X, d) be an ultrametric space and let $(\tilde{X}, \tilde{d})$ be its completion (which is also an ultrametric space). If A is a closed subset of X then the closure $\bar{A}$ of A in $\tilde{X}$ is its completion. So every uniformly continuous ultrapseudometric on A extends to an ultrapseudometric on $\bar{A}$. Hence, Theorems 2 and 3 are valid for not complete ultrametric spaces except that the extension operator in general maps partial uniformly continuous ultrametrics to uniformly continuous ultrapseudometrics on X. We do not know whether the completeness assumption of the space X is essential for the statement of Theorem 1 but it is used in our selection lemma.

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Department of Mechanics and Mathematics, Ivan Franko National University of Lviv, 79000 Universytetska St. 1, Lviv, Ukraine

E-mail: i_stasyuk@yahoo.com

Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon SK, S7N5E6, Canada

E-mail: tymchat@math.usask.ca

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