Radical decompositions of semiheaps

Ian Hawthorn, Tim Stokes

Abstract. Semiheaps are ternary generalisations of involuted semigroups. The first kind of semiheaps studied were heaps, which correspond closely to groups. We apply the radical theory of varieties of idempotent algebras to varieties of idempotent semiheaps. The class of heaps is shown to be a radical class, as are two larger classes having no involuted semigroup counterparts. Radical decompositions of various classes of idempotent semiheaps are given. The results are applied to involuted I-semigroups, leading to a radical-theoretic interpretation of the largest idempotent-separating congruence.

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1. Background on semiheaps

We begin with a review of some established definitions and results.

A **semiheap** H is a set with a ternary operation $[\,]$ satisfying the *para-associative* law:

$$[[abc]de] \approx [a[dcb]e] \approx [ab[cde]].$$

(Here, the symbol $\approx$ is used to denote an identity, holding for all possible substitutions of elements for the variables $a, b, \ldots$. We use this notation frequently in what follows.)

It seems that semiheaps were first considered by Vagner in [12]. Note that para-associativity is not the obvious ternary generalisation of associativity, in which the middle term above would be $[a[bcd]e]$: such "ternary semigroups" arise from semigroups if one defines $[abc] := abc$.

Various additional laws can hold in a semiheap. Historically, the most important have been the following:

1 $[aaa] \approx a$

2 $[aab] \approx b \approx [baa]$

3 $[aa[bbc]] \approx [bb[aac]]$ and $[[abb]cc] \approx [[acc]bb]$

A *heap* is a semiheap satisfying the second law above (and hence also the first and third). We call the one-element semiheap (which is a heap) the *trivial heap*. Every group gives a heap under the ternary operation $[abc] := ab^{-1}c$, a construction first considered in the setting of abelian groups by Prüfer in [10].

Conversely, a group arises from a heap H by choosing any element $e \in H$ and defining a binary operation $x * y := [xey]$; the element e becomes the identity of the constructed group and $[exe]$ the inverse of x. These constructions are mutually inverse up to isomorphism. Hence the varieties of groups and pointed heaps are term equivalent, as shown by Baer in [1].

A similar construction gives a semiheap when S is an *involuted semigroup*, that is, a semigroup equipped with a unary operation $'$ for which the following laws are satisfied:

- $a'' \approx a$, and
- $(ab)' \approx b'a'$.

Thus if S is an involuted semigroup, setting

$$[abc] := ab'c \quad \text{for all} \quad a, b, c \in S$$

gives a semiheap operation on S. Denote by $[S]$ the semiheap obtained from S in this way. Every semiheap can be embedded in $[S]$ for some involuted semigroup S: see Section 2 of [12].

A *generalised heap* is a semiheap satisfying the first and third laws above; these were considered by Vagner in [12]. Motivated by partial one-to-one functions from one set to another, generalised heaps also arise from certain kinds of involuted semigroups. An *inverse semigroup* is an involuted semigroup in which the additional laws

- $aa'a \approx a$, and
- $aa'bb' \approx bb'aa'$

hold. (Omitting the law $(ab)' \approx b'a'$, which follows from the others, gives Howie's definition as on page 145 of [7].) It can be shown that the set of idempotents of an inverse semigroup S is

$$E(S) = \{a'a \mid a \in S\},$$

and hence that $ef = fe$ for all $e, f \in E(S)$, so that $E(S)$ is a subsemigroup of S which is a semilattice. If S is an inverse semigroup then $[S]$ is a generalised heap, and all generalised heaps are subsemiheaps of generalised heaps of this form (see Section 3 of [12]).

Let us call a semiheap satisfying the idempotency law $[aaa] = a$ an *idempotent semiheap*. These are the main objects of interest in the current article. If S is an involuted semigroup, the semiheap $[S]$ is idempotent if and only if $aa'a = a$ for all $a \in S$, a condition which says that S is an *involuted I-semigroup* (see [7]).

Not all involuted I-semigroups are inverse semigroups (and indeed not all idempotent semiheaps are generalised heaps), as the following example shows. Let S be a set of symbols. Define the **rectangular band** on S to be $G = S \times S$ with binary operation $(a, b)(c, d) := (a, d)$ and involution $(a, b)' := (b, a)$. Then G is an involuted I-semigroup which is not an inverse semigroup, so the idempotent semiheap obtained from it is not a generalised heap, as is easily verified.

2. Semiheaps and radicals

In this section and the next, we shall concentrate entirely on idempotent semiheaps and their radicals, with interpretations for involuted I-semigroups considered in the final section. We begin with some quite general notions.

2.1 Background: radicals of idempotent algebras. Let $\mathcal{A}$ be any homomorphically closed class of algebras of some fixed signature. If for each $A \in \mathcal{A}$, there is an associated congruence ρ_A with the following properties:

- (Q) ρ_{A/ρ_A} is the equality congruence, and
- (F) if $f : A \to B$ is a surjection, then $f(\rho_A) \subseteq \rho_B$,

then we say the family of congruences ρ_A, $A \in \mathcal{A}$, determines a *Hoehnke radical* (see [4]). This notion of radical has proved useful in the study of semigroups (see [5]), and indeed wherever there is no natural notion of "normal subobject" available.

For rings and near rings, one can express the notion of a Hoehnke radical in terms of ideals (because of the correspondence between ideals and congruences), resulting in a notion strictly weaker than that of a Kurosh-Amitsur radical (developed earlier in the setting of associative ring theory; see [2]). The additional properties needed to give a Kurosh-Amitsur radical are that the assignment of the radical ideal to the algebra be idempotent and complete (the assigned ideal contains all "radical ideals", namely those ideals for which the associated radical congruence is the full congruence). Similar remarks apply to multioperator groups in general.

An *idempotent algebra* is one in which every singleton set is a subalgebra; equivalently, for each fundamental operation ρ of arity n, the law

$$\rho(x, x, \ldots, x) \approx x$$

holds. Thus every idempotent semiheap is an idempotent algebra, and indeed the idempotent semiheaps constitute a variety of idempotent algebras.

For any variety $\mathcal{V}$ of idempotent algebras, there is an enrichment of the Hoehnke radical notion available, with similar features to the Kurosh-Amitsur radical. Although idempotent algebras do not generally have "normal subalgebras" in any natural sense, this is compensated for by the fact that congruence classes in idempotent algebras are always subalgebras.

Let $\mathcal{V}$ be a variety of idempotent algebras (meaning algebras in which every singleton element is a subalgebra). The following definition is given in [3].

A non-empty subclass $\mathcal{R}$ of $\mathcal{V}$ is a *radical class* if

(R1) whenever δ, τ are congruences on $A \in \mathcal{V}$ for which all δ-classes are in $\mathcal{R}$, there exists a congruence λ on A/τ for which $(\delta \vee \tau)/\tau \leq \lambda$ and all λ-classes are in $\mathcal{R}$;

(R2) for every $A \in \mathcal{V}$ there is a congruence ρ_A on A for which all ρ_A-classes are in $\mathcal{R}$ and $\rho_A \geq \rho$ for every congruence ρ on A with all its classes in $\mathcal{R}$; and

(R3) $\rho_{A/\rho_A} = 0_{A/\rho_A}$ for every $A \in \mathcal{V}$.

(Here the notation 0_A refers to the trivial congruence of equality on A.) Call a congruence in A for which all classes are in $\mathcal{R}$ an *$\mathcal{R}$-congruence*. So (R2) says that every A has a largest $\mathcal{R}$-congruence, and (R3) then says that factoring this out leaves an algebra with no non-trivial $\mathcal{R}$-congruences. Obviously ρ_A above is unique for a given $\mathcal{R}$, and the ρ_A determine $\mathcal{R}$ since $A \in \mathcal{R}$ if and only if $|A/\rho_A| = 1$.

It follows from these axioms that the family of congruences ρ_A ($A \in \mathcal{V}$) determines a Hoehnke radical. Given a Hoehnke radical on $\mathcal{V}$ with associated congruence ρ, we say an algebra A is "radical" if $|A/\rho_A| = 1$. The definition of radical for varieties of idempotent algebras requires that in every A, each ρ_A-class is radical, and that ρ_A is largest amongst all such congruences; these conditions parallel similar conditions in the definition of a Kurosh-Amitsur radical, and are not even expressible in the general algebra setting to which the Hoehnke radical notion applies.

$A \in \mathcal{V}$ is *semisimple* if $\rho_A = 0_A$, and a *semisimple class* in $\mathcal{V}$ is a class $\mathcal{S}$ consisting of all semisimple algebras relative to some radical class in $\mathcal{V}$. There is an inclusion-reversing bijection between the collections of radical and semisimple classes in $\mathcal{V}$. (See Proposition 3.4 of [11] for a proof.)

A useful set of sufficient conditions is furnished by the following requirements on the subclass $\mathcal{R}$ of $\mathcal{V}$: it must satisfy (R1) together with the following.

(E) If σ is a congruence on $A \in \mathcal{V}$ for which all σ-classes of A and A/σ itself are in $\mathcal{R}$, then $A \in \mathcal{R}$.

(C) If $\{\sigma_\alpha \mid \alpha \in \Lambda\}$ is a chain of congruences on A such that each σ_α-class is in $\mathcal{R}$ for each α, then each $\bigvee_{\alpha\in\Lambda} \sigma_\alpha$-class is in $\mathcal{R}$.

(J) If σ and τ are congruences on $A \in \mathcal{V}$ such that all σ-classes and all τ-classes are in $\mathcal{R}$, then all $(\sigma \vee \tau)$-classes are in $\mathcal{R}$.

In [3], (E) and (C) are shown to be necessary conditions for $\mathcal{R}$ to be a radical class, and (R1), (E), (C) and (J) are shown to be sufficient. In fact it follows from the very general work of [8] that condition (J) is superfluous in this set of sufficient conditions, and so (R1), (E) and (C) provide a characterisation of radical classes. However, (J) is easily established in the examples to follow, and we include the arguments.

Proposition 4.2 in [3] states that a homomorphically closed class $\mathcal{R}$ is a radical class if and only if it satisfies (C), (E), (J), and condition (F) for a Hoehnke radical. If the class $\mathcal{R}$ is closed under taking subalgebras and ρ is an $\mathcal{R}$-congruence on $A \in \mathcal{A}$, and if δ is a congruence on A contained in ρ, then δ is obviously an $\mathcal{R}$-congruence on A also; it follows that $\mathcal{R}$ satisfies (R1) if and only if it satisfies the (usually stronger) condition

(R1′) whenever δ, τ are congruences on $A \in \mathcal{V}$ for which all δ-classes are in $\mathcal{R}$, then $(\delta \vee \tau)/\tau$ is an $\mathcal{R}$-congruence on A/τ.

The next result follows because (C) holds in any variety (see the proof of Theorem 3.1 in [3]).

Corollary 1. *The following are equivalent for the subvariety $\mathcal{R}$ of the variety $\mathcal{V}$ of idempotent algebras:*

- *$\mathcal{R}$ is a radical class in $\mathcal{V}$;*
- *$\mathcal{R}$ satisfies* (R1′), (E) *and* (J)*;*
- *$\mathcal{R}$ satisfies* (F), (E) *and* (J)*.*

2.2 Radicals of heaps. The radical theory just discussed immediately applies to any variety of idempotent semiheaps. A natural starting point is the variety of heaps, because of its close connection to the variety of groups (for which a conventional Kurosh-Amitsur radical theory exists). Now because every heap is, up to isomorphism, of the form $[S]$ for some group S, and because for a given heap H the group $\langle H \rangle$ is itself determined up to isomorphism, it follows that every isomorphically closed class of heaps has the form

$$[\mathcal{K}] = \{H \mid H \cong [S] \text{ for some } S \in \mathcal{K}\}$$

where $\mathcal{K}$ is an isomorphically closed class of groups.

Theorem 2. *The radicals of groups and heaps correspond: the usual Kurosh-Amitsur group radical class $\mathcal{R}$ corresponds to the heap radical class $[\mathcal{R}]$.*

Proof: This follows from the fact that the congruences on the group G and the heap $[G]$ correspond (easily checked) and the fact that heaps therefore have permuting congruences. In this case, (R1) can be replaced by the generally weaker condition of homomorphic closure (Corollary 1.8 of [3]), and it is easy to check that this together with the conditions (R2) and (R3) for a radical class of heaps correspond to the usual Kurosh-Amitsur conditions in the definition of a radical class of groups (interpreted in terms of congruences). □

2.3 Radicals of semiheaps. We now broaden the outlook, and show that the variety of heaps is a radical class within any variety of idempotent semiheaps. Indeed we shall consider three radical classes of idempotent semiheaps, two of which are strictly larger than the class of heaps. These two larger classes prove to be of interest only in the ternary setting (collapsing back down to the heap radical for involuted I-semigroups).

First, we introduce some useful pre-orders. Let H be an idempotent semiheap, with $a, b \in H$. We will say that $a \leq_l b$ if $a = [abb]$, $a \leq_r b$ if $a = [bba]$, and $a \leq b$ if $a = [abb] = [bba]$; hence $\leq \; = \; (\leq_l \cap \leq_r)$.

Proposition 3. *Let H be an idempotent semiheap. The relations $\leq_l$, $\leq_r$ and hence also $\leq$ are preorders* (*reflexive and transitive*) *on H.*

Proof: We consider $\leq_l$ only, the other cases following immediately. For each $a \in H$, we have $a \leq_l a$ since $[aaa] = a$ for all $a \in S$, so $\leq_l$ is reflexive.

Now assume $a, b, c \in H$ are such that $a \leq_l b$ and $b \leq_l c$. Then $[acc] = [[abb]cc] = [ab[bcc]] = [abb] = a$, so $a \leq_l c$ as claimed. □

Let us say that a semiheap H is a *right heap* if it satisfies $[aab] = b$, or equivalently $b \leq_r a$, for all $a, b \in H$. Let $\mathcal{R}H$ be the variety of right heaps. We define the variety $\mathcal{L}H$ of *left heaps* in the obvious way also. Let $\mathcal{H}$ be the variety of heaps; then of course $\mathcal{H} = \mathcal{L}H \cap \mathcal{R}H$.

The variety of right heaps properly contains the variety of heaps, since the variety of *right zero semiheaps*, $\mathcal{R}Z$, defined by the law $[xyz] = z$, is evidently a proper subvariety of the variety of right heaps intersecting trivially with the variety of heaps. Similarly for the variety of *left zero semiheaps*, $\mathcal{L}Z$, defined in the obvious manner.

Theorem 4. *$\mathcal{L}H$, $\mathcal{R}H$ and $\mathcal{H}$ are radical classes within any variety of idempotent semiheaps containing them.*

Proof: We check that (E), (R1) and (J) are satisfied by the variety of right heaps. For (E), suppose A is an idempotent semiheap and suppose σ is a congruence on A such that $A/\sigma \in \mathcal{R}H$ and every σ-class in A is in $\mathcal{R}H$; that is, for all $a, b \in A$, $[aab]\,\sigma\, b$ and $[bba]\,\sigma\, a$, and if $x\,\sigma\, y$ then $[xxy] = y$ and $[yyx] = x$. Hence for all $a, b \in A$, $[[aab][aab]b] = b$, $[aa[bba]] = [bba]$ and $[bb[aab]] = [aab]$, so

$$\begin{aligned} b &= [[aab][aab]b] \\ &= [[[aab]ba]ab] \\ &= [[aa[bba]]ab] \\ &= [[bba]ab] \\ &= [bb[aab]] \\ &= [aab]. \end{aligned}$$

To show (R1) for $\mathcal{R}H$, suppose σ is a congruence on A such that $a\,\sigma\, b$ implies $[aab] = b$ (that is, every σ-class is in $\mathcal{R}H$). Suppose τ is also a congruence on A. We shall show $\lambda = (\sigma \vee \tau)/\tau$ has all λ-classes in $\mathcal{R}H$. Thus we must show that if $a\,(\sigma \vee \tau)\, b$ then $[aab]\,\tau\, b$.

So suppose $a, b \in A$ are such that $a(\sigma \vee \tau)b$. This means that for $i = 1, 2, \ldots, k$, there are a_i, b_i (with $b_k = b$) such that

$$a\,\sigma\, a_1\,\tau\, b_1\,\sigma\, a_2\,\tau\, b_2\,\sigma\, \ldots \tau\, b.$$

Hence $[aaa_1] = a_1$, $a_1\,\tau\, b_1$, $[b_1b_1a_2] = a_2$, $a_2\,\tau\, b_2, \ldots, a_k\,\tau\, b_k = b$. Hence

$$[aab_1]\,\tau\, b_1, [b_1b_1b_2]\,\tau\, b_2, \ldots, [b_{k-1}b_{k-1}b_k]\,\tau\, b.$$

We show by induction that $[aab_i]\,\tau\, b_i$ for all $i = 1, 2, \ldots, k$. The $i = 1$ case is immediate, and if $[aab_j]\,\tau\, b_j$ for some j between 1 and $k-1$ inclusive, then

$$[aab_{j+1}]\,\tau\,[aa[b_jb_jb_{j+1}]] = [[aab_j]b_jb_{j+1}]\,\tau\,[b_jb_jb_{j+1}]\,\tau\, b_{j+1}.$$

Hence (R1) holds.

Finally, suppose σ, τ are congruences on A for which all classes are right heaps. So if $a\,\sigma\, b$ then $a = [bba]$ and $b = [aab]$, that is, $a \sim_r b$, and similarly for τ. The transitivity of the equivalence relation $\sim_r$ ensures that if $x\,(\sigma \vee \tau)\, y$, then $x \sim_r y$, that is, $x = [yyx]$ and $y = [xxy]$, and so each $(\sigma \vee \tau)$-class of A is a right heap. Hence (J) holds.

There are obvious variants of these arguments for $\mathcal{L}H$, and the $\mathcal{H}$ case follows by combining the two one-sided arguments. $\square$

In a number of important situations, the left heap, right heap and heap radical classes coincide. Here is one.

Proposition 5. *Let H be a generalised heap. Then $H \in \mathcal{R}H$ (resp. $\mathcal{L}H$) if and only if $H \in \mathcal{H}$.*

Proof: If H is a generalised heap which is a right heap, with $a, b \in H$, then

$$\begin{aligned} a &= [[abb][abb]a] \\ &= [[abb]bb]aa] \\ &= [[abb]aa] \\ &= [[aaa]bb] \\ &= [abb], \end{aligned}$$

and so $H \in \mathcal{H}$. $\square$

3. Partitions into maximal subheaps and radical theory

We have been interested in structure-preserving partitions of semiheaps into subheaps. However, it turns out that every idempotent semiheap can be decomposed into a disjoint union of its maximal subheaps. If this partition is a congruence (as is often the case), it will be the heap radical congruence. Similar remarks apply to maximal left/right heaps.

3.1 Disjoint unions of (left/right) heaps. It is obvious that those semiheaps admitting a partition into subheaps (or sub-left/right heaps) are exactly the idempotent semiheaps. (By contrast, involuted I-semigroups are unions of groups if and only if the law $xx' \approx x'x$ holds; so here is an example where the "binary" and "ternary" theories diverge.)

By making use of the preorder $\leq$ introduced earlier, far more can be said. Let H be an idempotent semiheap, with $a, b \in H$. Let $\sim$ be the equivalence relation generated by the preorder $\leq$ on H. For every $a \in H$, let $a^* = \{x \in H : x \sim a\}$.

Proposition 6. *Let H be an idempotent semiheap. For every $a \in H$, a^* is a subsemiheap of H which is a maximal subheap.*

Proof: Assume $x, y, z \in a^*$. Then $[aa[xyz]] = [[aax]yz] = [xyz]$ and similarly $[[xyz]aa] = [xy[zaa]] = [xyz]$, so $[xyz] \leq a$. As a^* is an equivalence class (with respect to $\sim$), it follows that $a^* = x^* = y^* = z^*$ and so $[xxy] = y$, $[zzy] = y$, and so forth. Hence

$$[[xyz][xyz]a] = [x[[xyz]zy]a] = [[x[xy[zzy]]a] = [x[xyy]a] = [zza] = a$$

and likewise $[a[xyz][xyz]] = a$, so $a \leq [xyz]$ as required. Hence a^* is a subsemiheap of H.

Furthermore, $x \leq y$ for all $x, y \in a^*$ and hence $[xyy] = [yyx] = x$ for all $x, y \in a^*$. So a^* is a heap. Clearly, if M is a subheap of H, then the elements of M are equivalent under $\sim$, so M is contained in a subheap of the form a^*. Hence each a^* is maximal. □

Corollary 7. *The idempotent semiheap H is a disjoint union of its maximal subheaps:*

$$H = \bigcup_{a \in H} a^*.$$

Of course $\leq$ induces a partial order on $S/\sim$, so the maximal subheaps of S are naturally partially ordered. Note however that $\sim$ is not in general a congruence. An inverse semigroup example quickly shows this: in the inverse semigroup $I(X)$ of all injective partial functions, the equivalence relation $\sim$ relates 1:1 partial maps with the same domain and range, and in general only the left and right positions in the ternary product are respected, not the central one. However, in important cases which we turn to shortly, the equivalence relation $\sim$ is always a congruence.

Lemma 8. *Let H be an idempotent semiheap on which $\sim$ is a congruence. Then $\sim$ is the $\mathcal{H}$-radical congruence on H.*

For an idempotent semiheap H, there are obvious definitions of the equivalence relations $\sim_l$ (resp. $\sim_r$) in terms of $\leq_l$ (resp. $\leq_r$). In general the $\sim_l$-classes will not be subsemiheaps, although of course they will be if $\sim_l$ is a congruence. Arguing as in the proof of Proposition 6, we obtain the following.

Proposition 9. *If $\sim_l$ (resp. $\sim_r$) is a congruence on the idempotent semiheap H, then it is the $\mathcal{L}H$-radical (resp. $\mathcal{R}H$-radical) congruence of H, and each $\sim_l$-class (resp. $\sim_r$-class) is a maximal sub-left heap (resp. sub-right heap).*

3.2 Radical decompositions of left and right heaps. The variety of heaps is a radical class within the variety of right heaps, and so the heap-semisimple class within the class of right heaps will be a measure of the difference between them.

Theorem 10. *Within the variety of right heaps (resp. left heaps), the semisimple class corresponding to the radical class of heaps is the variety of right zero heaps (resp. left zero heaps).*

Proof: Let H be a right heap. We first show that the equivalence relation $\sim$ is a congruence on H, from which it follows that $\sim$ is the $\mathcal{H}$-radical of H.

First note that if $a, b, x, y \in H$ are such that $a = [xyb]$ in H, then $b = [yxa]$. To see this, let $z \in H$ and then we have

$$\begin{aligned} b &= [[yxz][yxz]b] \\ &= [yx[z[yxz]b]] \\ &= [yx[zx[xyb]]] \\ &= [yx[zza]] \\ &= [y[zzx]a] \\ &= [yxa] \end{aligned}$$

as required.

Now suppose $a_i \sim b_i$, $i = 1, 2, 3$. Setting $x = [a_1a_2a_3]$, we have

$$[xa_3a_3] = [[a_1a_2a_3]a_3a_3] = [a_1a_2[a_3a_3a_3]] = [a_1a_2a_3] = x.$$

But because $x = [a_1a_2a_3]$, we also have that $a_3 = [a_2a_1x]$ from the previous argument, so it follows that $a_3 = [a_3xx]$. Of course we also have that $x = [a_3a_3x]$ and $a_3 = [xxa_3]$ because H is a semiheap. Hence $x \sim a_3$, that is, $[a_1a_2a_3] \sim a_3$. Similarly, $[b_1b_2b_3] \sim b_3$, and so because $a_3 \sim b_3$, it follows that $[a_1a_2a_3] \sim [b_1b_2b_3]$.

The above argument shows that for all $a_1, a_2, a_3 \in H$, $[a_1a_2a_3] \sim a_3$, and it is immediate that $H/\sim$ is a right zero semiheap.

The left-sided version is of course very similar. $\square$

Hence every right heap is a "right zero semiheap of heaps".

3.3 Rectangular semiheaps. It is well known that the class of rectangular bands is the variety of semigroups defined by the single law $xyx \approx x$; equivalently, it satisfies the two laws $xyz \approx xz$, $x^2 \approx x$ (Theorem 1.1.3 of [7]). A semiheap can be defined on any rectangular band S by setting $[abc] := ac$; if S is involuted, this coincides with the semiheap operation of $[S]$. It is clear that the law $[aba] = a$ is satisfied, and we call any semiheap satisfying this law a *rectangular semiheap*. Clearly a rectangular semiheap is idempotent.

Lemma 11. *For any $a, b, c, d \in H$, a rectangular semiheap, $[abc] = [adc]$.*

Proof: We have $a = [aba], c = [cbc], b = [b[adc]b]$, and so

$$[adc] = [[aba]d[cbc]] = [[aba][bcd]c] = [a[[bcd]ab]c] = [a[b[adc]b]c] = [abc].$$

$\square$

This result can readily be converted into a characterisation of rectangular semiheaps analogous to a familiar one for rectangular bands (see Chapter 1 of [7]).

Theorem 12. *The variety of rectangular semiheaps is term equivalent to the variety of rectangular bands, via the correspondences $[abc] \leftrightarrow abc$, $ab \leftrightarrow [abb]$.*

Proof: We must show the constructions are mutually inverse. In a rectangular semiheap H, the induced rectangular band induces a semiheap operation on $a, b, c \in H$ expressible as (by repeated use of the previous lemma)

$$abc = a(bc) = [a[bcc][bcc]] = [ab[bcc]] = [ab[bbc]] = [a[bbb]c] = [abc].$$

Conversely, starting with a rectangular band, the derived product of $a, b \in H$ looks like $[abb] = abb = ab$, completing the proof. □

No non-trivial semiheap can be both a heap and a rectangular semiheap.

Proposition 13. *If A is both a heap and a rectangular semiheap, then $|A| = 1$.*

Proof: Suppose A is both a heap and a rectangular semiheap. If $a, b \in A$ then $a = [abb] = [aab] = b$. □

An elementary fact about rectangular bands is that each is a direct product of a left zero semigroup with a right zero semigroup, and conversely each such direct product is a rectangular band; see Chapter 1 of [7] for instance. This fact has an immediate radical-theoretic interpretation in terms of radicals of rectangular bands (which form an idempotent variety). The semiheap form of this result can be given in terms of the left and right heap radicals.

Theorem 14. *Within the variety of rectangular semiheaps, the right heap radical class consists of right zero semiheaps, and has corresponding semisimple class all left zero semiheaps (and vice versa).*

Proof: Within rectangular semiheaps, $[aab] = [acb]$ for any a, b, c, by Lemma 11, so right heaps are exactly right zero semiheaps.

Let H be a rectangular semiheap. We first show the equivalence relation $\sim_r$ associated with the preorder $\leq_r$ (hence given by $a \sim_r b$ if $a = [bba], b = [aab]$) is a congruence. Suppose that in H, $a_i \sim_r b_i$, $i = 1, 2, 3$. Then by Lemma 11,

$$[a_1a_2a_3] = [a_1a_3a_3] = [[b_1b_1a_1]a_3a_3] = [b_1[a_3a_1b_1]a_3] = [b_1[a_3b_1b_1]a_3] = [b_1a_3a_3]$$

whereas $[b_1b_2b_3] = [b_1b_3b_3]$. Now

$$\begin{aligned}
[[b_1a_3a_3][b_1a_3a_3][b_1b_3b_3]] &= [[b_1a_3a_3]b[b_1b_3b_3]] \\
&= [b_1[b_1a_3a_3][b_1b_3b_3]] \\
&= [b_1b_1[b_1b_3b_3]] \\
&= [[b_1b_1b_1]b_3b_3] \\
&= [b_1b_3b_3],
\end{aligned}$$

so $[b_1b_3b_3] \leq_r [b_1a_3a_3]$. Similarly $[b_1a_3a_3] \leq_r [b_1b_3b_3]$, and so by symmetry we have $[b_1a_3a_3] \sim_r [b_1b_3b_3]$. Hence $\sim_r$ is indeed a congruence on H.

Hence by Proposition 9, $\sim_r$ is the right heap radical congruence on H. We show that $H/\sim_r$ is a left heap (equivalently, a left zero semiheap). Now for all

$a, b \in H$,

$$[[abb][abb]a] = [[abb]aa] = [a[abb]a] = [aaa] = a,$$

and also $[aa[abb]] = [[aaa]bb] = [abb]$, so by definition, $[abb] \sim_r a$. Moreover, it is clear that in any left zero semiheap, the right heap radical is trivial. Hence the $\mathcal{R}H$-semisimple rectangular semiheaps are exactly the left zero semiheaps. $\square$

Thus within the variety of rectangular semiheaps, both the left and right heap radicals (which reduce to the classes of left and right zero semiheaps) are SSR-classes in the sense of [3], that is, both semisimple and radical classes.

If S is an involuted rectangular band, then from Theorem 25, there is a largest $\mathcal{H}$-congruence ρ_S on $(S, \cdot, ')$, which is also its largest $\mathcal{L}H$-congruence and largest $\mathcal{R}H$-congruence. By Proposition 13, this is just equality, and so S is semisimple in the Hoehnke sense. Similarly, as a semiheap, $[S]$ is $\mathcal{H}$-semisimple.

However, letting S consist of the set of ordered pairs $(a, b) \in X \times X$ for some set X having at least two elements, with multiplication given by $(a, b)(c, d) = (a, d)$ and with $(a, b)' = (b, a)$, we see that S is an involuted rectangular band (in fact a typical one), but it is neither $\mathcal{L}H$-semisimple nor $\mathcal{R}H$-semisimple, since it is neither a left zero nor a right zero semiheap. In fact it is easy enough to identify the left heap radical congruence of $[S]$: it is ρ given by $(a, b)\, \rho\, (c, d)$ if and only if $a = c$. Factoring this out evidently gives a right zero semiheap, and each ρ-class is a left zero semiheap. On the other hand, ρ is not even a congruence on the involuted semigroup S.

By Proposition 24 an idempotent semiheap of the form $[S]$ (where S is an involuted I-semigroup) is in $\mathcal{H}$ if and only if it is in $\mathcal{R}H$, or equivalently $\mathcal{L}H$. However, the above shows that the $\mathcal{H}$-radical, $\mathcal{L}H$-radical, and $\mathcal{R}H$-radical congruences of $[S]$ can all be different, even though the associated Hoehnke radicals on S are necessarily all the same.

3.4 Near heaps. A further fairly well known class of involuted semigroups is the class of *Clifford semigroups*. Recall that a Clifford semigroup is an inverse semigroup that satisfies $aa' = a'a$ for all $a \in S$ (equivalently, $ea = ae$ for all $a \in S$ and $e \in E(S)$). As discussed in Section 3, these arise naturally in the setting of idempotent-separating congruences (that is, $\mathcal{H}$-congruences) on inverse semigroups.

If S is a Clifford semigroup, it is obvious that the following law is satisfied by $[S]$:

$$[aab] \approx [baa].$$

We call an idempotent semiheap H satisfying this law a *near heap*. As far as we know, this notion has not previously been studied.

All groups are Clifford semigroups, and obviously all heaps are near heaps. On the other hand, we have the following inclusion, analogous to the fact that all Clifford semigroups are inverse semigroups.

Proposition 15. *Every near heap is a generalised heap.*

Proof: If A is a near heap and $a, b, c \in A$, then

$$[[aa[bbc]] = [[bbc]aa] = [bb[caa]] = [bb[aac]],$$

and similarly $[[abb]cc] = [[acc]bb]$. □

The class of near heaps contains non-heaps; for example, it includes the semiheap analogs of semilattices. Viewing a semilattice L trivially as an involuted semigroup (in which $a' = a$ for all $a \in L$), the semiheap operation on $[L]$ is given by $[abc] = abc$ for all $a, b, c \in L$, and $[L]$ is easily seen to satisfy the following laws: $[abb] \approx [aab]$, and

$$[abc] \approx [acb] \approx [bac] \approx [bca] \approx [cab] \approx [cba].$$

We call a semiheap satisfying this last sequence of identities *commutative*. Conversely, if H is a commutative semiheap satisfying $[abb] \approx [aab]$, then the binary operation $*$ on H defined by $a * b := [aab] = [abb]$ is easily seen to be a semilattice operation (such that $[abc] = a * b * c$). We call a semiheap corresponding to a semilattice in this way a *semilattice semiheap*, or just a semilattice for short if the context is clear. Clearly the varieties of semilattices and semilattice semiheaps are term equivalent.

As for rectangular semiheaps, the classes of semilattices and heaps intersect minimally. The proof of the following is formally identical to that of Proposition 13, although the steps follow for different reasons.

Proposition 16. *If A is both a heap and a semilattice, then $|A| = 1$.*

Hence the class of semilattices is heap-semisimple; indeed semilattices are arguably the most "un-heap-like" of idempotent semiheaps. It is of interest to characterise those idempotent semiheaps which give semilattices when factored by their heap radical. Of course such a semiheap is a "semilattice of heaps" in the obvious sense: there is a congruence ρ for which the ρ-classes are heaps and the quotient semiheap is a semilattice. We now show that any such semilattice of heaps is a near heap: indeed we show something more general.

Proposition 17. *Suppose H is an idempotent semiheap with a heap-generating congruence ρ for which H/ρ is a near heap. Then H is a near heap.*

Proof: We must show that for $a, b \in H$, $[abb] = [bba]$. Writing $\bar{x}$ for the ρ-class containing $x \in H$, we see that for $a, b \in H$,

$$\overline{[aab]} = [\bar{a}\bar{a}\bar{b}] = [\bar{b}\bar{a}\bar{a}] = \overline{[baa]}$$

and so $[aab]\rho[baa]$. Hence, because ρ is heap-generating, we have that

$$\begin{aligned}
[aab] &= [[baa][baa][aab]] \\
&= [[baa]aa]b[aab]] \\
&= [[ba[aaa]]b[aab]] \\
&= [[baa]b[aab]].
\end{aligned}$$

By a dual argument, we see that $[baa]$ equals the same thing, and so $[aab] = [baa]$. Hence H is a near heap. $\square$

Of course, a special case of this last result arises by requiring that H/ρ be a semilattice.

Lemma 18. *In the near heap H, the equivalence relation $\sim$ is a congruence. Hence $\sim$ is the heap radical congruence on H.*

Proof: The near heap H can be embedded in a generalised heap of the form $[S]$ where S is an inverse semigroup; identify H with its image in $[S]$ under such an embedding. Then for all $a, b \in H$, the law $[xyy] \approx [yxx]$ tells us that $ab'b = bb'a$ (or equivalently $a'bb' = b'ba'$), as calculated in S.

Suppose that for all $a_i, b_i \in H$, $i = 1, 2, 3$, we have $a_i \sim b_i$. Then for each i, $a_i \leq b_i$, and so $a_i = [a_i, b_i, b_i] = [b_i, b_i, a_i]$, or in terms of S, $a_i = a_i b_i' b_i = b_i b_i' a_i$. We wish to show that $[a_1a_2a_3] \leq [b_1b_2b_3]$. But

$$\begin{aligned}
[[a_1a_2a_3][b_1b_2b_3][b_1b_2b_3]] &= a_1a_2'a_3(b_1b_2'b_3)'b_1b_2'b_3 \\
&= a_1a_2'a_3b_3'b_2b_1'b_1b_2'b_3 \\
&= a_1b_1'b_1a_2'a_3b_3'b_2b_2'b_3 \\
&= a_1a_2'a_3b_3'b_2b_2'b_3 \\
&= a_1a_2'b_2b_2'a_3b_3'b_3 \\
&= a_1(b_2b_2'a_2)'(a_3b_3'b_3) \\
&= a_1a_2'a_3 \\
&= [a_1a_2a_3],
\end{aligned}$$

as required. Similarly $[[b_1b_2b_3][b_1b_2b_3][a_1a_2a_3]] = [a_1a_2a_3]$, so $[a_1a_2a_3] \leq [b_1b_2b_3]$. Likewise, $[a_1a_2a_3] \geq [b_1b_2b_3]$, and so $[a_1a_2a_3] \sim [b_1b_2b_3]$.

The final part follows from Lemma 8. $\square$

This observation allows us to describe $H/\sim$ in such cases.

Theorem 19. *If H is a near heap, then $H/\sim$ is a semilattice.*

Proof: The near heap $K = H/\sim$ is heap-semisimple, so on K, $\sim$ is trivial, which means that the maximal subheaps of K are the singletons. This means that if $a \sim b$ in K, then $a = b$. Using arguments very similar to those used in the proof of Lemma 18 (involving viewing K as a subsemiheap of $[S]$ for some inverse semigroup S), it is not hard to show that $[[efg][abc][abc]] = [efg] = [[abc][abc][efg]]$ where e, f, g is any permutation of a, b, c, so that $[efg] \leq [abc]$, and hence by symmetry $[efg] \sim [abc]$ and so $[efg] = [abc]$. Similarly one can show that $[[abb][aab][aab]] = [abb] = [[aab][aab][abb]]$, and so on, to establish that $[abb] \sim [aab]$, so that $[abb] = [aab]$. Hence K is a (ternary) semilattice. $\square$

Corollary 20. *The class of near heaps consists of exactly those idempotent semiheaps H for which* (i) *the natural partial ordering $\leq$ on the $\sim$-classes is a semilattice ordering and* (ii) *$\sim$ itself is a congruence.*

Corollary 21. *The following are equivalent for an idempotent semiheap H:*

- *there is a heap-generating congruence ρ on H for which H/ρ is a near heap;*
- *there is a heap-generating congruence ρ on H for which H/ρ is a semilattice;*
- *H is a near heap.*

In particular then, near heaps are exactly "semilattices of heaps". This corresponds to the analogous fact from semigroup theory that a semilattice of groups (that is, a semigroup having a congruence partitioning it into subsemigroups which are groups and such that the factor semigroup is a semilattice) is exactly a Clifford semigroup; see Section 4.2 of [7] for an exposition.

In [3], Gardner cites the example of bands (semigroups in which all elements are idempotent) as a major motivation for his development of the theory of radicals of idempotent algebras: it is well known that every band is a semilattice of rectangular bands (satisfying $xyx = x$). The fact that every near heap is a semilattice of heaps can be interpreted similarly.

Corollary 22. *Within the variety of near heaps, $\mathcal{S}H$ is a semisimple class corresponding to the radical class of heaps.*

Proof: Any $\mathcal{H}$-semisimple near heap is a semilattice as we just saw. Conversely, because semilattices can only be heaps if they are trivial by Proposition 16, it follows that every semilattice is $\mathcal{H}$-semisimple (since it has no non-trivial subheaps). □

4. Consequences for involuted I-semigroups

4.1 Generalities. We now present a new general result. Let $\mathcal{A}$ be a class of algebras of some fixed type which is closed under homomorphic images. Suppose that some reduct of the algebras in $\mathcal{A}$ yields algebras in the variety $\mathcal{V}$ consisting of idempotent algebras; for $A \in \mathcal{A}$, let $[A]$ denote this idempotent reduct in $\mathcal{V}$. If $\mathcal{R}$ is a radical class in $\mathcal{V}$, and $A \in \mathcal{A}$, we say the congruence ρ of A is an *$\mathcal{R}$-congruence* if ρ is an $\mathcal{R}$-congruence on $[A]$. (Any congruence on A is of course one on $[A]$ as well.)

Note that the largest $\mathcal{R}$-congruence on $[A]$ may not even be a congruence on A, let alone an $\mathcal{R}$-congruence on it. However, we do have the following.

Theorem 23. *Let $\mathcal{A}$ and $\mathcal{V}$ be as above. Suppose $\mathcal{R}$ is a radical class in $\mathcal{V}$ that is closed under taking subalgebras. Then the congruence family ρ_A on $\mathcal{A}$ obtained by setting, for each $A \in \mathcal{A}$,*

$$\rho_A = \max\{\rho \mid \rho \ \textit{ is an } \mathcal{R}\textit{-congruence on } A\},$$

exists and is a Hoehnke radical in $\mathcal{A}$.

Proof: First we know $\delta_{[A]} = \max\{\rho \mid \rho$ is an $\mathcal{R}$-congruence on $[A]\}$ exists and is the radical congruence on $[A]$. Consider the following congruence on A:

$$\rho_A = \bigvee\{\rho \mid \rho \ \text{ is an } \mathcal{R}\text{-congruence on } A\}.$$

Of course this is also a congruence on $[A]$; moreover because each ρ in the join defining ρ_A is also one of those in the join defining $\delta_{[A]}$, and because the join is the same operation in both cases (namely equivalence relation join), it follows that $\rho_A \subseteq \delta_{[A]}$, and so the ρ_A-classes of $[A]$ are subalgebras of the $\delta_{[A]}$-classes of $[A]$, hence in $\mathcal{R}$ by the assumption that $\mathcal{R}$ is closed under taking subalgebras; hence ρ_A is indeed the maximum of all $\mathcal{R}$-congruences on A, as claimed.

We now show that the ρ_A, $A \in \mathcal{A}$, define a Hoehnke radical. Suppose δ is a congruence on A containing ρ_A, for which all δ/ρ_A-classes in $[A]/\rho_A$ are in $\mathcal{R}$. Let T be a δ-class in A. Then $[T]$ (defined in the obvious manner) is a subalgebra of $[A]$. Moreover $[T]/\rho_A$ is in $\mathcal{R}$ by assumption on δ since each δ/ρ_A-class is in $\mathcal{R}$, and also each ρ_A-class in T is in $\mathcal{R}$. Hence by (E), $T \in \mathcal{R}$. Hence $\delta = \rho_A$ by definition of ρ_A, and so only the trivial congruence (of equality) on A/ρ_A is such that all its classes (when viewed as in $\mathcal{V}$) are in $\mathcal{R}$.

For (F), it is obviously sufficient to prove that if $A \in \mathcal{A}$ and τ is a congruence on A, then for any $a, b \in A$, $(a, b) \in \rho_A$ implies $(a^\tau, b^\tau) \in \rho_{A/\tau}$ (where x^τ denotes the τ-class in A containing x). But if $(a, b) \in \rho_A$ then of course $(a, b) \in \rho_A \vee \tau$ and so of course $(a^\tau, b^\tau) \in (\rho_A \vee \tau)/\tau$ which, being a congruence on A/τ, is certainly also one on $[A]/\tau$, and necessarily an $\mathcal{R}$-congruence on $[A]/\tau$ since $\mathcal{R}$ is closed under subalgebras, by (R1′) (and using the fact that $\vee$ is the same in both congruence lattices). Hence it is an $\mathcal{R}$-congruence on A/τ as well, and so is contained in $\rho_{A/\tau}$. □

Although we shall only apply it to the varieties of involuted I-semigroups and idempotent semiheaps, Theorem 23 obviously has wide applicability, and could lead to consideration of new Hoehnke radicals for various structures having idempotent reducts.

4.2 The $\mathcal{H}$-radical for involuted I-semigroups. Recall that when restricted to the variety of generalised heaps, $\mathcal{L}H = \mathcal{R}H = \mathcal{H}$. Likewise, for involuted I-semigroups, all three radical classes are equal.

Proposition 24. *Suppose S an involuted I-semigroup. If $[S]$ is a left or right heap, then it is a heap.*

Proof: Let S be an involuted I-semigroup for which $[S]$ is a right heap. Then for all $a, b \in S$, we must have $aa'b = [aab] = b$. Then also, $(ba'a)' = a'(a')'b' = b'$, and so $[baa] = ba'a = b$, so $[S]$ is a heap. Similarly if $[S]$ is a left heap. □

Hence the variety of right heaps is different to most of the varieties of semiheaps considered previously in the literature: in contrast to the varieties of heaps, generalised heaps, idempotent semiheaps and semiheaps themselves (as discussed above), $\mathcal{R}H$ does not consist of exactly the semiheap subreducts of some variety (or even some class) of involuted semigroups $\mathcal{A}$. For if $\mathcal{R}H$ consisted of all subreducts of $\mathcal{A}$ for some class $\mathcal{A}$ of involuted semigroups, then by the previous result, the largest $\mathcal{A}$ could be is the class of all groups, whence $\mathcal{R}H$ would be the class of heaps, a contradiction, and so there is no such class $\mathcal{A}$. We return to consideration of $\mathcal{L}H$ and $\mathcal{R}H$ later.

Apart from its final observation, the following is immediate from Theorem 23.

Theorem 25. *Let S be an involuted I-semigroup, with $\mathcal{R}$ chosen to be one of the radical classes $\mathcal{H}$, $\mathcal{R}H$, $\mathcal{L}H$. Then S has a largest $\mathcal{R}$-congruence θ_S, and the only $\mathcal{R}$-congruence on S/θ_S is trivial. This congruence satisfies the functoriality property* (F) *and hence is a Hoehnke radical, and is independent of which of the three radical classes is chosen.*

Proof: Clearly every $\theta_{\mathcal{H}}$-congruence is a $\theta_{\mathcal{R}H}$-congruence. To see the converse, note that if ρ is a $\theta_{\mathcal{R}H}$-congruence on S, and $a\,\rho\,b$, then $aa'b = b$, but also $a'\,\rho\,b'$, and so $a'(a')'b' = b'$, and as before we see that $b = ba'a$ as well, so ρ is a $\theta_{\mathcal{H}}$-congruence. Similarly for $\theta_{\mathcal{L}H}$-congruences. □

Recall that a semigroup S is *regular* if for each $a \in S$ there is $b \in S$ such that $a = aba$. The class of regular semigroups includes all full transformation semigroups and all inverse semigroups, and is perhaps the most important single class of semigroups. *Idempotent-separating congruences* are of particular importance in the study of regular semigroups; these are congruences for which any two idempotents are congruent exactly when they are equal. It is well known that every regular semigroup S has a largest idempotent-separating congruence θ, and that S/θ is free of any non-trivial such congruences; see [7] for example. There is a structure theorem due to Munn in [9], describing S/θ when S is an inverse semigroup.

Proposition 26. *The following are equivalent for the congruence ρ on the involuted I-semigroup $(S,\cdot,')$:*

- *ρ is idempotent-separating;*
- *ρ is an $\mathcal{H}$-congruence.*

Proof: Let $(S,\cdot,')$ be as described, with ρ a congruence on it. Then for all $a \in S$, $(aa')(aa') = (aa'a)a' = aa'$, so aa' is idempotent.

If ρ is idempotent-separating and $(a,b) \in \rho$, then $(aa',bb') \in \rho$, and so $aa' = bb'$, so $[aab] = aa'b = bb'b = b$, and similarly $[baa] = b$. Conversely, assuming ρ is an $\mathcal{H}$-congruence, if $a,b \in S$ satisfy $a\,\rho\,b$, then $aa'\,\rho\,bb'$. But aa' and bb' are idempotent, so

$$bb' = [(aa')(aa')(bb')] = aa'aa'bb' = aa'bb' = aa'bb'bb' = [(aa')(bb')(bb')] = aa'.$$

Similarly $a'a = b'b$. If also a,b are idempotent, then so are a',b', and so

$$a = aa'a = (aa')(a'a) = (bb')(b'b) = bb'b = b,$$

so ρ is idempotent-separating. □

The following is now immediate from Theorem 25.

Corollary 27. *The heap radical congruence on an involuted I-semigroup is its largest idempotent-separating congruence.*

The previous corollary says that any involuted I-semigroup has a largest idempotent-separating congruence. This is not implied by the well-known fact regarding such congruences in regular semigroups, since such congruences may not respect the involution.

However, in an inverse semigroup, semigroup congruences automatically respect the inverse operation, and we recover the fact that every inverse semigroup has a largest idempotent-separating congruence. Indeed, if S is an inverse semigroup, the largest idempotent-separating congruence ρ_S admits an explicit description (see [6]):

$$(a, b) \in \rho \;\Leftrightarrow\; a'ea = b'eb \;\text{ for all }\; e \in E(S).$$

The resulting factor inverse semigroup S/ρ can be represented as a full subsemigroup of the semigroup $T_{E(S)}$ of isomorphisms between the principal ideals of $E(S)$. This can be interpreted as a structure theorem for “heap-semisimple” inverse semigroups.

Can this fact about inverse semigroups be given a Kurosh-Amitsur radical-theoretic interpretation, involving some kinds of “normal subobjects” in inverse semigroups (rather than just in terms of a Hoehnke radical)? The congruences on an inverse semigroup are all determined by *congruence pairs* (N, θ), where N is a *normal subsemigroup*, meaning an (inverse) subsemigroup containing $E(S)$ and closed under conjugation (defined in the obvious way, as for groups), θ is a congruence on the semilattice of idempotents $E(S)$, and there are various linking conditions between N and θ. The congruence δ on S can be defined via

$$(a, b) \in \delta \;\Leftrightarrow\; a'b \in N \;\text{ and }\; (aa', bb') \in \theta.$$

N then consists of those elements of S congruent to an idempotent (the *kernel* of S), while θ is the restriction of δ to $E(S)$ (the *trace* of δ), and all congruence pairs arise from congruences in this way. See [7] for the details.

In the case of an idempotent-separating congruence δ on the inverse semigroup S, the trace θ is of course equality on $E(S)$ (that is, the diagonal relation), while its kernel N is easily seen to be a *Clifford semigroup* (an inverse semigroup additionally satisfying the law $aa' \approx a'a$). Conversely, any pair (N, θ) in which θ is the diagonal relation on $E(S)$ and N is a normal Clifford subsemigroup is the congruence pair of an idempotent-separating congruence on S. So idempotent-separating congruences on S correspond one-to-one with normal Clifford subsemigroups of S.

It follows that any inverse semigroup has a largest normal Clifford subsemigroup which, when factored out, leaves an inverse semigroup with no non-trivial normal Clifford subsemigroups, that is, the only one is the semilattice $E(S)$ (and in particular, if S is itself a Clifford semigroup, the quotient is a copy of the semilattice $E(S)$). This is a radical-like fact, but no more than that because there is no general kind of subobject of an inverse semigroup that can play the role of the radical. Normal subsemigroups cannot do this job, since these do not define congruences in the absence of further information about the trace of the congruence. We could restrict to cases in which the trace is equality, but this merely gives us "radical" congruences in the current sense. A truly semiheap-theoretic view must be taken if one wishes to go beyond a Hoehnke radical when interpreting the largest idempotent-separating congruence.

References

[1] Baer R., *Zür Einfuhrung des Scharbegriffs*, J. Reine Angew. Math. **160** (1929), 199–207.

[2] Divinsky N., *Rings and Radicals*, Allen and Unwin, 1965.

[3] Gardner B.J., *Radical decompositions of idempotent algebras*, J. Austral Math. Soc. (Series A) **36** (1984), 213–236.

[4] Hoehnke H.-J., *Radikale in allgemeinen Algebren*, Math. Nachr. **32** (1966), 347–383.

[5] Hoehnke H.-J., *Einige neue Resultate über abstrakte Halbgruppen*, Colloq. Math. **14** (1966), 329–349.

[6] Howie J.M., *The maximum idempotent separating congruence on an inverse semigroup*, Proc. Edinburgh Math. Soc. **14** (1964/65), 71–79.

[7] Howie J.M., *Fundamentals of Semigroup Theory*, Oxford University Press, Oxford, 1995.

[8] Márki L., Mlitz R., Wiegandt R., *A General Kurosh-Amitsur radical theory*, Comm. Algebra **16** (1988), 249–305.

[9] Munn W.D., *Fundamental Inverse Semigroups*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 157–170.

[10] Prüfer H., *Theorie der Abelschen Gruppen*, Math. Z. **20** (1924), 165–187.

[11] Stokes T., *Radical classes of algebras with B-action*, Algebra Universalis **40** (1998), 73–85.

[12] Vagner V.V., *The theory of generalized heaps and generalized groups* (Russian), Mat. Sbornik N.S. **32** (1953), 545–632.

Department of Mathematics, The University of Waikato, Private Bag 3105, Hamilton, New Zealand

Email: stokes@math.waikato.ac.nz

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