On Butler $B(2)$-groups decomposing over two base elements

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Abstract. A $B(2)$-group is a sum of a finite number of torsionfree Abelian groups of rank 1, subject to two independent linear relations. We complete here the study of direct decompositions over two base elements, determining the cases where the relations play an essential role.

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Introduction

In this paper by *group* we mean *torsionfree Abelian group of finite rank*.

A Butler $B(n)$*-group* G is a finite sum of rank 1 groups, $G = \langle g_1 \rangle_* + \cdots + \langle g_m \rangle_*$, subject to n $(\leq m)$ independent relations. For basics we refer to [F II]; general results on $B(n)$-groups can be found in [A], [AV], [DVM 10], [DVM 11]; on $B(2)$, besides the decomposition result in [VWW], and the characterization of $B(2)$-groups that are a direct sum of two $B(1)$-groups in [DVM 8], the results from which we move are in [DVM 12], where we studied a particular kind of decomposition "over two base elements", that mimics the general case for $B(1)$-groups. We gave there necessary and sufficient conditions in two out of three cases, and a counterexample in the third. The third case, which is addressed here, turns out to be the most intricate; a complete solution depends on one of the main open problems for a $B(2)$-group, the determination of its typeset (the set of types of its pure rank one subgroups).

Viewing G as a quotient X/K, where $X = R_1x_1 \oplus \cdots \oplus R_mx_m$ and each $R_i \cong \langle g_i \rangle_*$ is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ of type t_i, we see that its structure is based on two features: a linear one, provided by the denominator K, purely generated in X by the two relations between base elements; and an order-theoretical one, determined by the numerator X, i.e. the isomorphism types t_i of the R_i, which form the *type-base* $(t_1, \ldots, t_m)$ of G.

We will show that the desired splitting depends on whether a certain type belongs to the typeset of G (Theorem 2.4). This determination will require different procedures, depending on the *tent* of G, an order structure determined by the base types: in some cases, the order structure yields an answer; in other cases, the linear part comes into play (Theorems 1.10 B, 3.3).

The numerous examples will also enlighten the importance of the *basic partition* of G (showing which base elements are not distinguished by the relations) and its interactions with the regularity of the representation (Examples 3.5 and 3.7).

Throughout, as is usual in this subject, we use as a basic equivalence *quasi-isomorphism* [F II] instead of isomorphism; we write "*isomorphic, indecomposable, direct decomposition, ...*" instead of "quasi-isomorphic, strongly indecomposable, quasi-direct decomposition, ...".

1. Notation and previous results

Lower case greek letters, with the exception of σ and τ, denote rational numbers; $I = \{1, \ldots, m\}$. We will keep notation and tools introduced in our previous papers (in particular [DVM 4], [DVM 11], [DVM 12]); we quote here the essential ones. $\mathbb{T}(\wedge, \vee)$ is the lattice of all *types* (= isomorphism classes of rank 1 groups), with the added maximum ∞ for the type of the 0 subgroup; if g is an element of a group G, $t_G(g)$ denotes the type in G of the pure subgroup $\langle g\rangle_*$; if G is a Butler group, typeset$(G) = \{t_G(g) \mid g \in G\}$ is a finite sub-$\wedge$-semilattice of $\mathbb{T}$, hence (having ∞ as a maximum) a lattice. $\mathcal{P}(I)$ denotes the set of parts of I; $\mathbb{P}(I)(\vee, \wedge)$ is the lattice of partitions $\mathcal{A} = \{A_1, \ldots, A_k\}$ of I under the ordering "bigger = coarser"; *blocks* A_i of partitions are nonempty by definition; their complements $I\backslash A_i$ are called *coblocks*.

If $E \subseteq I$ we set

$$E^{-1} = I\backslash E.$$

For a group $W = \langle w_1\rangle_* + \cdots + \langle w_m\rangle_*$, and $E \subseteq I$, let

$$\begin{aligned} w_E &= \sum\{w_i \mid i \in E\}, \\ W_E &= \langle w_i \mid i \in E\rangle_*. \end{aligned}$$

The partition $\mathcal{C} = \{C_1, \ldots, C_h\}$ associated to the element $w = \gamma_1 w_{C_1} + \cdots + \gamma_h w_{C_h}$ of W, where $\gamma_i \neq \gamma_j$ for $i \neq j$, is called *a partition of I into equal-coefficient blocks* for w, or shortly *a partition of* w, w.r.to the elements $w_1, \ldots, w_m$; when these elements are fixed, we set $\mathcal{C} = \mathrm{part}_W(w)$.

Definition 1.1 ([DVM 11])**.** If $(t_1, \ldots, t_m)$ is a fixed m-tuple of types and $E \subseteq I$, set

$$\tau(E) = \bigwedge\{t_i \mid i \in E\},$$

in particular, $\tau(\emptyset) = \infty$; if $\mathcal{E}$ is a set of subsets of I define

$$t(\mathcal{E}) = \bigvee\{\tau(E^{-1}) \mid E \in \mathcal{E}\};$$

the thus defined map $t : \mathcal{P}(\mathcal{P}(I)) \to T$ is called *tent* (details in [DVM 11]), and $(t_1, \ldots, t_m)$ is its *base*; we will often call tent the base itself. □

In the following, our $B(2)$-group

$$G = \langle g_1\rangle_* + \cdots + \langle g_m\rangle_*$$

will be *regular*, i.e. the *base elements* $g_1, \ldots, g_m$ will satisfy (in order to constitute a *regular base*) the two *basic relations*:

$$g_I = g_1 + \cdots + g_m = 0 \text{ (the diagonal relation), and}$$
$$g_0 = \alpha_1 g_{A_1} + \cdots + \alpha_k g_{A_k} = 0 \text{ (the second relation);}$$

here $k \geq 3$ (otherwise G is a direct sum of two $B(1)$-groups [DVM 8]), and for $j', j'' \in J = \{1, \ldots, k\}$ we have $\alpha_{j'} \neq \alpha_{j''}$ iff $j' \neq j''$. The partition

$$\mathcal{A} = \{A_1, \ldots, A_k\}$$

of I is called the *basic partition* of G, and its blocks A_j are called *sections*. Base elements indexed in the same section are called *adjacent*. Since in the following we will privilege A_1, we set

$$A = A_2 \cup \cdots \cup A_k.$$

An element $g \in G$ can be written in many ways as a linear combination of base elements; each such linear combination is a *representative* of g. If $G = X/K$, with $X = R_1x_1 \oplus \cdots \oplus R_m x_m$ and $K = \langle x_I, x_0 \rangle_*$ purely generated by the two relations, the representatives of $g = x + K$ are the elements $x + y$ of X with $y \in K$; when we look for types, w.l.og. $y = \lambda x_I + \mu x_0$. For representatives of 0 we have

Lemma 1.2. *If* $0 = \gamma_1 g_{C_1} + \cdots + \gamma_h g_{C_h}$ *with* $\gamma_i \neq \gamma_j$ *if* $i \neq j$ *then either* $h = 1$ *or* $\mathcal{C} = \mathcal{A}$. □

For all $i \in I$, $t_i = t_G(g_i)$ is a *base type* of G, and $(t_1, \ldots, t_m)$ the *type-base* of G.

The type in G of an element $g = x + K$ is the supremum of the types in X of its representatives $x + y$ $(y \in K)$. The type in X of an element $x = \gamma_1 x_1 + \cdots + \gamma_m x_m \in X$ is the infimum of the types of the base elements of X effectively occurring in x. Setting

$$\begin{aligned}
&\mathrm{supp}(x) = \{i \in I \mid \gamma_i \neq 0\} \text{ and} \\
&Z(x) = \{i \in I \mid \gamma_i = 0\} \text{ (the \textit{zero-block} of } x\text{); we then have} \\
&t_X(x) = \wedge\{t_i \mid i \in \mathrm{supp}(x)\} = \tau(\mathrm{supp}(x)) = \tau((Z(x))^{-1}), \text{ hence} \\
&t_G(g) = \vee\{t_X(x+y) \mid y \in K\} = \vee\{\tau((Z(x+y))^{-1}) \mid y \in K\}.
\end{aligned}$$

We call *zero-blocks of* g the zero-blocks of its representatives $x + y$; $\underline{\mathrm{fam}_G(g)} = \{Z(x) \mid x + K = g\}$, *the set of zero-blocks of* g; $\underline{\mathrm{maxfam}_G(g)}$ *the set of maximal elements of* $\mathrm{fam}_G(g)$; $\underline{\mathrm{Maxfam}(G)} = \{\mathrm{maxfam}_G(g) \mid g \in G\}$.

Lemma 1.3. $t_G(g) = t(\mathrm{fam}_G(g)) = t(\mathrm{maxfam}_G(g))$;
$\mathrm{typeset}(G) = t(\mathrm{Maxfam}(G))$;
thus the elements of $\mathrm{typeset}(G)$ *are suprema of infima of base types.* □

A useful special case is the following:

Lemma 1.4. *If* $\emptyset \neq E \subseteq A_1$ *we have* [DVM 12]*:*

$$\begin{aligned} t_G(g_E) &= \tau(E) \vee \tau(E^{-1} \cap A_2^{-1}) \vee \cdots \vee \tau(E^{-1} \cap A_k^{-1}) \\ &= \tau(E) \vee [\tau(A_1 \backslash E) \wedge (\vee\{\wedge\{\tau(A_{j'}) \mid j' \neq 1, j\} \mid j = 2, \ldots, k\})] \\ &= \tau(E) \vee [\tau(A_1 \backslash E) \wedge (\tau(A \backslash A_2) \vee \cdots \vee \tau(A \backslash A_k))]. \qquad \square \end{aligned}$$

Following [DVM 4] and [DVM 11], without loss of generality *we will take the base types to consist of all zeros but a finite number of infinities*; they form a finite table (called *tent* as well) where sections are marked; its *columns* are also called *primes*, and we consider each type as a product of its primes: e.g., writing primes instead of infinities and dots instead of zeros:

Example 1.5.

A_1	t_1	=	p_1	p_2	.	.	.	.	.	.
	t_2	=	.	.	p_3	p_4	.	.	.	.
A_2	t_3	=	.	p_2	.	p_4	s_3	.	.	.
A_3	t_4	=	.	p_2	p_3	.	.	s_4	.	.
A_4	t_5	=	p_1	.	.	p_4	.	.	s_5	.
A_5	t_6	=	p_1	.	p_3	.	.	.	.	s_6

(This is in fact the tent of the main counterexample 4.8 in [DVM 12]). The partition $\{A_1, \ldots, A_5\} = \{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$ adds a linear information, i.e. it tells that the second relation is of the form

$$\alpha_1 g_{\{1,2\}} + \alpha_2 g_3 + \alpha_3 g_4 + \alpha_4 g_5 + \alpha_5 g_6 = 0$$

with $\alpha_i \neq \alpha_j$ for $i \neq j$.

If the type σ is a product of primes among which there is p, we say p *divides* σ ($p \mid \sigma$), or σ *has the prime* p; p *is a prime of* g, or *divides* g, if p divides $t_G(g)$. *Each prime* p *has a zero-block* $\underline{Z(p)}$*, with* $\underline{\mathrm{supp}(p)} = Z(p)^{-1}$: e.g., $Z(p_1) = \{2,3,4\}$, $\mathrm{supp}(s_3) = \{3\}$. For a type σ, the set of zero-blocks of primes dividing σ is

$$\underline{ZB_t(\sigma)} = \{Z(p) \mid p \text{ divides } \sigma\}.$$

A prime p with $Z(p) = I \backslash \{i\}$ (like the primes s_i above) is called a *locking prime*; by Lemma 1.3 the base type t_i where p occurs (a *locked type*, e.g. type t_3 above) is the only such type, and is bound to belong to every type-base of G. Here we will not consider the empty prime (one that does not divide any base type).

Lemma 1.6. (a) (Regularity, [DVM 12]) *In the tent of a regular* $B(2)$*-group there are no primes with only one zero in a section, and all other zeros in another section. If such a situation should occur, the single hole will be filled by the prime (regularization).*

(b) *The subset* E *of* I *contains the zero-block* $Z(p)$ *if and only if* p *divides* $\tau(E^{-1})$*;* p *is a prime of* g *(equivalently:* p divides g*, or* g covers p*) if and only if some zero-block* E *of* g *contains* $Z(p)$*; a type* σ *divides* g *if and only if every set in* $ZB_t(\sigma)$ *is contained in a set of* $\mathrm{maxfam}_G(g)$. $\square$

We quote now from [DVM 12, Proposition 1.12] a useful description of the pure subgroups G_E of a $B(2)$-group G *viewed as* $B(n)$*-groups*, when E or E^{-1} is contained in a section.

Proposition 1.7 ([DVM 12])**.** *Let G be a $B(2)$-group, $E \subseteq A_j$ for some $j \in J$. Then*

(i) $G_E = \sum\{\langle g_i\rangle_* \mid i \in E\} + \langle g_{E^{-1}}\rangle_*$ *is a $B(1)$-group of rank $|E|$, with the diagonal relation, and*

(ii) $G_{E^{-1}} = \sum\{\langle g_i\rangle_* \mid i \in E^{-1}\} + \langle g_E\rangle_*$ *is a $B(2)$-group of rank $|E^{-1}| - 1$, with the same basic relations as G.* □

We get now to $B(2)$-groups decomposing over two base elements.

Definition 1.8. Let $G = G' \oplus G''$ be a $B(2)$-group. We say G *splits over d base elements* if exactly d of its base elements do not belong to the set $G' \cup G''$. □

If $d = 2$, the following hold:

Lemma 1.9 ([DVM 12])**.** *If a $B(2)$-group splits over 2 base elements then it splits into the direct sum of two $B(2)$-groups.* □

Theorem 1.10 ([DVM 12])**.** *Let the $B(2)$-group G split over the two base elements g_i and g_j $(i \neq j)$. Then*

(1) *if g_i and g_j are not adjacent, there is a partition $\{\{i, j\}, E, F\}$ of I, such that $G = G_E \oplus G_F$, and we have necessary and sufficient conditions on the tent of G for this to happen* [DVM 12, 3.5]*;*

(2) *if g_i and g_j are adjacent, let w.l.o.g. $i = 1$, $j = 2$, with $1, 2 \in A_1$. Let $G = G' \oplus G''$, $\{\{1, 2\}, E, F\}$ the partition of I such that $G' \geq G_E$, $G'' \geq G_F$. Then $E \subseteq A_1$, and one of two situations occurs:*

- (A) *$G' = G_E$ and $G'' > G_F$. Here again we have necessary and sufficient conditions on the tent of G for this splitting* [DVM 12, 4.6]*.*
- (B) *$G' > G_E$ and $G'' = G_F$. The tent in Example 1.5 yields two $B(2)$-groups, one with second relation $3g_3 - 3g_4 + 6g_5 - 2g_6 = 0$, where G splits over g_1 and g_2; another with second relation $g_3 - g_4 + 2g_5 - 2g_6 = 0$, where G does not split.* □

Our goal in this paper is to examine Case (B), determining when conditions on the tent suffice to cause the splitting, and when instead the second relation comes into play.

2. The setting

Given $G = G' \oplus G''$, decompose accordingly each base element: $g_i = g_i' + g_i''$. Set $E = \{i \in I \mid g_i'' = 0\}$, $F = \{i \in I \mid g_i' = 0\}$; if G splits over two base elements, we have $m = |E| + |F| + 2$, $\mathrm{rk}(G) = |E| + |F|$.

Since $g_i = g_i' + g_i''$ implies $t_i = t_G(g_i') \wedge t_G(g_i'')$, we have $\langle g_i\rangle_* = \langle g_i'\rangle_* + \langle g_i''\rangle_*$; then $G = \sum\{\langle g_i'\rangle_* + \langle g_i''\rangle_* \mid i \in I\}$, hence G' is the sum of $|E| + 2$ rank 1 groups, G'' is the sum of $|F| + 2$ rank 1 groups.

For case (B), let $G' > G_E$ and $G'' = G_F$. Then G'' – in its form as G_F – is by Proposition 1.7 a $B(2)$-group of rank $|F|-1$ (and since $E \subseteq A_1$, hence $A \subseteq F$, the second relation of G holds in G_F). Then $\mathrm{rk}(G') = |E|+1$ (thus $\mathrm{rk}(G'/G_E) = 1$); since G' is the sum of $|E|+2$ rank 1 groups, G' is a $B(1)$-group. Again from Proposition 1.7, it is easy to see that

Proposition 2.1. *In the above setting, G splits over g_1 and g_2 (with $1, 2 \in A_1$) into $G' \oplus G_F$ if and only if $\{\{1,2\}, E, F\}$ is a partition of I with $E \subseteq A_1$ such that*

(1a) *G/G_E splits over $g_1 + G_E$ and $g_2 + G_E$ into $(G'/G_E) \oplus (G_{E\cup F}/G_E)$;*
(1b) *$G_{E\cup F}$ splits over its base element $g_{\{1,2\}}$ into $G_E \oplus G_F$.* □

Conditions (1a) and (1b) are independent: to see that (1b) does not imply (1a), let $G = \langle g_1\rangle_* + \cdots + \langle g_5\rangle_*$ with second relation $\alpha_4 g_4 + \alpha_5 g_5 = 0$, $\mathcal{A} = \{\{1,2,3\}, \{4\}, \{5\}\}$, and tent

$$\begin{array}{cccccc} A_1 & t_1 & = & p & . \\ & t_2 & = & . & q \\ & t_3 & = & . & . \\ \hline A_2 & t_4 & = & . & . \\ \hline A_3 & t_5 & = & . & . \end{array}$$

(hence G is also $B(1)$ with base $(h_1 = g_1, h_2 = g_2, h_3 = g_3, h_4 = \frac{\alpha_5-\alpha_4}{\alpha_5} g_4)$). For $E = \{3\}$, $F = \{4,5\} = A$, (1b) holds, since $G_{E\cup F} = \langle h_3, h_4\rangle_* = \langle h_3\rangle_* \oplus \langle h_4\rangle_*$; but G/G_E cannot split over $g_1 + G_E$, $g_2 + G_E$, whose types are locked. Note also that G has no element of type $t_1 \vee t_2 = p\, q$.

To see that (1a) does not imply (1b), let G, E, F be as above with tent

$$\begin{array}{ccccccc} A_1 & t_1 & = & p & q & . \\ & t_2 & = & p & . & . \\ & t_3 & = & . & . & r \\ \hline A_2 & t_4 & = & . & q & . \\ \hline A_3 & t_5 & = & . & q & . \end{array}$$

The tent of $G_{E\cup F}$ is

$$\begin{array}{ccccccc} A_1 & t_{1,2} & = & p & . & . \\ & t_3 & = & . & . & r \\ \hline A_2 & t_4 & = & . & q & . \\ \hline A_3 & t_5 & = & . & q & . \end{array}$$

Here $G_{E\cup F}$ cannot split into $G_E \oplus G_F$, because $t_{1,2}$ is locked; while G/G_E splits over $g_1 + G_E$, $g_2 + G_E$, since it becomes homogeneous by regularization (see below).

The next corollary makes condition (1b) easy to check first:

Corollary 2.2. *Condition* (1b) *is equivalent to*

$$(*) \qquad t_G(g_{\{1,2\}}) \le t_G(g_E).$$

Moreover, there is a maximum $E \subseteq A_1 \backslash \{1,2\}$ satisfying $(*)$.

PROOF: The first assertion is in [DVM 10, 2.4]. Let then t' be the tent of the $B(1)$-group $G' = G_{A_1} = \sum\{\langle g_i\rangle_* \mid i \in A_1\} + \langle g_{A_1}\rangle_*$ (Proposition 1.7), with $t'_i = t_i$ for $i \in A_1$, and

$$t'_0 = t_G(g_{A_1}) = \tau(A_1) \vee \tau(A\backslash A_2) \vee \cdots \vee \tau(A\backslash A_k) \quad \text{(Lemma 1.4)}.$$

Then a prime p divides t'_0 if and only if it either divides $\tau(A_1)$ (hence its zero-block $Z'(p)$ is contained in A), or it divides all but one of the $\tau(A_{j'})$ with $j' \neq 1$ (that is $Z'(p) \subseteq A_1 \cup A_j$ for some $j = 2, \dots, k$).

Let $\text{part}_{t'}(t_1 \wedge t_2) = \mathcal{C} = \{\{1\}, \{2\}, C_0, \dots, C_s\}$, where $s \geq 0$ and C_0 is the block containing 0. Set

$$E = \bigcup\{C_i \mid i = 1, \dots, s\}.$$

We show that E is maximum satisfying $(*)$.

Computing types in the $B(1)$-group G' (pure in G) we have

$$t_{G'}(g_{\{1,2\}}) = (t_1 \wedge t_2) \vee \tau(C_0 \cup E) = (t_1 \wedge t_2) \vee (\tau(C_0) \wedge \tau(E)),$$
$$t_{G'}(g_E) = (t_1 \wedge t_2 \wedge \tau(C_0)) \vee \tau(E).$$

A prime p dividing $g_{\{1,2\}}$ either divides $\tau(C_0 \cup E)$, hence $\tau(E)$, hence g_E; or it divides $t_1 \wedge t_2$; if p does not divide $\tau(E)$, it has a hole – say – in C_1. But then all of its holes are in C_1, since $\mathcal{C} = \text{part}_{t'}(t_1 \wedge t_2)$; thus it divides $\tau(C_0)$, hence g_E. Therefore $t_{G'}(g_{\{1,2\}}) \le t_{G'}(g_E)$.

To show maximality, let $E' \subseteq A_1 \backslash \{1,2\}$ with $C_0 \cap E' \neq \emptyset$; note that $0 \notin E'$, thus $E' \neq C_0$. Since C_0 is a block of $\mathcal{C}$, there must be a prime dividing $t_1 \wedge t_2$ (hence $g_{\{1,2\}}$) that has a hole in E' and a hole in $C_0 \backslash E'$. But then p does not divide $\tau(E')$ nor $\tau(C_0)$, hence does not divide $t_{G'}(g_{E'})$; thus $(*)$ does not hold. □

Our check on G then starts with (1b): when it holds, we can operate a first reduction, modding out G_E: a simple operation, reducing I to $\{1,2\} \cup C_0 \cup A = \{1,2\} \cup F$.

Lemma 2.3. *The tent of G/G_E is obtained from the tent of G by eliminating the base types indexed in E and then regularizing.*

PROOF: $G/G_E = \sum\{\langle g_i + G_E\rangle_* \mid i \in \{1,2\} \cup F\}$, with the relations of G inherited by the cosets, hence in particular with the same second relation. A surviving base type t_i ($i \notin E$) might change only if a prime p that did not divide a base element g_i will divide $g_i + G_E$: this means that

- no representative $g_i + \lambda g_I + \mu g_0$ of g_i covers p, while
- there is a representative of $g_i + G_E$: $g = g_i + \sum\{\beta_r g_r \mid r \in E\} + \lambda g_I + \mu g_0$, covering p; that is, $Z(p)\backslash E \subseteq \{i\} \cup A_j$ for some $j \in J$, with $i \notin A_j$.

But this is the situation described in Lemma 1.6, where the hole of p in $\{i\}$ will be filled by regularization, hence t_i would have the prime p, against our hypothesis. □

After modding out G_E, the remaining condition (1a) is reduced to the solution of the following

Theorem 2.4. *Given the $B(2)$-group G, let $I = \{1,2\} \cup F$ with $A_1 \supseteq \{1,2\}$. Then G splits over g_1 and g_2 into $\langle g' \rangle_* \oplus G_F$ if and only if there is an element g' such that*

(a) $g' = \sum\{\beta_i g_i \mid i \in I\}$ *with* $\beta_1 \neq \beta_2$;
(b) $t_G(g') = t_1 \vee t_2$.

If in particular $t_1 \leq t_2$, G splits over g_1.

Proof: To recover g_1 linearly inside $\langle g' \rangle_* \oplus G_F$ we must have in G a relation $\gamma g_1 = \gamma' g' - g''$, with $g'' \in G_F$, that is $\gamma' g' = \gamma g_1 + g''$, where $\gamma \neq 0$: here the coefficient of g_2 is 0. As a consequence, no representative of g' has both 1 and 2 in the same zero-block; but since a prime p not dividing $t_1 \vee t_2$ (i.e. not dividing either t_1 or t_2) has $\{1,2\} \subseteq Z(p)$, p cannot be covered by g'; therefore $t_G(g') \leq t_1 \vee t_2$. Finally, to recover the type of g_1 from $t_G(g_1) = t_G(\gamma' g') \wedge t_G(g'')$, we must have $t_G(g') \geq t_1$; same for g_2; hence we get $t_G(g') \geq t_1 \vee t_2$. □

Here is an example showing that condition (a) does not depend from (b):

Example 2.5. Let

$$
\begin{array}{cccccc}
A_1 & t_1 & = & . & q \\
 & t_2 & = & p & . \\
\hline
A_2 & t_3 & = & p & q \\
\hline
A_3 & t_4 & = & . & . \\
\hline
A_4 & t_5 & = & . & .
\end{array}
$$

Here the only element of maximum type is g_3, and all of its representatives have $\beta_1 = \beta_2$. □

3. Elements of given type

Condition (b) asks for the existence in G of an element of a given type. This is a version of the open question of the determination of typeset(G). The typeset of a $B(1)$-group is the image of the restriction of the map $t : \mathcal{P}(\mathcal{P}(I)) \to \mathbb{T}$ to $\mathbb{P}(I)$; it is well understood and easy to work on. The typeset of a $B(2)$-group is the image of the restriction of the same map $t : \mathcal{P}(\mathcal{P}(I)) \to \mathbb{T}$ to Maxfam(G), which depends on the linear part of G; the typeset shows no clear structure at this point of research.

What we are required by our problem is, *given a type σ as a product of primes of the tent, whether or not $\sigma \in$* typeset(G). The answer – which always depends on the tent – may in some cases also depend on the linear structure, that is, on the second relation of G.

Starting with our $B(2)$-group G, and with a product of primes of $G : \sigma = p_1 p_2 \dots p_n$, we look for an element $\rho g'$ (defined up to a scalar multiple ρ, with $g' = \sum\{\beta_i g_i \mid i \in I\}$) of type σ; that is, such that, for every prime p_r of σ, there is a representative $\rho g' + \lambda g_I + \mu g_0$ of $\rho g'$ whose zero-block contains $Z(p_r)$. We are thus looking for an element

$$\begin{aligned}
&\rho g' + \lambda g_I + \mu g_0 = \\
&\quad = \sum\{\rho\beta_i g_i \mid i \in I\} + \lambda \sum\{g_i \mid i \in I\} + \mu \sum\{\alpha_j g_{A_j} \mid j \in J\} = \\
&\quad = \sum\{(\rho\beta_i + \lambda + \mu\alpha_{j(i)})g_i \mid i \in I \cap A_{j(i)}\}
\end{aligned}$$

(where $j(i)$ *is the index of the section containing* i) that covers p_r for $r = 1, \dots, n$; that is,

Proposition 3.1. *To find an element of type σ we must solve the following system of equations* (0) *in the unknowns ρ, λ_r, μ_r, β_i:*

$$\rho\beta_i - \lambda_r - \mu_r\alpha_{j(i)} = 0, \tag{0}$$

for all $i \in Z(p_r)$ and all $r = 1, \dots, n$, looking for solutions different from the trivial ones $(\rho,\ \lambda_r = \lambda,\ \mu_r = \mu,\ \beta_i \mid r = 1, \dots, n,\ i = 1, \dots, m)$ *which yield the zero element.* □

This search does not involve primes of the tent not dividing σ, hence

Main reduction. *Cancel from the tent all primes not dividing σ.* □

Our problem now amounts to asking whether the tent of G – thus reduced – has a proper ($\neq \infty$) maximum σ. Recalling the definition of the classical fully invariant subgroup $G(\sigma) = \{g \in G \mid t_G(g) \geq \sigma\}$, the question can also be refined by looking for the rank of $G(\sigma)$.

An immediate observation is the following:

Lemma 3.2. *If a tent has a locked type, either this type is maximum, or there is no maximum.* □

System (0) can be visualized from the tent, as in the following example:

Example 1.5 (continued). Consider the tent of Example 1.5; say we look for $\sigma = p_1 p_2 p_3 p_4$; then with the main reduction we may simplify the tent into:

A_1	t_1	$=$	p_1	p_2	.	.
	t_2	$=$	.	.	p_3	p_4
A_2	t_3	$=$	.	p_2	.	p_4
A_3	t_4	$=$	.	p_2	p_3	.
A_4	t_5	$=$	p_1	.	.	p_4
A_5	t_6	$=$	p_1	.	p_3	.

Attach to the tent a *grid* (1) where the holes are represented by dots:

(1)

		$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$
A_1	t_1			•	•
	t_2	•	•		
A_2	t_3	•		•	
A_3	t_4	•			•
A_4	t_5		•	•	
A_5	t_6		•		•

Then, viewing the linear system (0) in the unknowns λ_1, μ_1, …, λ_4, μ_4 (although the $\rho\beta$'s should be considered unknowns as well), its matrix is shaped by the above grid in the following way:

$$\begin{array}{rc}
 & \begin{array}{cccccccc} \lambda_1 & \mu_1 & \lambda_2 & \mu_2 & \lambda_3 & \mu_3 & \lambda_4 & \mu_4 \end{array} \\
\begin{array}{r} \rho\beta_1 \\ \rho\beta_1 \\ \rho\beta_2 \\ \rho\beta_2 \\ \rho\beta_3 \\ \rho\beta_3 \\ \rho\beta_4 \\ \rho\beta_4 \\ \rho\beta_5 \\ \rho\beta_5 \\ \rho\beta_6 \\ \rho\beta_6 \end{array} &
\left[\begin{array}{cc|cc|cc|cc}
0 & 0 & 0 & 0 & \mathbf{1} & \alpha_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \alpha_1 \\
\hline
\mathbf{1} & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \alpha_1 & 0 & 0 & 0 & 0 \\
\hline
\mathbf{1} & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \alpha_2 & 0 & 0 \\
\hline
\mathbf{1} & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \alpha_3 \\
\hline
0 & 0 & \mathbf{1} & \alpha_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \alpha_4 & 0 & 0 \\
\hline
0 & 0 & \mathbf{1} & \alpha_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & \alpha_5 &
\end{array}\right]
\end{array}$$

E.g., the first dot in the third row refers to the equation $\rho\beta_2 + \lambda_1 + \mu_1\alpha_1 = 0$, which looks for a representative $\rho g' + \lambda_1 g_I + \mu_1 g_0$ that is 0 on the hole that p_1 has on t_2. We see how the tent (an order structure) determines the system (a linear structure). □

We start by looking for conditions independently from the partition, hence we *redenote the second relation* as

$$\alpha_1 g_1 + \cdots + \alpha_m g_m = 0;$$

the section equalities $\alpha_i = \alpha_j$ will then represent additional linear conditions to be satisfied. We have a simple sufficient condition:

Theorem 3.3. *Let t be a tent where no prime has two holes in the same section, $Z(t)$ the number of holes of t. Then a sufficient condition for the existence of an element of maximum type is*

$$Z(t) - 2n \leq m - 3. \tag{2}$$

Proof: Let the prime p_r have its $z_r = |Z(p_r)|$ holes on – say – $t_1, t_2, \dots, t_{z_r}$ ($z_r \le m$); note that, in our hypothesis, $\alpha_1, \alpha_2, \dots, \alpha_{z_r}$ are pairwise distinct. The part of system (0) determined by p_r is

$$\begin{aligned} \rho\beta_1 + \lambda_r + \mu_r\alpha_1 &= 0 \\ \rho\beta_2 + \lambda_r + \mu_r\alpha_2 &= 0 \\ \dots\dots\dots\dots\dots\dots & \\ \rho\beta_{z_r} + \lambda_r + \mu_r\alpha_{z_r} &= 0, \end{aligned}$$

a homogeneous system in the unknowns ρ, λ_r, μ_r which in our hypothesis has a nonzero solution if and only if the determinants $\begin{vmatrix} 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_i & \beta_i \end{vmatrix}$ are zero for all $i = 3, \dots, z_r$. We have thus a system of $|Z(p_r)| - 2$ linear equations for the prime p_r:

$$(3) \qquad (\alpha_2 - \alpha_i)\beta_1 + (\alpha_i - \alpha_1)\beta_2 + (\alpha_1 - \alpha_2)\beta_i = 0, \quad i = 3, \dots, z_r$$

homogeneous in the unknowns β. On the whole, we have $\sum\{|Z(p_r)| - 2 \mid r = 1, \dots, n\} = |Z(t)| - 2n$ homogeneous linear equations in the m unknowns $\beta_1, \dots, \beta_m$.

A nonzero solution of this system determines a single representative of our $\rho g'$. In order to have all representatives we need three independent solutions, and this happens whenever $Z(t) - 2n \le m - 3$. □

Observation 3.4. In general, system (3) consists of equations *in the differences* $\alpha_i - \alpha_j$; therefore such will also be the compatibility conditions for the system. There will thus always be solutions, at least those of the form $\alpha_i = \alpha_j$ for suitable $i \ne j$; but this does not mean that, given an m-tuple of base types, there will always be second conditions guaranteeing, for the group they define, the existence of a maximum type. The fact is, the sections defined by these conditions may collide with regularity, as shown both in the next and especially in the last example. □

What may happen if a prime has two holes in the same section is shown in

Example 3.5. Consider the following tent:

$$\begin{array}{cccc} A_1 & t_1 & = & . \quad q \\ & t_2 & = & . \quad q \\ \hline A_2 & t_3 & = & . \quad q \\ \hline A_3 & t_4 & = & p \quad . \\ & t_5 & = & . \quad . \end{array}$$

Here $Z(t) = 6$, $n = 2$, $m = 5$, hence (2) is satisfied (but p has two holes in the same section); t_4 is locked, thus by Lemma 3.2 there is no maximum. Observe that, if we replaced the partition $\{\{1, 2\}, \{3\}, \{4, 5\}\}$ of the above tent with the partition $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$, removing the obstacle to Theorem 3.3, the numbers in (2) would not change, so the conditions would seemingly be satisfied, provoking an apparent contradiction with Lemma 3.2. What really happens is that the new tent

A_1	t_1	$=$	.	q
A_2	t_2	$=$	.	q
A_3	t_3	$=$	.	q
A_4	t_4	$=$	p	.
A_5	t_5	$=$	.	.

is not regular any more; by regularity, q divides both t_4 and t_5, hence t_4 is maximum, as predicted by Lemma 3.2. $\square$

A practical rule in the case of a prime with ≥ 2 holes in the same section is found in [DVM 14]:

Proposition 3.6 ([DVM 14])**.** *If, for a prime p, $E = Z(p) \cap A_j$ for some $j = 1, \ldots, k$, the maximum of G can be found in the tent obtained by replacing the types indexed in E with the type $\tau(E)$ and regularizing.* $\square$

We can now conclude the analysis of our initial example:

Example 1.5 (completed). Without the partition, the grid (1) becomes

(1′)

		$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$
A_1	t_1			•	•
A_2	t_2	•	•		
A_3	t_3	•		•	
A_4	t_4	•			•
A_5	t_5		•	•	
A_6	t_6		•		•

yielding (since every prime has three holes in different sections) the *transposed* matrix of system (3):

$$(3) \quad \begin{array}{c c} & \begin{array}{cccc} \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 & \mathbf{P}_4 \end{array} \\ \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{array} & \left[\begin{array}{c|c|c|c} 0 & 0 & \alpha_3-\alpha_5 & \alpha_4-\alpha_6 \\ \hline \alpha_3-\alpha_4 & \alpha_5-\alpha_6 & 0 & 0 \\ \hline \alpha_4-\alpha_2 & 0 & \alpha_5-\alpha_1 & 0 \\ \hline \alpha_2-\alpha_3 & 0 & 0 & \alpha_6-\alpha_1 \\ \hline 0 & \alpha_6-\alpha_2 & \alpha_1-\alpha_3 & 0 \\ \hline 0 & \alpha_2-\alpha_5 & 0 & \alpha_1-\alpha_4 \end{array}\right] \end{array}$$

Here $Z(t) = 12$, $2n = 8$, $m = 6$; $Z(t) - 2n = 4$ is not $\leq m - 3 = 3$, thus the sufficient condition does not hold.

We need three independent solutions of the system: the rank of the matrix must be ≤ 3, which sets a condition on the α's. In order to simplify the problem, w.l.o.g. place P_1 in the origin, setting $\lambda_1 = \mu_1 = 0$, hence $\beta_2 = \beta_3 = \beta_4 = 0$; we are left with three equations:

$$(2') \qquad \begin{aligned} (\alpha_6-\alpha_2)\beta_5+(\alpha_2-\alpha_5)\beta_6 &= 0,\\ (\alpha_3-\alpha_5)\beta_1+(\alpha_1-\alpha_3)\beta_5 &= 0,\\ (\alpha_4-\alpha_6)\beta_1+(\alpha_1-\alpha_4)\beta_6 &= 0, \end{aligned}$$

whose determinant needs to be 0:

$$\det\begin{bmatrix} 0 & \alpha_6-\alpha_2 & \alpha_2-\alpha_5\\ \alpha_3-\alpha_5 & \alpha_1-\alpha_3 & 0\\ \alpha_4-\alpha_6 & 0 & \alpha_1-\alpha_4 \end{bmatrix} =$$

$$= (\alpha_1-\alpha_3)(\alpha_4-\alpha_6)(\alpha_2-\alpha_5)+(\alpha_1-\alpha_4)(\alpha_3-\alpha_5)(\alpha_6-\alpha_2)=0.$$

The solution then exists when the 6-tuple $(\alpha_1,\dots,\alpha_6)$ lies on the above hypersurface of $PS(5)$ and $\alpha_i\neq\alpha_j$ whenever $i\neq j$ are in some $Z(p_r)$.

Bringing back our initial section A_1 by requiring $\alpha_2=\alpha_1$ does not eliminate the condition, which thus remains in place, as we knew from the start. □

A last example clarifies Observation 3.4:

Example 3.7. Consider a group G with the following tent:

A_1	t_1	$=$	.	.	.	p	q
	t_2	$=$	p_1	p_2	p_3	.	.
A_2	t_3	$=$	.	p_2	p_3	.	q
	t_4	$=$	p_1	.	p_3	p	.
A_3	t_5	$=$	.	p_2	p_3	p	.
	t_6	$=$	p_1	p_2	.	.	q
A_4	t_7	$=$	p_1	.	p_3	.	q
	t_8	$=$	p_1	p_2	.	p	.

(by which we mean that $\alpha_1=\alpha_2$, $\alpha_3=\alpha_4$, $\alpha_5=\alpha_6$, $\alpha_7=\alpha_8$ and α_1, α_3, α_5, α_7 are pairwise distinct). We look for an element $g=\sum\{\beta_i g_i\mid i=1,\dots,8\}$ of maximum type, hence divisible by all the 5 primes above. Starting with $Z(p)$, we set $\beta_2=\beta_3=\beta_6=\beta_7=0$. Then divisibility by p_1 requires $\det\begin{vmatrix} 1 & \alpha_1 & \beta_1\\ 1 & \alpha_3 & 0\\ 1 & \alpha_5 & \beta_5\end{vmatrix}=0$; proceeding analogously for the other primes we obtain the system

$$\begin{aligned} \beta_1(\alpha_5-\alpha_3)+\beta_5(\alpha_3-\alpha_1) &= 0 \quad (\text{for } p_1)\\ \beta_1(\alpha_7-\alpha_3)+\beta_4(\alpha_1-\alpha_7) &= 0 \quad (\text{for } p_2)\\ \beta_1(\alpha_7-\alpha_5)+\beta_8(\alpha_5-\alpha_1) &= 0 \quad (\text{for } p_3)\\ \beta_4(\alpha_1-\alpha_5)+\beta_5(\alpha_3-\alpha_1) &= 0 \quad (\text{for } q)\\ \beta_4(\alpha_1-\alpha_7)+\beta_8(\alpha_3-\alpha_1) &= 0 \quad (\text{for } q) \end{aligned}$$

which is solvable (in our hypotheses on the α's) if and only if

$$(\alpha_3-\alpha_1)(\alpha_3-\alpha_5)(\alpha_1-\alpha_5)=0,$$

whose solutions are forbidden by our hypotheses: G has no element of maximum type.

If on the other hand we consider e.g. $\alpha_3 = \alpha_1$, not only the partition but also the base-types change (by regularity) into

$$\begin{array}{llcccccc}
A_1 & t_1 & = & . & . & . & p & q \\
 & t_2 & = & p_1 & p_2 & p_3 & . & . \\
 & t_3 & = & . & p_2 & p_3 & . & q \\
 & t_4 & = & p_1 & . & p_3 & p & . \\
\hline
A_3 & t_5 & = & \mathbf{p_1} & p_2 & p_3 & p & . \\
 & t_6 & = & p_1 & p_2 & . & . & q \\
\hline
A_4 & t_7 & = & p_1 & \mathbf{p_2} & p_3 & . & q \\
 & t_8 & = & p_1 & p_2 & . & p & .
\end{array}$$

An element of maximum type for such a new group G' would have $\beta_1 = \beta_3 = \beta_4 (= 0)$; divisibility by p_3 would require $\beta_8(\alpha_5 - \alpha_1) = 0$, thus either $\beta_8 = 0$ (no nonzero solution) or again a change of group. In fact, setting $\alpha_5 = \alpha_1$ we get a G'' with tent

$$\begin{array}{llcccccc}
A_1 & t_1 & = & . & . & . & p & q \\
 & t_2 & = & p_1 & p_2 & p_3 & . & . \\
 & t_3 & = & . & p_2 & p_3 & . & q \\
 & t_4 & = & p_1 & . & p_3 & p & . \\
 & t_5 & = & \mathbf{p_1} & p_2 & p_3 & p & . \\
 & t_6 & = & p_1 & p_2 & . & . & q \\
\hline
A_4 & t_7 & = & p_1 & \mathbf{p_2} & p_3 & \mathbf{p} & q \\
 & t_8 & = & p_1 & p_2 & \mathbf{p_3} & p & \mathbf{q}
\end{array}$$

which is the tent of a degenerate $B(2)$-group: the direct sum of a rank 1 group (finally, of maximum type) and a $B(1)$-group of rank 5 without maximum. $\square$

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