On π-metrizable spaces, their continuous images and products

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Abstract. A space X is said to be π-metrizable if it has a σ-discrete π-base. The behavior of π-metrizable spaces under certain types of mappings is studied. In particular we characterize strongly d-separable spaces as those which are the image of a π-metrizable space under a perfect mapping. Each Tychonoff space can be represented as the image of a π-metrizable space under an open continuous mapping. A question posed by Arhangel'skii regarding if a π-metrizable topological group must be metrizable receives a negative answer.

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1. Introduction

By $\mathbb{N}$ we mean the set of all natural numbers. Recall that for a space X, a collection of nonempty open sets Θ, is called a π-base if for every nonempty open set O, there exists $U \in \Theta$ such that $U \subset O$. Recall that for a Tychonoff space X, $\pi w(X)$ is defined to be the least cardinal τ such that X has a π-base Γ with $|\Gamma| = \tau$. Recall that a collection of sets Γ π-refines a collection of sets Θ if for each $O \in \Theta$ there exists $U \in \Gamma$ such that $U \subset O$ and $\emptyset \notin \Gamma$. It is clear that π-metrizability is preserved by open subspaces, closures of open subspaces, and dense subspaces. A space X is said to be weakly π-metrizable if it has a σ-disjoint π-base. Weak π-metrizability is preserved by open subspaces, closures of open subspaces, and dense subspaces in both directions. Some examples of π-metrizable spaces are: $\beta\mathbb{N}$, the Sorgenfrey Line and $K^{\aleph_1}$ where K is uncountable discrete (as we shall later see). The space $[0,1]^2_\tau$ where τ is the topology induced by lexicographic ordering is one of many weakly π-metrizable, not π-metrizable spaces. Recall that a space X is called d-separable if there exists $\{K_n : n \in \mathbb{N}\}$ such that each K_n is a discrete (in itself) subset of X and $\bigcup\{K_n : n \in \mathbb{N}\}$ is dense in X. For more on d-separable spaces see [2]. A space X is called strongly d-separable if there exists $\{K_n : n \in \mathbb{N}\}$ such that each K_n is a closed discrete subset of X and $\bigcup\{K_n : n \in \mathbb{N}\}$ is dense in X.

A σ-discrete π-base was first observed as a necessary condition for being the absolute of a metrizable space (see [7]). First countable spaces with σ-disjoint

π-bases (weakly π-metrizable) were studied by H.E. White in [8]. In this paper he has also shown that a first countable space has a dense metrizable subspace if and only if it is π-metrizable. Also Fearnley has constructed a Moore space with a σ-discrete π-base which does not densely embed into any Moore space having the Baire property [5]. This paper will be an attempt to examine the behavior of π-metrizable spaces under products and mappings.

All spaces are assumed to be Tychonoff.

2. Continuous mappings

Lemma 2.1. *Every locally finite collection of open sets in a space X has a discrete π-refinement (of open sets) of the same cardinality if the collection is infinite.*

Proof: Let Ψ be a locally finite collection of nonempty open sets in X. For each $O \in \Psi$ choose $x_O \in O$. Put $F = \{x_O : O \in \Psi\}$. Well order F: that is for some cardinal κ write $F = \{x_\alpha : \alpha < \kappa\}$ where the indexing is faithful. Clearly F is closed and discrete, thus there exists an open set U_α such that $\mathrm{cl}(U_\alpha) \cap F = \{x_\alpha\}$ and $W_\alpha \subset \bigcap\{V \in \Psi : x_O \in V\}$ for each $\alpha < \kappa$. Put $\Gamma = \{U_\alpha : \alpha < \kappa\}$. Then clearly Γ is a π-refinement of Ψ. Now Γ is also locally finite so the set $V_\alpha = U_\alpha \setminus \bigcup\{\mathrm{cl}(U_\beta) : \beta < \alpha\}$ is an open set containing x_α. Thus $\{V_\alpha : \alpha < \kappa\}$ is disjoint and locally finite. Finally use regularity to choose an open set H_α such that $x_\alpha \in H_\alpha$ and $\mathrm{cl}(H_\alpha) \subset V_\alpha$. Then $\{H_\alpha : \alpha < \kappa\}$ is a discrete π-refinement of Ψ and it has of course the same cardinality. $\square$

It is well known from metrizability criterion that the existence of a σ-locally finite base is equivalent to existence of a σ-discrete base. Analogous to this is the following result.

Theorem 2.2. *A space X is π-metrizable if and only if it has a σ-locally finite π-base.*

Proof: This follows from Lemma 2.1 and the fact that a π-refinement of a π-base is a π-base. $\square$

A collection of sets Γ in a space X each with nonempty interior is called a π_*-base if for each open set O there exists $B \in \Gamma$ with $B \subset O$. It is typically clear that the existence of a π_*-base with a finiteness type property implies the existence of a π-base with the same property.

Proposition 2.3. *Open perfect mappings preserve π-metrizability.*

Proof: Let $f : X \longrightarrow Y$ be perfect onto and open and X be π-metrizable. Let $\bigcup\{\Psi_n : n \in \mathbb{N}\}$ be a π-base for X with each Ψ_n discrete. For each set $B \in \bigcup\{\Psi_n : n \in \mathbb{N}\}$ there exists a closed set $C_B \subset B$ with nonempty interior (using regularity). Then $\{C_B : B \in \bigcup\{\Psi_n : n \in \mathbb{N}\}\}$ is a π_*-base and it is of course σ-discrete. Since f is closed and open, $\{f(C_B) : B \in \bigcup\{\Psi_n : n \in \mathbb{N}\}\}$ is a π_*-base for Y consisting of closed sets. Let us show that this collection is σ-locally finite. $\bigcup\{C_B : B \in \Psi_n\}$ is the union of a discrete collection of closed sets so it is closed. Let $y \in Y$. The set $f^{-1}(y)$ is compact so $(\bigcup\{C_B : B \in \Psi_n\}) \cap f^{-1}(y)$ is compact. But Ψ_n is an open cover of this set. So we have a finite subcover. But the cover is pairwise disjoint, so $f^{-1}(y)$ must intersect only finitely many elements of Ψ_n and thus of $\{C_B : B \in \Psi_n\}$. Let $H = \{C_B : B \in \Psi_n \text{ and } f^{-1}(y) \cap C_B = \emptyset\}$ and let $Z = \bigcup H$. Since H is a discrete collection of closed sets, Z is closed and $f^{-1}(y) \cap Z = \emptyset$. Thus $Y \backslash f(Z)$ is an open set containing y and intersecting only finitely many elements of $\{f(C_B) : B \in \Psi_n\}$ (only those not in H). Therefore $\{f(C_B) : B \in \Psi_n\}$ is locally finite and so Y has a σ-locally finite π_*-base and thus is π-metrizable. □

Corollary 2.4. *If $X \times Y$ is π-metrizable and Y is compact then X is π-metrizable.*

Proof: The projection map $\pi : X \times Y \longrightarrow X$ is perfect and open so this follows by Proposition 2.3. □

Proposition 2.5. *Irreducible perfect mappings preserve π-metrizability in both directions.*

Proof: Let $f : X \longrightarrow Y$ be perfect, onto and irreducible and X be π-metrizable. Let $\bigcup\{\Psi_n : n \in \mathbb{N}\}$ be a π-base for X with each Ψ_n discrete. Take as a π-base in Y, the family $\bigcup\{\Gamma_n : n \in \mathbb{N}\}$ where $\Gamma_n = \{Y \backslash f(X \backslash B) : B \in \Psi_n\}$. First note that for each $B \in \Gamma_n$ since $B \neq \emptyset$ then by the irreducibility of f we have $Y \backslash f(X \backslash B) \neq \emptyset$ and that each $Y \backslash f(X \backslash B)$ is open. So now let O be open in Y, then there exists $B \in \bigcup\{\Gamma_n : n \in \mathbb{N}\}$ such that $B \subset f^{-1}(O)$, thus $Y \backslash f(X \backslash B) \subset Y \backslash f(X \backslash f^{-1}(O)) \subset O$ so it is a π-base.

Now to see that Γ_n is locally finite: Let $y \in Y$. For each $x \in f^{-1}(y)$, there exists an open set O_x such that $x \in O_x$ and O_x intersects at most one element of Ψ_n. By compactness of $f^{-1}(y)$ there exist $O_{x_1}, \ldots, O_{x_k}$ such that $f^{-1}(y) \subset \bigcup\{O_{x_i} : i = 1, \ldots, k\}$. So then let $U = \bigcup\{O_{x_i} : i = 1, \ldots, k\}$. Then $f^{-1}(y) \subset U$ and U intersects only finitely many elements of Ψ. Now if $Y \backslash f(X \backslash U) \cap Y \backslash f(X \backslash B) \neq \emptyset$ then $U \cap B \neq \emptyset$. It follows that $Y \backslash f(X \backslash U)$ intersects only finitely many elements of Γ_n. Furthermore since $f^{-1}(y) \subset U$, it follows that $y \in Y \backslash f(X \backslash U)$. So Γ_n is locally finite which implies that Y is π-metrizable.

That π-metrizability is preserved by irreducible perfect continuous inverse images follows by a standard argument. □

Theorem 2.6. *A space Y is the image of a π-metrizable space X under a perfect mapping if and only if Y is strongly d-separable.*

Proof: Every π-metrizable space is strongly d-separable and strong d-separability is preserved by closed mappings.

Now assume Y is strongly d-separable. Let $\{D_n : n \in \mathbb{N}\}$ be a collection of closed discrete subspaces of Y with $\bigcup\{D_n : n \in \mathbb{N}\}$ dense in Y. Let $E_n = \bigcup\{D_i :$

$i = 1, \dots, n\}$. Then E_n is closed and discrete for each n and $\bigcup\{E_n : n \in \mathbb{N}\}$ is dense in Y. Now consider the following subspace of $\mathbb{N}_* \times Y$, where $\mathbb{N}_* = \mathbb{N} \cup \{p\}$ is the Alexandroff compactification of $\mathbb{N}$: The space $X = (\bigcup\{\{n\} \times E_n : n \in \mathbb{N}\}) \cup (\{p\} \times Y)$.

We shall show that X is π-metrizable. Let $\Gamma_n = \{\{(n,d)\} : d \in E_n\}$. Then Γ_n is discrete, for if $(a,b) \in X$ with $a \neq n$ then $(X\backslash\{n\}) \times E_n$ is an open set containing (a,b) and intersecting no element of Γ_n. Now if $a = n$ then $\{(a,b)\}$ is open. Furthermore $\bigcup\{\Gamma_n : n \in \mathbb{N}\}$ is a π-base. Let O be a nonempty open set in X. It will be sufficient to show O intersects $\bigcup\{\{n\} \times E_n : n \in \mathbb{N}\}$. If $O \cap (\{p\} \times Y) = \emptyset$ then this is trivial. So otherwise let us assume we have $O = U \times N_m$ for an open set $U \subset Y$ and $N_m = \mathbb{N}_* \backslash \{1, 2, \dots, m-1\}$. Now since $\bigcup\{E_n : n \in \mathbb{N}\}$ is dense in Y there exists $d \in E_n$ for some n such that $d \in U$. Now if $n \geq m$ then we have $\{(n,d)\} \subset O$ where $\{(n,d)\} \in \Gamma_n$. If instead $n < m$ then $E_n \subset E_m$ and thus $d \in E_m$ and so $\{(m,d)\} \subset O$ where $\{(m,d)\} \in \Gamma_m$. Hence $\bigcup\{\Gamma_n : n \in \mathbb{N}\}$ is a π-base for X and so X is π-metrizable.

Now take $f : X \longrightarrow Y$ to be the projection map. The projection of $\mathbb{N}_* \times Y$ onto Y is a closed mapping as $\mathbb{N}_*$ is compact and X is a closed subspace of $\mathbb{N}_* \times Y$ thus f is a closed mapping. That $f^{-1}(y)$ is compact for all $y \in Y$ follows as $f^{-1}(y)$ is homeomorphic to a subspace of $\mathbb{N}_*$ containing the limit point p. Thus f is a perfect mapping. $\square$

In fact we need only that the mapping be closed in order that the image be strongly d-separable and thus we get another characterization.

Corollary 2.7. *A space Y is the image of a π-metrizable space X under a closed mapping if and only if Y is strongly d-separable.*

Proposition 2.8. *If X is π-metrizable and $f : X \longrightarrow Y$ is an onto open continuous mapping such that each fiber is compact, then Y has a σ-point-finite π-base.*

Proof: Let $\bigcup\{\Psi_n : n \in \mathbb{N}\}$ be a π-base for X with Ψ_n discrete. Let $\Gamma_n = \{f(B) : B \in \Psi_n\}$, for $y \in Y$, the $f^{-1}(y)$ is compact and thus intersects only finitely many members of Ψ_n. Thus $y \in f(B)$ for only finitely many $B \in \Psi_n$ and so Γ_n is point-finite. That $\bigcup\{\Gamma_n : n \in \mathbb{N}\}$ is a π-base follows trivially as f is an open continuous mapping. $\square$

Theorem 2.9. *If Y has an open dense π-metrizable subspace then there exists a π-metrizable space X and $f : X \longrightarrow Y$ such that f is onto, open, continuous and each fiber is compact.*

Proof: Let O be the subspace. Let $\bigcup\{\Psi_n : n \in \mathbb{N}\}$ be a π-base for O with Ψ_n discrete in O. Now consider subspace of $\mathbb{N}_* \times Y$, where $\mathbb{N}_* = \mathbb{N} \cup \{p\}$ is the Alexandroff compactification of $\mathbb{N}$: The space $X = (\mathbb{N} \times O) \cup (\{p\} \times Y)$. Now let $\Gamma_{n,m} = \{\{n\} \times B : B \in \Psi_m\}$. Then $\Gamma_{n,m}$ is discrete. For if $(a,b) \in X$ and $a \neq n$ then $X\backslash\{n\} \times O$ is an open set containing (a,b) and intersecting no element of $\Gamma_{n,m}$. If $a = n$ then there exists and open set $U \in O$ with $b \in U$ and U intersecting at most one element of Ψ_m. Then $(a, b) \in \{n\} \times U$ and $\{n\} \times U$ intersects at most one element of $\Gamma_{n,m}$. Thus $\Gamma_{n,m}$ is discrete.

Now let U be a basic open set in X. Then $U = (V \times W) \cap X$ where V is open in $\mathbb{N}_*$ and W is open in Y. Then there exists $n \in \mathbb{N}$ such that $n \in V$, and $W \cap O$ is a nonempty open subset of Y so there exists $B \in \Psi_m$ for some m, such that $B \subset W \cap O$. Thus $\{n\} \times B \subset U$ and $\{n\} \times B \in \Gamma_{n,m}$. Thus $\bigcup\{\Psi_{n,m} : n, m \in \mathbb{N}\}$ is a π-base for X and thus X is π-metrizable.

Now consider $f : X \longrightarrow Y$ to be the projection mapping. Let U be a basic open set in X. Then $U = (V \times W) \cap X$ where V is open in $\mathbb{N}_*$ and W is open in Y. Then $f(U) = W$ and so f is an open mapping. That $f^{-1}(y)$ is compact for all $y \in Y$ follows as $f^{-1}(y)$ is homeomorphic to a subspace of $\mathbb{N}_*$ containing the limit point p. $\square$

Problem 2.10. *How might the class of spaces described in the previous two propositions be further characterized?*

3. Products

We now turn our attention to the question of when products are (weakly) π-metrizable. First a standard observation.

Proposition 3.1. *If X_n is π-metrizable for $n \in \mathbb{N}$, then $\prod\{X_n : n \in \mathbb{N}\}$ is π-metrizable.*

Proof: Let $\Psi_n = \bigcup\{\Psi_{n,m} : m \in \mathbb{N}\}$ be a π-base for X_n with $\Psi_{n,m}$ discrete. There are countably many ways to select a finite subset $a_1, \ldots, a_k \in \mathbb{N}$. Then there are countably many ways to select $n_1, \ldots, n_k \in \mathbb{N}$. Now let $P(a_1, \ldots, a_k, n_1, \ldots, n_k) = \{\prod\{O_n : n \in \mathbb{N}\} : O_{n_i} \in \Psi_{a_i,n_i}$ for $i = 1, \ldots, k$ and $O_n = X_n$ otherwise$\}$. Then there are countably many such $P(a_1, \ldots, a_k, n_1, \ldots, n_k)$. Let $x \in \prod\{X_n : n \in \mathbb{N}\}$ for each Ψ_{a_i,n_i} there exists U_i open in X_i such that $x(i) \in U_i$ and U_i intersect at most one member of Ψ_{a_i,n_i}. Now define $U_i = X_i$ for all $i \neq 1, \ldots, k$. Then $x \in \prod\{U_n : n \in \mathbb{N}\}$ and this is an open set intersecting at most one element of $P(a_1, \ldots, a_k, n_1, \ldots, n_k)$. Therefore each $P(a_1, \ldots, a_k, n_1, \ldots, n_k)$ is discrete.

Let $\prod\{U_n : n \in \mathbb{N}\}$ be a basic open set (U_n open in X_n). Let k be such that $U_n = X_n$ for all $n > k$. Then we can find $O_{i,j(i)} \in \Psi_{i,j(i)}$ such that $O_{i,j(i)} \subset U_i$ for each $i \leq k$. Then $O_{1,j(1)} \times \cdots \times O_{k,j(k)} \times X_{k+1} \times X_{k+2} \times \cdots \subset \prod\{U_n : n \in \mathbb{N}\}$ and this is in $P(1, \ldots, k, j(1), \ldots, j(k))$. Thus this is a π-base and so $\prod\{X_n : n \in \mathbb{N}\}$ is π-metrizable. $\square$

The proof of the corresponding result for weakly π-metrizable spaces follows by a similar argument.

Proposition 3.2. *If X_n is weakly π-metrizable for each $n \in \mathbb{N}$, then $\prod\{X_n : n \in \mathbb{N}\}$ is weakly π-metrizable.*

It is not at all obvious (at this point) whether we can have $X \times Y$ being π-metrizable without both X and Y being so. We will see that in fact much more is true.

Lemma 3.3. *If X_n has a discrete collection of κ open sets for all $n \in \mathbb{N}$ and $\pi w(Y), \pi w(X_n) \leq \kappa$ for all n, then $Y \times (\prod\{X_n : n \in \mathbb{N}\})$ is π-metrizable.*

Proof: Let $X_0 = Y$ and let $\mathbb{N}_* = \mathbb{N} \cup \{0\}$. Now let Ψ_n be a π-base for X_n for each $n \in \mathbb{N}_*$ with $|\Psi_n| = \kappa$. Now let Γ_n be a discrete collection open sets with $|\Gamma_n| = \kappa$ for all $n \in \mathbb{N}$. We essentially want to construct "almost all" of the products where n factors are nontrivial: the trick is to do it for $\mathbb{N}_* \backslash \{n\}$. So we observe that there are $\aleph_0$ ways to choose $A \subset \mathbb{N}_* \backslash \{n\}$ such that $|A| = n$. For each $k \in A$ there are κ ways to choose $B_k \in \Psi_k$. Thus there are κ ways to choose a set A and $\{B_k : k \in A\}$ where $B_k \in \Psi_k$. Now $|\Gamma_n| = \kappa$, so for each $A \subset \mathbb{N}_* \backslash \{n\}$ such that $|A| = n$ and $\{B_k : k \in A\}$ where $B_k \in \Psi_k$, we can associate a unique $f(A, \{B_k | k \in A\}) \in \Gamma_n$. So f is a one to one function. Now let $O(A, \{B_k : k \in A\}) = \prod\{O_n : n \in \mathbb{N}_*\}$ where $O_k = B_k$ for all $k \in A$, $O_n = f(A, \{B_k : k \in A\})$ and $O_m = X_m$ for all $m \in \mathbb{N}_* \backslash (A \cup \{n\})$. Then $O(A, \{B_k : k \in A\})$ is open. So let $\Delta_n = \{O(A, \{B_k | k \in A\}) : A \subset \mathbb{N}_* \backslash \{n\}$ with $|A| = n$ and $B_k \in \Psi_k$ for each $k \in A\}$. Then Δ_n is discrete. Let $g \in \prod\{X_n : n \in \mathbb{N}_*\}$. Since Γ_n is discrete, there exists $g(n) \in O$ open in X_n such that O intersects at most one elements of Γ_n. Then $\prod\{U_n : n \in \mathbb{N}_*\}$ where $U_m = X_m$ for $n \neq m$ and $U_n = O$, is an open set containing g. Furthermore $\prod\{U_n : n \in \mathbb{N}_*\}$ intersects $B \in \Delta_n$ only if O intersects $\pi_n(B) = f(A, \{B_k : k \in A\}) \in \Gamma_n$. Since O intersects at most one element of Γ_n and f is one to one, it follows that $\prod\{U_n : n \in \mathbb{N}_*\}$ intersects at most one element of Δ_n.

Now to see that $\bigcup\{\Delta_n : n \in \mathbb{N}\}$ is a π-base choose a basic open set $\prod\{U_n : n \in \mathbb{N}_*\}$, that is, U_n is open for all n and $U_n = X_n$ for all but finitely many n. Let $B = \{n \in \mathbb{N}_* : U_n \neq X_n\}$. Since $|B| = n$ is finite, there exists $m \in \mathbb{N}$ such that $n < m$ and $m \notin B$. Now let $A \subset \mathbb{N}_*$ be such that $|A| = m$, $m \notin A$ and $B \subset A$. Now for $k \in A$ choose $B_k \in \Psi_k$ such that $B_k \subset U_k$. Then $O(A, \{B_k | k \in A\}) = \prod\{O_n : n \in \mathbb{N}_*\} \subset \prod\{U_n : n \in \mathbb{N}_*\}$ as $O_k = B_k$ for $k \in A$ so $O_k \subset U_k$ and $U_k = X_k$ for $k \notin A$ so $O_k \subset U_k$ automatically. Thus $\bigcup\{\Delta_n : n \in \mathbb{N}\}$ is a σ-discrete π-base so $\prod\{X_n : n \in \mathbb{N}_*\} = Y \times (\prod\{X_n : n \in \mathbb{N}\})$ is π-metrizable. $\square$

Theorem 3.4. *For every space X there exists a space Y such that $X \times Y$ is π-metrizable.*

Proof: Let D be a discrete space with $|D| = \pi w(X)$. Now let $Y = D^{\aleph_0}$. Then $X \times Y$ is π-metrizable by Lemma 3.3. $\square$

Corollary 3.5. *Every space is the open continuous image of a π-metrizable space.*

Proof: Let Y be a space, and X be such that $X \times Y$ is π-metrizable. Now take $\pi : X \times Y \longrightarrow Y$ to be the projection map. $\square$

One further consequence of this is a solution to a problem posed in [1].

Example 3.6. There exists a π-metrizable topological group that is not metrizable (and therefore not first countable).

PROOF: Let K be a discrete space with $|K| = \aleph_1$, then K is a topological group, as is $K^{\aleph_1}$. Furthermore $K^{\aleph_1}$ is π-metrizable. However $K^{\aleph_1}$ is not metrizable. □

Theorem 3.7. *Let $\{X_\alpha : \alpha \in I\}$ with $(\mathbb{N} \subset I)$ be a collection of not more than κ spaces, with $\pi w(X_\alpha) \leq \kappa$. If $\{\lambda_n\}$ is a sequence of cardinal numbers converging to κ (in the topology induced by the usual ordering) and X_n has a discrete collection of λ_n open sets for all $n \in \mathbb{N}$, then $\prod\{X_\alpha : \alpha \in I\}$ is π-metrizable.*

PROOF: From elementary set theory, there exist a partition $\mathbb{N} = \bigcup\{N_n : n \in \mathbb{N}\}$, such that $N_i \cap N_j = \emptyset$ if $i \neq j$, and $|N_n| = \omega$ for all $n \in \mathbb{N}$. Then $\{\lambda_n : n \in N_i\}$ converges to κ. $\prod\{X_n : n \in N_i\}$ has a discrete collection of κ open sets. Write $N_i = \{i_n : n \in \mathbb{N}\}$. Without loss of generality assume $\lambda_{i_1} \geq \aleph_0$. Now let Γ_n be a discrete collection of open sets of X_{i_n} of cardinality λ_{i_n}. Choose $\{O_n : n \in \mathbb{N}\} \subset \Gamma_1$. Now for each $U \in \Gamma_n$ with $n > 1$ let $h(U) = \prod\{O_{i_n} : n \in \mathbb{N}\}$ where $O_{i_1} = O_n$ and $O_{i_n} = U$ and for $k \neq 1, n$ put $O_{i_k} = X_{i_k}$. Now let $\Delta_n = \{h(U) : U \in \Gamma_n\}$ and $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. Then Δ is discrete and $|\Delta| = \kappa$. So let $Y_n = \prod\{X_k : k \in N_n\}$. Let $Y = \prod\{X_\alpha : \alpha \in I \backslash \mathbb{N}\}$. It is known that $\pi w(Y) \leq \kappa$.

Then $\prod\{X_\alpha : \alpha \in I\} = \prod\{X_\alpha : \alpha \in I\backslash\mathbb{N}\} \times \prod\{X_k : k \in N_n, n \in \mathbb{N}\} = Y \times \prod\{Y_n : n \in \mathbb{N}\}$ is π-metrizable by Lemma 3.3. □

Theorem 3.8. *Let $\{X_\alpha : \alpha \in I\}$ be a collection of not more that κ spaces with $\pi w(X_\alpha) \leq \kappa$ for all $\alpha \in I$. Assume that whenever $\{\lambda_n\}$ is a sequence of cardinal numbers converging (in the topology induced by the usual ordering) to κ, there exist $\{X_n : n \in \mathbb{N}\}$ such that X_n has a collection of pairwise disjoint open sets of cardinality λ_n. Then $\prod\{X_\alpha : \alpha \in I\}$ is weakly π-metrizable.*

The proof is similar in spirit to that of Theorem 3.7.

Lemma 3.9. *For every space X there exists a compact space Y such that $X \times Y$ is weakly-π-metrizable.*

PROOF: Let A be a discrete space with $|A| = \pi w(X)$, let B be the Alexandroff compactification of A and declare $Y = B^{\aleph_0}$. Then $X \times Y$ is weakly-π-metrizable by Theorem 3.8. □

Theorem 3.10. *Every space is the image of a weakly π-metrizable space under an open perfect mapping.*

We now present a result of a different kind: one which provides an upper bound on the number of factors in a weakly π-metrizable product.

Theorem 3.11. *Let κ and λ be cardinal numbers. If Y is the product of κ factors each with at least two points and density less than or equal to λ where $\lambda < \kappa$, then Y is not weakly π-metrizable.*

Proof: Let $p(\beta) = \prod\{O_\alpha : \alpha \in I\}$ where $O_\alpha = X_\alpha$ for all $\alpha \neq \beta$ and $O_\beta = X_\beta \backslash \{x\}$ for some $x \in X_\beta$. Let $\Gamma = \{p(\beta) : \beta \in I\}$. Put $\Delta = \bigcup\{\Psi_n | n \in \mathbb{N}\}$ to be a π-base with Ψ_n pairwise disjoint. For each element U of Γ there is an element B of Δ such that $B \subset U$. Furthermore for each $B \in \Delta$ there can exist only finitely many $U \in \Gamma$ such that $B \subset U$. Thus $|\Delta| = |\Gamma| = \kappa$. Therefore there exists n such that $|\Psi_n| > \lambda$. But Ψ_n is pairwise disjoint and it is known that $c(\prod\{X_\alpha : \alpha \in I\}) \leq \lambda$. So we have a contradiction. Thus $\prod\{X_\alpha : \alpha \in I\}$ is not weakly π-metrizable. $\square$

As an application of the product theorem offered earlier, the premise on the above theorem cannot be weakened to "If Y is the product of κ factors each with at least two points and density less than κ then Y is not weakly π-metrizable". To see this take $A(n)$ to be a discrete space of size $\aleph_n$. Then take $Y = \prod\{A(n)^{\aleph_n} : n \in \mathbb{N}\}$. There are $\aleph_\omega$ factors each with density less than $\aleph_\omega$ but the product is π-metrizable as evident from Theorem 3.8 by using the sequence $\{\aleph_n\}$ which converges to $\aleph_\omega$.

Theorem 3.12. *If $\pi w(X)$ is a cardinal with countable cofinality or a successor, and X^κ is π-metrizable, then $X^{\aleph_0}$ is π-metrizable.*

Proof: We shall begin with the simple case where $\pi w(X) = \tau$, with τ is a successor and X^τ is π-metrizable. Let $\bigcup\{\Psi_n : n \in \mathbb{N}\}$ be a π-base for X^τ. We have $\pi w(X^\tau) \geq \pi(X) = \tau$ so $|\bigcup\{\Psi_n : n \in \mathbb{N}\}| \geq \tau$ thus $|\Psi_n| \geq \tau$ for some n as τ is a successor. Thus X^τ has a discrete collection of nonempty open subsets: Γ such that $|\Gamma| = \tau$ and without loss of generality we may assume all elements of Γ are basic open sets: that is, the product of open sets. Now again since τ is a successor, there must exist n such that $|\{O \in \Gamma : \pi_\alpha(O) \neq X_\alpha$ for n values of $\alpha\}| = \tau$. Let Θ be this set. So Θ is discrete. Let $P : \Theta \longrightarrow I^m$ be defined by $P(U) = \{\alpha : \pi_\alpha(U) \neq X_\alpha\}$. Now let $O \in \Theta$, for all $U \in \Theta$ we have $P(O) \cap P(U) \neq \emptyset$ else $O \cap U \neq \emptyset$. Thus there must exist $\alpha_1 \in P(O)$ such that $|\{U \in \Theta : \alpha_1 \in P(U)\}| = \tau$. Let Θ_1 be this set and α the corresponding coordinate. Now suppose in the set Θ_i if there is an $\alpha_{i+1} \neq \alpha_1, \ldots, \alpha_i$ such that $|\{U \in \Theta_i : \alpha_{i+1} \in P(U)\}| = \tau$, then let Θ_{i+1} be this set. Since each set has only n elements, this must terminate at some finite point. That is, there exists Θ_m such that for all $\alpha \neq \alpha_1, \ldots, \alpha_m$ we have $|\{U \in \Theta_m : \alpha \in P(U)\}| < \tau$. Now define $Q : \Theta_m \longrightarrow I^{n-m}$ by $Q(U) = P(U) \backslash \{\alpha_1, \ldots, \alpha_m\}$. So then for all $\alpha \in I$ we have $|\{U \in \Theta_m : \alpha \in Q(U)\}| < \tau$.

Now well order Θ_m. Construct the set Ω as follows: let $O_1 \in \Omega$ and for O_α, if there exists $\beta < \alpha$ such that $Q(O_\beta) \cap Q(O_\alpha) \neq \emptyset$ then $O_\alpha \notin \Omega$ otherwise $O_\alpha \in \Omega$. Then by construction Ω is pairwise disjoint. We will see that $|\Omega| = \tau$.

Let us define $s : \Omega \longrightarrow \operatorname{Pow}(\Theta_m)$ by $s(O) = \{U \in \Theta_m : Q(U) \cap Q(O) \neq \emptyset\}$. If $|s(O)| = \tau$ then there exists $\zeta \in Q(O)$ such that $|\{U \in \Theta_m : \zeta \in Q(U)\}| = \tau$ a contradiction. So $|s(O)| \leq \tau - 1$ for all $O \in \Omega$. Now $\Theta_m = \bigcup\{s(O) : O \in \Omega\}$. Thus $\tau = |\Theta_m| \leq \sum\{|s(O)| : O \in \Omega\} \leq |\Omega|(\tau - 1)$. Hence $|\Omega| = \tau$.

Now assume (for contradiction) that X^n does not have a discrete collection of open sets of cardinality τ for all $n \in \mathbb{N}$. Then by Lemma 2.1, X^n does not have a locally finite collection of open sets of cardinality τ for all $n \in \mathbb{N}$. So there exists a point in $(x_{\alpha_1}, \ldots, x_{\alpha_m}) \in X_{\alpha_1} \times \cdots \times X_{\alpha_m}$ such that every open set containing $(x_{\alpha_1}, \ldots, x_{\alpha_m})$ intersects infinitely many members of $\{\pi_{\alpha_1,\ldots,\alpha_m}(O) : O \in \Omega\}$ where $\pi_{\alpha_1,\ldots,\alpha_m}$ is the projection onto the coordinates $\alpha_1, \ldots, \alpha_m$ and the collection is not taken faithfully so that it has cardinality τ. Now define the point f as follows: $f(\alpha_i) = x_{\alpha_i}$ for $i = 1, \ldots, m$, if $x_\alpha \in Q(O)$ (for some $O \in \Omega$) then $f(\alpha)$ is chosen so that $f(\alpha) \in O$, otherwise choose $f(\alpha)$ arbitrarily.

Let $\prod\{O_\alpha : \alpha \in I\}$ be an open set containing z. So O_α is open and $O_\alpha = X$ for all but finitely many values of α. Now $O_{\alpha_1} \times \cdots \times O_{\alpha_m}$ is an open set containing $(x_{\alpha_1}, \ldots, x_{\alpha_m})$ so it intersects infinitely many elements of $\{\pi_{\alpha_1,\ldots,\alpha_m}(O) : O \in \Omega\}$. Let Δ be this infinite set. If $\{\beta_1, \ldots, \beta_k\} = \{\alpha \in I : O_\alpha \neq X\}$. Since $\{Q(O) : O \in \Delta\}$ is an infinite collection of pairwise disjoint sets, there exists $V, U \in \Delta$ such that $\beta_i \notin Q(V) \cup Q(U)$ for all $i = 1, \ldots, k$. All that is left is to show that $\prod\{O_\alpha : \alpha \in I\}$ intersects V and U. $\pi_\alpha(V) = X$ for each $\alpha \notin P(V)$. So we know that $O_\alpha \cap \pi_\alpha(V) \neq \emptyset$ for each $\alpha \notin P(V)$. Now $P(V) = \{\alpha_1, \ldots, \alpha_m\} \cup Q(V)$. The set $O_{\alpha_1} \times \cdots \times O_{\alpha_m}$ intersects $\pi_{\alpha_1,\ldots,\alpha_m}(V)$ by virtue of $V \in \Delta$, and since $O_\alpha = X$ for each $\alpha \in Q(V)$ it follows that $O_\alpha \cap \pi_\alpha(V) \neq \emptyset$ for each $\alpha \in Q(V)$. Thus $\pi_\alpha(V) \cap O_\alpha \neq \emptyset$ for all $\alpha \in I$. Since V is the product of open sets, this implies that $V \cap \prod\{O_\alpha : \alpha \in I\} \neq \emptyset$. Similarly, $U \cap \prod\{O_\alpha : \alpha \in I\} \neq \emptyset$ thus Ω is not discrete. Thus Γ is not discrete: a contradiction.

Therefore X^n does have a discrete collection of open sets of cardinality τ for some $n \in \mathbb{N}$. Since $\pi w(X^n) = \tau$, we get $(X^n)^{\aleph_0} = X^{\aleph_0}$ is π-metrizable.

In the case of $\pi w(X) = \tau$ not a successor but with countable cofinality. There exists an increasing sequence $\lambda_n \longrightarrow \tau$ such that each λ_n is a successor. Then we may repeat the above argument to see that since X^τ must have a discrete collection of λ_n open sets, there exists $m_n \in \mathbb{N}$ such that X^{m_n} has a discrete collection of λ_n open sets. Thus $\prod\{X^{m_n} : n \in \mathbb{N}\} = X^{\aleph_0}$ is π-metrizable.

Finally for the most general case where X^κ is π-metrizable. By Theorem 3.11 we get $\kappa \leq d(X) \leq \pi w(X)$. Thus $\pi w(X^\kappa) = \pi w(X)$. So $(X^\kappa)^{\pi w(X)} = X^{\pi w(X)}$ is π-metrizable, thus $X^{\aleph_0}$ is π-metrizable from above. $\square$

Many of these results can be summarized in the following corollary.

Corollary 3.13. *If $\pi w(X)$ is a cardinal with countable cofinality or a successor and X^κ is π-metrizable (for some κ), then X^τ is π-metrizable for all $\aleph_0 \leq \tau \leq \pi w(X)$.*

Problem 3.14. *Is it true that for any non-π-metrizable spaces X and Y, we have that $X \times Y$ is also non-π-metrizable?*

Problem 3.15. *Does there exist a non-π-metrizable space X such that X^n is π-metrizable for some $n \in \mathbb{N}$?*

Problem 3.16. *If X^κ is π-metrizable is $X^{\aleph_0}$ π-metrizable as well?*

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