The fixed points and iterated order of some differential polynomials

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Abstract. This paper is devoted to considering the iterated order and the fixed points of some differential polynomials generated by solutions of the differential equation

$$f^{''} + A_1(z)f^{'} + A_0(z)f = F,$$

where $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are meromorphic functions of finite iterated p-order.

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1. Introduction and statement of results

In this paper, it is assumed that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [5], [10]). For the definition of the iterated order of a meromorphic function, we use the same definition as in [6], [2, p. 317], [7, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1 (see [6], [7]). Let f be a meromorphic function. Then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \varlimsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} \quad (p \geq 1 \ \text{ is an integer}), \tag{1.1}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [5], [10]). For $p = 1$, this notation is called order and for $p = 2$ hyper order.

Remark 1.1. If f is an entire function, then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log_{p+1} M(r,f)}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.2}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.2 (see [6], [7]). The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0, & \text{for } f \text{ rational,} \\ \min\left\{ j \in \mathbb{N} : \rho_j(f) < +\infty \right\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } j \in \mathbb{N} \text{ with } \rho_j(f) < +\infty \text{ exists,} \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.3}$$

Definition 1.3 (see [6]). Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f})}{\log r} \quad (p \geq 1 \text{ is an integer}), \tag{1.4}$$

where $\overline{N}(r, \frac{1}{f})$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$. For $p = 1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p = 2$ hyper exponent of convergence of the sequence of distinct zeros.

Definition 1.4 (see [8]). Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_p(f) = \overline{\lambda}_p(f - z) = \overline{\lim_{r\to+\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f-z})}{\log r} \quad (p \geq 1 \text{ is an integer}). \tag{1.5}$$

For $p = 1$, this notation is called exponent of convergence of the sequence of distinct fixed points and for $p = 2$ hyper exponent of convergence of the sequence of distinct fixed points (see [9]). Thus $\overline{\tau}_p(f) = \overline{\lambda}_p(f - z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Consider the linear differential equation

$$f^{''} + A_1(z)f^{'} + A_0(z)f = F, \tag{1.6}$$

where $A_1(z), A_0(z) \not\equiv 0$, F are meromorphic of finite iterated p-order. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [12]). However, there are a few studies on the fixed points of solutions of differential equations. It was in year 2000 that Z.X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [4]). In [11], Wang and Yi investigated fixed points and hyper order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [8], Laine and Rieppo gave improvement of the results of [11] by considering fixed points and iterated order.

Recently, the author has studied the relation between solutions and their derivatives of the differential equation

$$f^{(k)} + A(z)f = 0, \tag{1.7}$$

where $k \geq 2$, $A(z)$ is a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ and have obtained the following result.

Theorem A ([1]). *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$. Suppose, moreover, that either*

(i) *all poles of f are of uniformly bounded multiplicity or that*

(ii) *$\delta(\infty, f) > 0$.*

If $\varphi(z) \not\equiv 0$ is a meromorphic function with finite iterated p-order $\rho_p(\varphi) < +\infty$, then every meromorphic solution $f(z) \not\equiv 0$ of (1.7) satisfies

$$\overline{\lambda}_p(f-\varphi) = \overline{\lambda}_p(f' - \varphi) = \cdots = \overline{\lambda}_p(f^{(k)} - \varphi) = \rho_p(f) = +\infty, \tag{1.8}$$

$$\overline{\lambda}_{p+1}(f-\varphi) = \overline{\lambda}_{p+1}(f' - \varphi) = \cdots = \overline{\lambda}_{p+1}(f^{(k)} - \varphi) = \rho_{p+1}(f) = \rho. \tag{1.9}$$

We know that a differential equation bear a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations.

The first main purpose of this paper is to study the growth and the oscillation of some differential polynomials generated by solutions of second order linear differential equation (1.6). We obtain some estimates of their iterated order and fixed points.

Before we state our results, we denote by

$$\alpha_0 = d_0' - d_1 A_0, \quad \alpha_1 = d_1' + d_0 - d_1 A_1, \tag{1.10}$$

$$h = d_1\alpha_0 - d_0\alpha_1 \tag{1.11}$$

and

$$\psi = \frac{d_1(\varphi' - b' - d_1 F) - \alpha_1(\varphi - b)}{h}, \tag{1.12}$$

where $A_1(z), A_0(z) \not\equiv 0$, F, d_j $(= 0, 1)$, b and φ are meromorphic functions with finite iterated p-order.

Theorem 1.1. *Let $A_1(z), A_0(z) \not\equiv 0$, F be meromorphic functions of finite iterated p-order. Let $d_0(z)$, $d_1(z)$, $b(z)$ be meromorphic functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$ and that $h \not\equiv 0$. Let $\varphi(z)$ be a meromorphic function with finite iterated p-order such that $\psi(z)$ is not a solution of (1.6). If f is an infinite iterated p-order meromorphic solution of (1.6) with $\rho_{p+1}(f) = \rho < +\infty$, then the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty, \tag{1.13}$$

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho. \tag{1.14}$$

Theorem 1.2. *Let $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F be meromorphic functions of finite iterated p-order such that all meromorphic solutions of equation (1.6) are of infinite iterated p-order. Let $d_0(z)$, $d_1(z)$, $b(z)$ be meromorphic functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$ and that $h \not\equiv 0$. Let φ be a finite iterated p-order meromorphic function. If f is a meromorphic solution of equation (1.6) with $\rho_{p+1}(f) = \rho < +\infty$, then the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies (1.13) and (1.14).*

Applying Theorem 1.2 for $\varphi(z) = z$, we obtain the following result.

Corollary 1.1. *Let $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F be meromorphic functions of finite iterated p-order such that all meromorphic solutions of equation (1.6) are of infinite iterated p-order. Let $d_0(z)$, $d_1(z)$, $b(z)$ be meromorphic functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ $(j = 0, 1)$, $\rho_p(b) < \infty$ and that $h \not\equiv 0$. If f is a meromorphic solution of equation (1.6) with $\rho_{p+1}(f) = \rho < +\infty$, then the differential polynomial $g_f = d_1 f' + d_0 f + b$ satisfies $\overline{\tau}_p(g_f) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\overline{\tau}_{p+1}(g_f) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$.*

The second main purpose of this paper is to investigate the relation between infinite iterated p-order solutions of higher order linear differential equations with meromorphic coefficients and meromorphic functions of finite iterated p-order. We will prove the following theorem.

Theorem 1.3. *Let $A_0, A_1, \ldots, A_{k-1}$, F be finite iterated p-order meromorphic functions, and let φ be a finite iterated p-order meromorphic function which is not a solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = F. \tag{1.15}$$

If f is an infinite iterated p-order meromorphic solution of equation (1.15) *with $\rho_{p+1}(f) = \rho < +\infty$, then we have $\overline{\lambda}_p(f-\varphi) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f-\varphi) = \rho_{p+1}(f) = \rho$.*

Applying Theorem 1.3 for $\varphi(z) = z$, we obtain the following result.

Corollary 1.2. *Let $A_0, A_1, \ldots, A_{k-1}$, F be finite iterated p-order meromorphic functions such that $zA_0 + A_1 \not\equiv F$. If f is an infinite iterated p-order meromorphic solution of equation* (1.15) *with $\rho_{p+1}(f) = \rho < +\infty$, then we have $\overline{\tau}_p(f) = \rho_p(f) = \infty$ and $\overline{\tau}_{p+1}(f) = \rho_{p+1}(f) = \rho$.*

Corollary 1.3. *Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions such that*

$$i(A_0) = p\ (1 \le p < \infty),\ \max\left\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\right\} < \rho_p(A_0) = \rho \tag{1.16}$$

and

$$\max\left\{\lambda\left(\frac{1}{A_j}\right) : j = 0, 1, \ldots, k-1\right\} < \rho(A_0). \tag{1.17}$$

Let $\varphi(z)$ ($\not\equiv 0$) be a meromorphic function with finite iterated p-order. Then every meromorphic solution $f(z) \not\equiv 0$ whose poles are of uniformly bounded multiplicity of the equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = 0 \tag{1.18}$$

satisfies $\overline{\lambda}_p(f-\varphi) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f-\varphi) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. In particularly every solution $f(z) \not\equiv 0$ of equation (1.18) *satisfies $\overline{\tau}_p(f) = \rho_p(f) = \infty$ and $\overline{\tau}_{p+1}(f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$.*

2. Auxiliary lemmas

We need the following lemmas in the proofs of our theorems.

Lemma 2.1 (see Remark 1.3 of [6])**.** *If f is a meromorphic function with $i(f) = p \ge 1$, then $\rho_p(f) = \rho_p(f')$.*

Lemma 2.2 ([8])**.** *If f is a meromorphic function with $0 < \rho_p(f) < \rho$ $(p \ge 1)$, then $\rho_{p+1}(f) = 0$.*

Lemma 2.3. *Let $A_0, A_1, \dots, A_{k-1}$, F $(\not\equiv 0)$ be finite iterated p-order meromorphic functions. If f is a meromorphic solution with $\rho_p(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.1}$$

then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty$.

Proof: By equation (2.1), we can write

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{2.2}$$

If f has a zero at z_0 of order $\alpha(> k)$ and if $A_0, A_1, \dots, A_{k-1}$ are all analytic at z_0, then F has a zero at z_0 of order at least $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \le k\, \overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} n\left(r, A_j\right) \tag{2.3}$$

and

$$N\left(r, \frac{1}{f}\right) \le k\, \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N\left(r, A_j\right). \tag{2.4}$$

By (2.2), we have

$$m\left(r, \frac{1}{f}\right) \le \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k-1} m\left(r, A_j\right) + m\left(r, \frac{1}{F}\right) + O(1). \tag{2.5}$$

Applying the lemma of the logarithmic derivative (see [5]), we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O(\log T(r,f) + \log r) \quad (j = 1, \dots, k), \tag{2.6}$$

holds for all r outside a set $E \subset (0, +\infty)$ with a finite linear measure $m(E) < +\infty$. By (2.4), (2.5) and (2.6), we get

$$\begin{aligned} T(r,f) &= T(r, \frac{1}{f}) + O(1) \\ &\le k\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=0}^{k-1} T\left(r, A_j\right) + T(r,F) + O(\log(rT(r,f))) \\ &\qquad\qquad (|z| = r \notin E). \end{aligned} \tag{2.7}$$

Since $\rho_p(f)=+\infty$, there exists $\{r_n'\}$ $(r_n'\to+\infty)$ such that

$$\lim_{r_n'\to+\infty}\frac{\log_p T(r_n',f)}{\log r_n'}=+\infty. \tag{2.8}$$

Denoting the linear measure of E, $m(E)=\gamma<+\infty$, there exists a point $r_n\in[r_n',r_n'+\gamma+1]-E$. From

$$\frac{\log_p T(r_n,f)}{\log r_n}\geq\frac{\log_p T(r_n',f)}{\log(r_n'+\gamma+1)}=\frac{\log_p T(r_n',f)}{\log r_n'+\log(1+(\gamma+1)/r_n')}, \tag{2.9}$$

it follows that

$$\lim_{r_n\to+\infty}\frac{\log_p T(r_n,f)}{\log r_n}=+\infty. \tag{2.10}$$

Set $\sigma=\max\{\rho_p(A_j)\ (j=0,\dots,k-1),\rho_p(F)\}$. Then for a given arbitrary large $\beta>\sigma$,

$$T(r_n,f)\geq\exp_{p-1}\left\{r_n^{\beta}\right\} \tag{2.11}$$

holds for sufficiently large r_n. On the other hand, for any given ε with $0<2\varepsilon<\beta-\sigma$, we have

$$\begin{aligned}T\left(r_n,A_j\right)&\leq\exp_{p-1}\left\{r_n^{\sigma+\varepsilon}\right\}\quad(j=0,\dots,k-1),\\ T(r_n,F)&\leq\exp_{p-1}\left\{r_n^{\sigma+\varepsilon}\right\}\end{aligned} \tag{2.12}$$

for sufficiently large r_n. Hence, we have

$$\begin{aligned}&\max\left\{\frac{T(r_n,F)}{T(r_n,f)},\frac{T(r_n,A_j)}{T(r_n,f)}\quad(j=0,\dots,k-1)\right\}\\ &\qquad\leq\frac{\exp_{p-1}\left\{r_n^{\sigma+\varepsilon}\right\}}{\exp_{p-1}\left\{r_n^{\beta}\right\}}\to0,\quad r_n\to+\infty.\end{aligned} \tag{2.13}$$

Therefore,

$$T(r_n,F)\leq\frac{1}{k+3}T(r_n,f),\quad T\left(r_n,A_j\right)\leq\frac{1}{k+3}T(r_n,f)\quad(j=0,\dots,k-1) \tag{2.14}$$

holds for sufficiently large r_n. From

$$O(\log r_n+\log T(r_n,f))=o(T(r_n,f)), \tag{2.15}$$

we obtain that

$$O(\log r_n + \log T(r_n, f)) \le \frac{1}{k+3} T(r_n, f) \tag{2.16}$$

also holds for sufficiently large r_n. Thus, by (2.7), (2.14), (2.16), we have

$$T(r_n, f) \le k(k+3)\, \overline{N}\left(r_n, \frac{1}{f}\right). \tag{2.17}$$

It yields $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = +\infty$. □

Lemma 2.4 ([1])**.** *Let $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ be finite iterated p-order meromorphic functions. If f is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of equation (2.1), then $\overline{\lambda}_p(f) = \rho_p(f) = +\infty$ and $\overline{\lambda}_{p+1}(f) = \rho_{p+1}(f) = \rho$.*

Lemma 2.5. *Suppose that $A_1(z)$, $A_0(z)$ ($\not\equiv 0$), F are meromorphic functions of finite iterated p-order. Let $d_0(z)$, $d_1(z)$, $b(z)$ be meromorphic functions such that at least one of $d_0(z)$, $d_1(z)$ does not vanish identically with $\rho_p(d_j) < \infty$ ($j = 0, 1$), $\rho_p(b) < \infty$ and that $h \not\equiv 0$, where h is defined in (1.11). If f is an infinite iterated p-order meromorphic solution of (1.6) with $\rho_{p+1}(f) = \rho < +\infty$, then the differential polynomial*

$$g_f = d_1 f' + d_0 f + b \tag{2.18}$$

satisfies

$$\rho_p(g_f) = \rho_p(f) = \infty, \quad \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho. \tag{2.19}$$

PROOF: Suppose that f is a meromorphic solution of equation (1.6) with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$. First we suppose that $d_1 \not\equiv 0$. Differentiating both sides of equation (2.18) and replacing f'' with $f'' = F - A_1 f' - A_0 f$, we obtain

$$g_f' - b' - d_1 F = (d_1' + d_0 - d_1 A_1) f' + (d_0' - d_1 A_0) f. \tag{2.20}$$

Then by (1.10), (2.18) and (2.20), we have

$$d_1 f' + d_0 f = g_f - b, \tag{2.21}$$

$$\alpha_1 f' + \alpha_0 f = g_f' - b' - d_1 F. \tag{2.22}$$

Set

$$h = d_1\alpha_0 - d_0\alpha_1 = d_1(d_0' - d_1 A_0) - d_0(d_1' + d_0 - d_1 A_1). \tag{2.23}$$

By $h \not\equiv 0$ and (2.21)–(2.23), we obtain

$$f = \frac{d_1(g_f^{'} - b^{'} - d_1 F) - \alpha_1(g_f - b)}{h}. \tag{2.24}$$

If $\rho_p(g_f) < \infty$, then by (2.24) and Lemma 2.1, we get $\rho_p(f) < \infty$ and this is a contradiction. Hence $\rho_p(g_f) = \infty$.

Finally, if $d_1 \equiv 0$, $d_0 \not\equiv 0$, then we have $g_f = d_0 f + b$ and by $\rho_p(d_0) < \infty$, $\rho_p(b) < \infty$, we get $\rho_p(g_f) = \infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. By (2.18), Lemma 2.1 and Lemma 2.2, we get $\rho_{p+1}(g_f) \leq \rho_{p+1}(f)$ and by (2.24) we have $\rho_{p+1}(f) \leq \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. □

Remark 2.1. In Lemma 2.5, if we do not have the condition $h \not\equiv 0$, then the differential polynomial can be of finite iterated p-order. For example, if $d_0^{'} - d_1 A_0 \equiv 0$ and $d_1^{'} + d_0 - d_1 A_1 \equiv 0$, then $h \equiv 0$ and $g_f^{'} - b^{'} - d_1 F \equiv 0$. It follows that $\rho_p(g_f) = \rho_p(g_f^{'}) < +\infty$.

Lemma 2.6 ([3]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions such that*

$$i(A_0) = p \ (1 \leq p < \infty), \max\left\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\right\} < \rho_p(A_0) = \rho \tag{2.25}$$

and

$$\max\left\{\lambda\left(\frac{1}{A_j}\right) : j = 0, 1, \ldots, k-1\right\} < \rho(A_0). \tag{2.26}$$

Then every meromorphic solution $f(z) \not\equiv 0$ whose poles are of uniformly bounded multiplicity of equation (1.18) *satisfies $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.*

3. Proof of Theorem 1.1

Suppose that f is a meromorphic solution of equation (1.6) with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho < +\infty$. Set $w(z) = d_1 f^{'} + d_0 f + b - \varphi$. Since $\rho_p(\varphi) < \infty$, then by Lemma 2.5 we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. In order to prove $\overline{\lambda}_p(g_f - \varphi) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho$, we need to prove only $\overline{\lambda}_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \rho$. By $g_f = w + \varphi$, we get from (2.24)

$$f = \frac{d_1 w^{'} - \alpha_1 w}{h} + \psi, \tag{3.1}$$

where α_1, h, ψ are defined in (1.10)–(1.12). Substituting (3.1) into equation (1.6), we obtain

$$\begin{aligned} &\frac{d_1}{h} w^{'''} + \phi_2 w^{''} + \phi_1 w^{'} + \phi_0 w \\ &= F - (\psi^{''} + A_1(z)\psi^{'} + A_0(z)\psi) = A, \end{aligned} \tag{3.2}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho_p(\phi_j) < \infty$ $(j = 0, 1, 2)$. Since $\psi(z)$ is not a solution of (1.6), it follows that $A \not\equiv 0$. Then by Lemma 2.3 and Lemma 2.4, we obtain $\overline{\lambda}_p(w) = \rho_p(w) = \infty$, $\overline{\lambda}_{p+1}(w) = \rho_{p+1}(w) = \rho$, i.e., $\overline{\lambda}_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$.

4. Proof of Theorem 1.2

By the hypotheses of Theorem 1.2 all meromorphic solutions of equation (1.6) are of infinite iterated p-order. From (1.12), we see that $\psi(z)$ is a meromorphic function of finite iterated p-order, hence $\psi(z)$ is not a solution of (1.6). By Theorem 1.1, we obtain Theorem 1.2.

5. Proof of Theorem 1.3

Suppose that f is a meromorphic solution of equation (1.15) with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho < +\infty$. Set $w = f - \varphi$. Then by $\rho_p(\varphi) < \infty$, we have $\rho_p(w) = \rho_p(f - \varphi) = \rho_p(f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho$. Substituting $f = w + \varphi$ into equation (1.15), we obtain

$$\begin{aligned} \text{(4.1)} \qquad & w^{(k)} + A_{k-1}w^{(k-1)} + \cdots + A_1 w' + A_0 w \\ &= F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi) = W. \end{aligned}$$

Since φ is not a solution of equation (1.15), we have $W \not\equiv 0$. By Lemma 2.3 and Lemma 2.4, we get $\overline{\lambda}_p(w) = \rho_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \rho_{p+1}(w) = \rho$, i.e., $\overline{\lambda}_p(f - \varphi) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho$.

6. Proof of Corollary 1.3

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.18). Then by Lemma 2.6, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$. By using Theorem 1.3, we obtain Corollary 1.3.

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References

[1] Belaïdi B., *Oscillation of fixed points of solutions of some linear differential equations*, Acta. Math. Univ. Comenianae **77** (2008), no. 2, 263–269.
[2] Bernal L.G., *On growth k-order of solutions of a complex homogeneous linear differential equations*, Proc. Amer. Math. Soc. **101** (1987), 317–322.
[3] Cao T.B., Yi H.X., *On the complex oscillation of higher order linear differential equations with meromorphic functions*, J. Syst. Sci. Complex. **20** (2007), no. 1, 135–148.
[4] Chen Z.X., *The fixed points and hyper order of solutions of second order complex differential equations*, Acta Math. Sci. Ser. A Chin. Ed. **20** (2000), no. 3, 425-432 (in Chinese).
[5] Hayman W.K., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[6] Kinnunen L., *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. **22** (1998), no. 4, 385–405.

[7] Laine I., *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, New York, 1993.

[8] Laine I., Rieppo J., *Differential polynomials generated by linear differential equations*, Complex Var. Theory Appl. **49** (2004), no. 12, 897–911.

[9] Liu M.S., Zhang X.M., *Fixed points of meromorphic solutions of higher order linear differential equations*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 191–211.

[10] Nevanlinna R., *Eindeutige Analytische Funktionen*, Zweite Auflage, reprint, Die Grundlehren der mathematischen Wissenschaften, 46, Springer, Berlin-New York, 1974.

[11] Wang J., Yi H.X., *Fixed points and hyper order of differential polynomials generated by solutions of differential equation*, Complex Var. Theory Appl. **48** (2003), no. 1, 83–94.

[12] Zhang Q.T., Yang C.C., *The Fixed Points and Resolution Theory of Meromorphic Functions*, Beijing University Press, Beijing, 1988 (in Chinese).

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