Hausdorff and packing dimensions for ergodic invariant measures of two-dimensional Lorenz transformations

Franz Hofbauer

Abstract. We extend the notions of Hausdorff and packing dimension introducing weights in their definition. These dimensions are computed for ergodic invariant probability measures of two-dimensional Lorenz transformations, which are transformations of the type occuring as first return maps to a certain cross section for the Lorenz differential equation. We give a formula of the dimensions of such measures in terms of entropy and Lyapunov exponents. This is done for two choices of the weights using the recurrence time of a set and equilibrium states respectively.

Keywords: Hausdorff dimension, packing dimension, Lorenz transformation, ergodic measure

Classification: 37D50, 28A78, 37C45, 37A35

Introduction

We extend the notions of Hausdorff and packing dimension, replacing the sum $\sum_A |A|^t$ occuring in the definition of the dimension by $\sum_A w(A)|A|^t$, where $|A|$ denotes the *diameter* of the set A and w is called the *weight function* which assigns to each Borel subset of $\mathbb{R}^n$ a number in $[0,\infty]$. The definitions of these extended versions of Hausdorff and packing dimension and some of their properties are given in Section 1. Hausdorff and packing dimensions of this type were first introduced in [20] for the case where the weight function is a power of a measure. The basic properties in the more general case we consider in this paper are the same as in [20]. A theory of dimension structures can be found in [21].

In this paper we do not consider the dimension of a set but the dimension of a probability measure, which is defined as the infimum of the dimensions of all sets of measure one. The dimension of a measure allows a local approach. This is considered in Section 2. Let μ be a probability measure on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ define $\ell_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$ and $\ell_w(x) = \lim_{r\to 0} \frac{\log w(B_r(x))}{\log r}$, provided these limits exist. As usual, $B_r(x)$ denotes the ball of radius r around x. We have the following theorem. If there are numbers $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\ell_\mu(x) = \alpha$ and $\ell_w(x) = \beta$ hold for μ-almost all $x \in \mathbb{R}^n$, then both the extended Hausdorff dimension and the extended packing dimension of the measure μ, which have w as weight function, are equal to $\alpha - \beta$.

The dynamical systems we investigate in this paper are injective skew product transformations $F : [0,1]^2 \to [0,1]^2$ defined by $F(x_1, x_2) = (T(x_1), g(x_1, x_2))$, where T is a piecewise monotone interval transformation and $x_2 \mapsto g(x_1, x_2)$ is a contraction for each x_1. Transformations of this kind can serve as geometric models for first return maps to certain cross sections of the Lorenz differential equation and are therefore called Lorenz transformations. Under certain regularity conditions, for an ergodic F-invariant probability measure μ, whose entropy $h_\mu(F)$ is positive, it is shown in [25], that $\ell_\mu(x)$ equals $h_\mu(F)(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$ for μ-almost all x, where $u(x_1, x_2) = \log|T'(x_1)|$ and $v(x_1, x_2) = -\log|\partial_2 g(x_1, x_2)|$. Therefore it remains to compute $\ell_w(x)$. In Section 3 we use the method of [25] to show that $\ell_w(x)$ can be computed as a limit which uses cylinder sets around x instead of balls $B_r(x)$. This is then used in the subsequent sections for the computation of $\ell_w(x)$. In Section 3 we give also the definition of a Lorenz transformation and its regularity properties used in this paper.

In Section 4 we investigate the recurrence dimension, which was first considered in [1]. It is the extended dimension with weight function $w(A) = e^{-s\tau(A)}$, where $s \in \mathbb{R}$ and $\tau(A)$ is the recurrence time of the set A, which is defined by $\tau(A) = \inf\{n \geq 1 : F^n(A) \cap A \neq \emptyset\}$. Under regularity conditions on the Lorenz transformation F we show for an ergodic F-invariant probability measure μ with positive entropy that $\ell_w(x) = s(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$ for μ-almost all x and hence the extended dimension of μ equals $(h_\mu(F) - s)(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$.

A formula of this type has been proved in [2] for ergodic measures on a twosided shift space with a weak specification property and a certain choice of the metric, and in [24] for ergodic measures of certain diffeomorphisms on two-dimensional manifolds. Related formulas for dynamical systems on the interval can be found in [23] and in [11].

Finally, in Section 5 we consider the extended dimension with weight function w defined by $w(A) = \varrho(A)^s$, where ϱ is a probability measure and $s \in \mathbb{R}$. This is the dimension introduced in [20]. In this paper we assume that ϱ is an equilibrium state of a piecewise Hölder continuous function $\varphi : [0,1]^2 \to \mathbb{R}$, which satisfies $\sup\varphi - \inf\varphi < h_{\mathrm{top}}(F)$ and has pressure zero. Under regularity conditions on the Lorenz transformation F we show for an ergodic F-invariant probability measure μ with positive entropy that $\ell_w(x) = s\int\varphi\,d\mu\,(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$ for μ-almost all x and hence the extended dimension of μ equals $(h_\mu(F) - s\int\varphi\,d\mu)(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$. A related formula for dynamical systems on the interval can be found in [15].

1. Extended Hausdorff and packing dimension

We introduce extended versions of Hausdorff and packing dimension for subsets E of $\mathbb{R}^n$. These are numbers $\dim_w(E)$ and $\mathrm{Dim}_w(E)$ depending on a weight function w, which is a function from the Borel subsets of $\mathbb{R}^n$ to $[0,\infty]$.

Let γ be a function defined on all subsets of $\mathbb{R}^n$ with values in $[0,\infty]$. We call γ *monotone*, if $G \subset H$ implies $\gamma(G) \le \gamma(H)$ for all subsets G and H of $\mathbb{R}^n$. We call γ *subadditive*, if $\gamma(\bigcup_{j=1}^{\infty} G_j) \le \sum_{j=1}^{\infty} \gamma(G_j)$ holds for all subsets G_j of $\mathbb{R}^n$. If γ is monotone and subadditive, it is called an *outer measure*.

For $G \subset \mathbb{R}^n$ and $\delta > 0$ we call $\mathcal{A}$ a *centered δ-cover* of G, if it is a finite or countable set of balls $B_r(x)$ with $x \in G$ and $r \le \delta$ which cover G. Denote by $\mathcal{C}_\delta(G)$ the collection of all centered δ-covers of G. For $t \in \mathbb{R}$ and $G \subset \mathbb{R}^n$ define

$$\tilde{\nu}_t(G) = \lim_{\delta\to 0} \inf_{\mathcal{A}\in\mathcal{C}_\delta(G)} \sum_{A\in\mathcal{A}} w(A)|A|^t.$$

The limit always exists, but can be infinite. In general, $\tilde{\nu}_t$ is not an outer measure on $\mathbb{R}^n$. It is subadditive, but not monotone. Therefore define for $E \subset \mathbb{R}^n$

$$\nu_t(E) = \sup_{G\subset E} \tilde{\nu}_t(G).$$

By standard arguments one shows that ν_t is monotone and subadditive, and hence an outer measure on $\mathbb{R}^n$.

Suppose $t < s$. We show that $\nu_t(E) < \infty$ implies $\nu_s(E) = 0$. For any $\mathcal{A} \in \mathcal{C}_\delta(G)$ we have $\sum_{A\in\mathcal{A}} w(A)|A|^s \le (2\delta)^{s-t} \sum_{A\in\mathcal{A}} w(A)|A|^t$. Taking the infimum over all $\mathcal{A} \in \mathcal{C}_\delta(G)$ and then the limit $\delta \to 0$, we get that $\tilde{\nu}_t(G) < \infty$ implies $\tilde{\nu}_s(G) = 0$. If we have now $\nu_t(E) < \infty$ then $\tilde{\nu}_t(G) < \infty$ for all $G \subset E$, which implies $\tilde{\nu}_s(G) = 0$ for all $G \subset E$ and hence $\nu_s(E) = 0$ follows.

Therefore, there is a number $t_0 \in [-\infty,\infty]$ such that $\nu_t(E) = \infty$ for $t < t_0$ and $\nu_t(E) = 0$ for $t > t_0$. We denote t_0 by $\dim_w(E)$ and call it the *extended Hausdorff dimension* of the set E with weight function w.

Now we define the extended packing dimension. For a subset G of $\mathbb{R}^n$ and $\delta > 0$ we call $\mathcal{R}$ a *centered δ-packing* of G, if $\mathcal{R}$ consists of pairwise disjoint balls $B_r(x)$ with $x \in G$ and $r \le \delta$. Denote by $\mathcal{P}_\delta(G)$ the collection of all centered δ-packings of G. For $t \in \mathbb{R}$ and $G \subset \mathbb{R}^n$ define

$$\tilde{\kappa}_t(G) = \lim_{\delta\to 0} \sup_{\mathcal{R}\in\mathcal{P}_\delta(G)} \sum_{A\in\mathcal{R}} w(A)|A|^t.$$

The limit always exists, but can be infinite. In general, $\tilde{\kappa}_t$ is not an outer measure on $\mathbb{R}^n$. It is monotone, but not subadditive. Therefore define for $E \subset \mathbb{R}^n$

$$\kappa_t(E) = \inf_{\mathcal{G}\in\mathcal{Q}(E)} \sum_{G\in\mathcal{G}} \tilde{\kappa}_t(G)$$

where $\mathcal{Q}(E)$ is the set of all finite or countable covers of E by arbitrary subsets of $\mathbb{R}^n$. Again one easily shows that κ_t is an outer measure on $\mathbb{R}^n$.

Suppose that $t < s$. We show that $\kappa_t(E) < \infty$ implies $\kappa_s(E) = 0$. For any $\mathcal{R} \in \mathcal{P}_\delta(G)$ we have $\sum_{A\in\mathcal{R}} w(A)|A|^s \le (2\delta)^{s-t}\sum_{A\in\mathcal{R}} w(A)|A|^t$. Taking the supremum over all $\mathcal{R} \in \mathcal{P}_\delta(G)$ and then the limit $\delta \to 0$, we get that $\tilde{\kappa}_t(G) < \infty$ implies $\tilde{\kappa}_s(G) = 0$. If we have now $\kappa_t(E) < \infty$ then there is $\mathcal{G} \in \mathcal{Q}(E)$ with $\sum_{G\in\mathcal{G}} \tilde{\kappa}_t(G) < \infty$. This implies $\tilde{\kappa}_t(G) < \infty$ and hence also $\tilde{\kappa}_s(G) = 0$ for all $G \in \mathcal{G}$, and $\kappa_s(E) \le \sum_{G\in\mathcal{G}} \tilde{\kappa}_s(G) = 0$ follows.

Therefore, there is a number $t_0 \in [-\infty, \infty]$ such that $\kappa_t(E) = \infty$ for $t < t_0$ and $\kappa_t(E) = 0$ for $t > t_0$. We denote t_0 by $\mathrm{Dim}_w(E)$ and call it the *extended packing dimension* of the set E with weight function w.

Finally, one can show that the Hausdorff dimension is less or equal to the packing dimension.

Proposition 1. *For any subset E of $\mathbb{R}^n$ we have* $\dim_w(E) \le \mathrm{Dim}_w(E)$.

PROOF: Set $q = (3n)^n$. We show first that $\tilde{\nu}_t(G) \le q\tilde{\kappa}_t(G)$ holds for all bounded subsets G of $\mathbb{R}^n$. To this end set $\tilde{\nu}_{t,\delta}(G) = \inf_{\mathcal{A}\in\mathcal{C}_\delta(G)} \sum_{A\in\mathcal{A}} w(A)|A|^t$ and $\tilde{\kappa}_{t,\delta}(G) = \sup_{\mathcal{R}\in\mathcal{P}_\delta(G)} \sum_{A\in\mathcal{R}} w(A)|A|^t$. Fix $\delta > 0$ and set $\gamma = \frac{\delta}{\sqrt{n}}$. For all $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ define $W_k = \prod_{j=1}^n [k_j\gamma, k_j\gamma + \gamma)$. Since G is bounded, the set $M = \{k \in \mathbb{Z}^n : W_k \cap G \ne \emptyset\}$ is finite. For each $k \in M$ choose $x \in W_k \cap G$ and set $B_k = B_\delta(x)$. Then $\mathcal{H} = \{B_k : k \in M\}$ covers G and hence $\mathcal{H} \in \mathcal{C}_\delta(G)$. This implies $\tilde{\nu}_{t,\delta}(G) \le \sum_{A\in\mathcal{H}} w(A)|A|^t$. Set $a = 3n$. Then $B_k \cap B_l = \emptyset$, if $\|k - l\|_\infty \ge a$. Set $D = \{0, 1, \dots, a-1\}^n$. For $d \in D$ let M_d be the set of all $k \in M$ such that a divides all coordinates of $k - d$ and set $\mathcal{H}_d = \{B_k : k \in M_d\}$. By the choice of a the sets in $\mathcal{H}_d$ are pairwise disjoint and hence $\mathcal{H}_d \in \mathcal{P}_\delta(G)$. This implies $\tilde{\kappa}_{t,\delta}(G) \ge \sum_{A\in\mathcal{H}_d} w(A)|A|^t$ for all $d \in D$. Because of $\mathcal{H} = \bigcup_{d\in D} \mathcal{H}_d$ we have $\sum_{A\in\mathcal{H}} w(A)|A|^t \le \sum_{d\in D}\sum_{A\in\mathcal{H}_d} w(A)|A|^t$. Since q is the cardinality of D, this implies $\tilde{\nu}_{t,\delta}(G) \le q\tilde{\kappa}_{t,\delta}(G)$. Taking the limit $\delta \to 0$ we have $\tilde{\nu}_t(G) \le q\tilde{\kappa}_t(G)$.

Next we show that $\nu_t(E) \le q\kappa_t(E)$ holds for all bounded subsets E of $\mathbb{R}^n$. If $\kappa_t(E) = \infty$ nothing is to show. Otherwise fix $c > \kappa_t(E)$. Then there is $\mathcal{G} \in \mathcal{Q}(E)$ with $\sum_{G\in\mathcal{G}} \tilde{\kappa}_t(G) < c$. Let $H \subset E$ be arbitrary. Since $\tilde{\kappa}_t$ is monotone, we have $\sum_{G\in\mathcal{G}} \tilde{\kappa}_t(H \cap G) \le \sum_{G\in\mathcal{G}} \tilde{\kappa}_t(G) < c$. As H is bounded and $\tilde{\nu}_t$ is subadditive, we get $\tilde{\nu}_t(H) \le \sum_{G\in\mathcal{G}} \tilde{\nu}_t(H \cap G) \le q\sum_{G\in\mathcal{G}} \tilde{\kappa}_t(H \cap G) < qc$. Since $H \subset E$ is arbitrary, we have $\nu_t(E) \le qc$. Since $c > \kappa_t(E)$ is arbitrary we have $\nu_t(E) \le q\kappa_t(E)$.

Finally, for arbitrary $E \subset \mathbb{R}^n$ we show that $\kappa_t(E) = 0$ implies $\nu_t(E) = 0$. To this end suppose that $\kappa_t(E) = 0$. There are bounded subsets E_j of E with $E = \bigcup_{j=1}^\infty E_j$. Since κ_t is monotone, we have $\kappa_t(E_j) = 0$ for $j \ge 1$, and since ν_t is subadditive, we get $\nu_t(E) \le \sum_{j=1}^\infty \nu_t(E_j)$. Since we have $\nu_t(E_j) \le q\kappa_t(E_j)$ for all $j \ge 1$, this implies $\nu_t(E) = 0$.

For any subset E of $\mathbb{R}^n$ we have shown that $\kappa_t(E) = 0$ implies $\nu_t(E) = 0$ proving $\dim_w(E) \le \mathrm{Dim}_w(E)$. $\square$

2. Extended dimensions of probability measures

Denote the collection of all Borel subsets of $\mathbb{R}^n$ by $\mathcal{B}$ and let $w : \mathcal{B} \to [0, \infty]$ be a weight function. For a Borel probability measure μ on $\mathbb{R}^n$ define

$$\dim_w(\mu) = \inf_{E \in \mathcal{B}, \mu(E)=1} \dim_w(E) \quad \text{and} \quad \mathrm{Dim}_w(\mu) = \inf_{E \in \mathcal{B}, \mu(E)=1} \mathrm{Dim}_w(E).$$

We call $\dim_w(\mu)$ the *extended Hausdorff dimension* and $\mathrm{Dim}_w(\mu)$ the *extended packing dimension of the measure* μ with weight function w. Since we have $\dim_w(E) \leq \mathrm{Dim}_w(E)$ for all subsets E of $\mathbb{R}^n$, we get also $\dim_w(\mu) \leq \mathrm{Dim}_w(\mu)$.

The dimension of a measure allows a local approach. For $x \in \mathbb{R}^n$ define

$$\ell_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} \quad \text{and} \quad \ell_w(x) = \lim_{r \to 0} \frac{\log w(B_r(x))}{\log r}$$

provided these limits exists. Often $\ell_\mu(x)$ is called the local or pointwise dimension of the measure μ. The following theorem gives a connection between these local quantities and dimension. For the usual Hausdorff dimension theorems of this type are well known. See for example [3], [28] and [5].

Theorem 1. *Let μ be a Borel probability measure on $\mathbb{R}^n$ and let w be a function on the Borel subsets of $\mathbb{R}^n$ with values in $[0, \infty]$. Suppose there are numbers $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\ell_\mu(x) = \alpha$ and $\ell_w(x) = \beta$ hold for μ-almost all $x \in \mathbb{R}^n$. Then we have* $\dim_w(\mu) = \mathrm{Dim}_w(\mu) = \alpha - \beta$.

Proof: By assumption, there exists a Borel subset M of $\mathbb{R}^n$ of μ-measure one, such that $\lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} = \alpha$ and $\lim_{r \to 0} \frac{\log w(B_r(x))}{\log r} = \beta$ hold for all $x \in M$.

In order to prove $\mathrm{Dim}_w(\mu) \leq \alpha - \beta$ choose an arbitrary $t > \alpha - \beta$ and set $\varepsilon = \frac{1}{2}(t - \alpha + \beta)$. For $k \in \mathbb{N}$ we define

$$M_k = \left\{ x \in M : \mu(B_r(x))^{\frac{t-\varepsilon}{\alpha}} \geq r^t \text{ and } \mu(B_r(x))^{\frac{\beta-\varepsilon}{\alpha}} \geq w(B_r(x)) \text{ for } r \in (0, \tfrac{1}{k}) \right\}.$$

For all $x \in M$ there exists $r(x) \in (0, 1)$ with

$$\frac{t-\varepsilon}{\alpha} \frac{\log \mu(B_r(x))}{\log r} \leq t \quad \text{and} \quad \frac{\beta-\varepsilon}{\alpha} \frac{\log \mu(B_r(x))}{\log r} \leq \frac{\log w(B_r(x))}{\log r}$$

for all $r \in (0, r(x))$. Hence there is $k \in \mathbb{N}$ with $x \in M_k$. This shows $M = \bigcup_{k=1}^\infty M_k$.

Fix $k \in \mathbb{N}$ and $\delta \in (0, \frac{1}{k})$, and let $\mathcal{R}$ be a centered δ-packing of M_k. By the definition of ε and M_k we get

$$\sum_{A \in \mathcal{R}} w(A)|A|^t \leq 2^t \sum_{A \in \mathcal{R}} \mu(A)^{\frac{\beta-\varepsilon}{\alpha}} \mu(A)^{\frac{t-\varepsilon}{\alpha}} = 2^t \sum_{A \in \mathcal{R}} \mu(A) \leq 2^t.$$

This implies $\kappa_t(M_k) \le \tilde{\kappa}_t(M_k) < \infty$. It follows that $\kappa_s(M_k) = 0$ holds for all $s > t$. Since κ_s is an outer measure, we get $\kappa_s(M) = 0$ for all $s > t$, which implies $\mathrm{Dim}_w(M) \le t$. Since $t > \alpha - \beta$ was arbitrary, we have $\mathrm{Dim}_w(M) \le \alpha - \beta$. The definition of the dimension of a measure gives now $\mathrm{Dim}_w(\mu) \le \alpha - \beta$, since M has μ-measure one.

Next we prove $\alpha - \beta \le \dim_w(\mu)$. By the definition of the dimension of a measure it suffices to show $\mu(L) = 0$ for every Borel subset L of $\mathbb{R}^n$ with $\dim_w(L) < \alpha - \beta$.

Let L be a Borel subset of $\mathbb{R}^n$ satisfying $\dim_w(L) < \alpha - \beta$. Choose t with $\dim_w(L) < t < \alpha - \beta$ and set $\varepsilon = \frac{1}{2}(\alpha - \beta - t)$. For $k \in \mathbb{N}$ we define

$$L_k = \big\{x \in L : \mu(B_r(x))^{\frac{t+\varepsilon}{\alpha}} \le r^t \text{ and } \mu(B_r(x))^{\frac{\beta+\varepsilon}{\alpha}} \le w(B_r(x)) \text{ for } r \in (0, \tfrac{1}{k})\big\}.$$

If $x \in L \cap M$, then there exists $r(x) \in (0,1)$ with

$$\frac{t+\varepsilon}{\alpha}\,\frac{\log \mu(B_r(x))}{\log r} \ge t \quad \text{and} \quad \frac{\beta+\varepsilon}{\alpha}\,\frac{\log \mu(B_r(x))}{\log r} \ge \frac{\log w(B_r(x))}{\log r}$$

for all $r \in (0, r(x))$. Therefore there exists $k \in \mathbb{N}$ with $x \in L_k$. This shows that we have $L \cap M \subset \bigcup_{k=1}^{\infty} L_k$.

Fix $k \in \mathbb{N}$ and $\eta > 0$. Because of $\dim_w(L) < t$ we have $\nu_t(L) = 0$ and hence also $\nu_t(L_k) = 0$, since $L_k \subset L$ and ν_t is an outer measure. This implies $\tilde{\nu}_t(L_k) = 0$ and there are $\delta \in (0, \frac{1}{k})$ and a centered δ-cover $\mathcal{A}$ of L_k with

$$\sum_{A \in \mathcal{A}} w(A)|A|^t < 2^t \eta.$$

By the definition of ε and L_k we get now

$$\mu(L_k) \le \sum_{A \in \mathcal{A}} \mu(A) = \sum_{A \in \mathcal{A}} \mu(A)^{\frac{\beta+\varepsilon}{\alpha}} \mu(A)^{\frac{t+\varepsilon}{\alpha}} \le \frac{1}{2^t} \sum_{A \in \mathcal{A}} w(A)|A|^t < \eta.$$

As $\eta > 0$ was arbitrary we get $\mu(L_k) = 0$. This implies $\mu(L \cap M) = 0$. Since M has μ-measure one, $\mu(L) = 0$ is shown.

We have shown that $\mathrm{Dim}_w(\mu) \le \alpha - \beta$ and $\alpha - \beta \le \dim_w(\mu)$ hold. Since $\dim_w(\mu) \le \mathrm{Dim}_w(\mu)$ holds for all probability measures μ, we get the desired result $\dim_w(\mu) = \mathrm{Dim}_w(\mu) = \alpha - \beta$. □

3. Lorenz transformations

We investigate the same class of transformations on $[0,1]^2$ as in [25], which are called Lorenz transformations, since transformations of this kind can serve as geometric models for first return maps to a certain cross section of the Lorenz differential equation. A *Lorenz transformation* $F : [0,1]^2 \to [0,1]^2$ is defined by

$F(x_1, x_2) = (T(x_1), g(x_1, x_2))$, where T and g satisfy the following properties:
(a) $T : [0,1] \to [0,1]$ is *piecewise monotone*, which means that there are points $0 = c_0 < c_1 < c_2 < \ldots < c_N = 1$, such that $T|(c_{j-1}, c_j)$ is continuous and strictly monotone for $1 \le j \le N$, and *piecewise differentiable*, which means that $T|(c_{j-1}, c_j)$ is C^1 for $1 \le j \le N$.
(b) $g : [0,1]^2 \to [0,1]$ is C^1 on the set $\bigcup_{j=1}^{N} (c_{j-1}, c_j) \times [0,1]$ with $\sup |\partial_1 g| < \infty$ and $\sup |\partial_2 g| < 1$, and $\partial_2 g(x) \ne 0$ for all $x \in \bigcup_{j=1}^{N} (c_{j-1}, c_j) \times [0,1]$.
(c) The sets $F((c_{i-1}, c_i) \times [0,1])$ for $1 \le i \le N$ are pairwise disjoint and their closure is contained in $[0,1] \times (0,1)$.

Set $\mathcal{Z} = \{(c_{j-1}, c_j) : 1 \le j \le N\}$. We call $\mathcal{Z}$ a finite partition of $[0,1]$ into open intervals, although it covers $[0,1]$ only up to finitely many points, and say that T is piecewise monotone with respect to $\mathcal{Z}$. Notice that we can add finitely many further partition points, and T is then also piecewise monotone with respect to the partition we get in this way.

Now let $\mathcal{Z}$ be any finite partition of $[0,1]$ into open intervals, with respect to which T is piecewise monotone. For $n \ge 0$ set $\mathcal{Z}_n = \bigvee_{j=0}^{n} T^{-j}\mathcal{Z}$, which is again a finite partition of $[0,1]$ into open intervals. Define $P_{\mathcal{Z}} = \bigcup_{Z \in \mathcal{Z}} Z$ and $R_{\mathcal{Z}} = \bigcap_{j=0}^{\infty} T^{-j}(P_{\mathcal{Z}})$. For every $n \ge 0$ and every $t \in R_{\mathcal{Z}}$ there is a unique element in $\mathcal{Z}_n$ which contains t. We denote it by $Z_n(t)$. Set $\mathcal{Y} = \{Z \times [0,1] : Z \in \mathcal{Z}\}$, which is a finite partition of $[0,1]^2$ up to finitely many vertical lines. For $k \ge 0$ and $l \ge 0$ set $\mathcal{Y}_{k,l} = \bigvee_{j=-l}^{k} F^{-j}\mathcal{Y}$. Define $Q_{\mathcal{Y}} = \bigcup_{Y \in \mathcal{Y}} Y$ and $X_{\mathcal{Y}} = \bigcap_{j=-\infty}^{\infty} F^{-j}(Q_{\mathcal{Y}})$. For every $k \ge 0$ and $l \ge 0$ and every $x \in X_{\mathcal{Y}}$ there is a unique element in $\mathcal{Y}_{k,l}$ which contains x. We denote it by $Y_{k,l}(x)$.

We say that a measure has *no atoms*, if it assigns measure zero to all single points. Furthermore, let $\pi : [0,1]^2 \to [0,1]$ be the projection to the first coordinate. Suppose that μ is an ergodic F-invariant probability measure on $[0,1]^2$ with positive entropy. Then its image μ_π under π is an ergodic T-invariant probability measure on $[0,1]$ which is not concentrated on a periodic orbit, because otherwise μ would also be concentrated on a periodic orbit. This implies that μ_π has no atoms. Since $[0,1] \setminus P_{\mathcal{Z}}$ is a finite set, we get $\mu_\pi(P_{\mathcal{Z}}) = 1$, which gives $\mu(Q_{\mathcal{Y}}) = 1$ and $\mu(X_{\mathcal{Y}}) = 1$.

We define the functions u and v from $Q_{\mathcal{Y}}$ to $\mathbb{R}$ by $u(x) = \log |T'(\pi(x))|$ and $v(x) = -\log |\partial_2 g(x)|$. They play an important role throughout the paper.

In order to determine the quantities $\ell_\mu(x)$ and $\ell_w(x)$ of Theorem 1, we proceed as in [25]. We say that the Lorenz transformation $F(x_1, x_2) = (T(x_1), g(x_1, x_2))$ is *regular*, if T' has onesided limits and satisfies $0 < \inf |T'| \le \sup |T'| < \infty$, and if $x_2 \mapsto \log |\partial_2 g(x_1, x_2)|$ is a uniformly equicontinuous family of maps for $x_1 \in P_{\mathcal{Z}}$. These conditions are slightly different from those in [25]. They are chosen to have a finite partition in Theorem 2 below. One can get all results of this paper also under the assumptions of [25], but this needs generalizations of results in [6] for countable partitions.

We need some lemmas.

Lemma 1. *Let F be a Lorenz transformation and let μ be an F-invariant probability measure such that μ_π has no atoms. Then we have $h_{\mu_\pi}(T) = h_\mu(F)$.*

PROOF: We follow the definition of entropy (see [27]). Let $\mathcal{Z}$ be any finite partition of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, and set $\mathcal{Y} = \{Z \times [0,1] : Z \in \mathcal{Z}\}$. Then $\mu_\pi(R_\mathcal{Z}) = \mu(X_\mathcal{Y}) = 1$. It is easy to see that $H_{\mu_\pi}(\mathcal{Z}_n) = H_\mu(\mathcal{Y}_{n,0})$ for $n \geq 0$. Because of $h_{\mu_\pi}(T, \mathcal{Z}) = \lim_{n\to\infty} \frac{1}{n} H_{\mu_\pi}(\mathcal{Z}_n)$ and $h_\mu(F, \mathcal{Y}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{Y}_{n,0})$ we get $h_{\mu_\pi}(T, \mathcal{Z}) = h_\mu(F, \mathcal{Y})$. If now $\mathcal{Z}$ runs through a sequence of partitions, which become finer and finer with interval lengths going to zero, then $h_{\mu_\pi}(T, \mathcal{Z})$ tends to $h_{\mu_\pi}(T)$ and $h_\mu(F, \mathcal{Y})$ tends to $h_\mu(F)$. This gives $h_{\mu_\pi}(T) = h_\mu(F)$. □

In particular, we can apply Lemma 1 to any ergodic F-invariant probability measure with positive entropy, since then μ_π has no atoms.

Lemma 2. *Let $T : [0,1] \to [0,1]$ be piecewise monotone and piecewise differentiable with $\sup |T'| < \infty$. For $t \in R_\mathcal{Z}$ let $r_n(t)$ be the distance of $T^n(t) \in T^n(Z_n(t))$ to the nearer endpoint (to one endpoint, if both have equal distance) of the interval $T^n(Z_n(t))$. If η is an ergodic T-invariant probability measure with positive entropy, then $\limsup_{n\to\infty} \frac{1}{n} \log r_n(t) = 0$ for η-almost all $t \in R_\mathcal{Z}$.*

PROOF: This is a special case of Proposition 2 in [14]. A different definition of $\mathcal{Z}_n$ is used there, so we have $Z_n(t)$ instead of $Z_{n+1}(t)$. □

For $x \in X_\mathcal{Y}$ and for integers k, $l \geq 0$ we can write $Y_{k,l}(x)$ as $Y_{k,0}(x) \cap Y_{0,l}(x)$, which will be used below to compare the set $Y_{k,l}(x)$ with a ball $B_r(x)$. We have $Y_{k,0}(x) = Z_k(\pi(x)) \times [0,1]$, which is a rectangle, and $Y_{0,l}(x) = F^l(Y_{l,0}(F^{-l}(x)))$, which is a set of the form $\{(t,s) : t \in T^l(Z_l(\pi \circ F^{-l}(x))), \alpha(t) \leq s \leq \beta(t)\}$, where $T^l(Z_l(\pi \circ F^{-l}(x)))$ is an interval, since T^l is monotone and continuous on the intervals in $\mathcal{Z}_l$, and α and β are continuous functions satisfying $\alpha < \beta$, since g is C^1 and $\partial_2 g \neq 0$ on $Q_\mathcal{Y}$. We call the functions α and β the lower and upper boundary of $Y_{0,l}(x)$.

Lemma 3. *Let F be a Lorenz transformation and let μ be an ergodic F-invariant probability measure satisfying $\int u\, d\mu > 0$. Furthermore, suppose that we have $\sup_{y,z\in Y} |u(y) - u(z)| < \int u\, d\mu$ for all $Y \in \mathcal{Y}$. Then for μ-almost all $x \in X_\mathcal{Y}$ there is a constant $C(x) < \infty$ with $\sup_{n\geq 0} \sup_{t\in\pi(Y_{0,n}(x))} |\gamma_n'(t)| \leq C(x)$, where γ_n is either the upper or the lower boundary of $Y_{0,n}(x)$.*

PROOF: Let $n \geq 0$ and $t \in \pi(Y_{0,n}(x))$ be arbitrary. By differentiation rules we get

$$\gamma_n'(t) = \sum_{i=1}^{n} \frac{\partial_1 g(F^{-i}(\overline{x}))}{T'(\pi(F^{-i}(\overline{x})))} \prod_{j=1}^{i-1} \frac{\partial_2 g(F^{-j}(\overline{x}))}{T'(\pi(F^{-j}(\overline{x})))} \quad \text{with} \quad \overline{x} = (t, \gamma_n(t)).$$

For a detailed computation see Lemma 6 and the first part of the proof of Proposition 2 in [25]. Because of $d := \sup|\partial_1 g| < \infty$ and $q := \sup|\partial_2 g| < 1$ we get

$$|\gamma_n'(t)| \le d \sum_{i=1}^{n} q^{i-1} \exp(-\sum_{j=1}^{i} u(F^{-j}(\overline{x}))).$$

Set $c = \max_{Y\in\mathcal{Y}} \sup_{y,z\in Y} |u(y) - u(z)|$. We have $c < \int u\,d\mu$ by assumption. Set $\theta(x) = \sup_{i\ge 1} ic - \sum_{j=1}^{i} u(F^{-j}(x))$. Since both x and $\overline{x}$ are in $Y_{0,n}(x)$, for $0 \le j \le n$ we have that $F^{-j}(\overline{x})$ and $F^{-j}(x)$ are in the same element of $\mathcal{Y}$, which implies $|u(F^{-j}(\overline{x})) - u(F^{-j}(x))| \le c$. From this we get $|\gamma_n'(t)| \le C(x)$ if we set $C(x) = d\sum_{i=1}^{\infty} q^{i-1} e^{\theta(x)} = d\frac{e^{\theta(x)}}{1-q}$. By the ergodic theorem, for μ-almost all x we have $\lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} u(F^{-j}(x)) = \int u\,d\mu$, which implies $\theta(x) < \infty$ because of $c < \int u\,d\mu$. This shows that $C(x)$ is finite for μ-almost all x and the lemma is proved. □

For $k, l \ge 0$ and for any $x \in X_{\mathcal{Y}}$ let $D_k^1(x)$ be the length of the interval $\pi(Y_{k,0}(x)) = Z_k(\pi(x))$ and $d_k^1(x)$ the distance of $\pi(x) \in Z_k(\pi(x))$ to the nearer endpoint of the interval $Z_k(\pi(x))$. We have $D_k^1(x) \ge d_k^1(x)$ by definition, and $d_k^1(x) > 0$ since $Z_k(\pi(x))$ is open. If α and β are the lower and upper boundary of $Y_{0,l}(x)$, set $D_l^2(x) = \beta(\pi(x)) - \alpha(\pi(x))$ and $d_l^2(x) = \min\{\beta(\pi(x)) - p, p - \alpha(\pi(x))\}$, where p is the second coordinate of x. We have $D_l^2(x) \ge d_l^2(x)$ by definition. The closure of the sets $F(Z \times [0,1])$ for $Z \in \mathcal{Z}$ is assumed to be contained in $[0,1] \times (0,1)$, which means that each point in $F(Z \times [0,1])$ has a positive distance to the lines $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$. Since an $x \in X_{\mathcal{Y}}$ is the image under F^l of a point in one of the sets $F(Z \times [0,1])$ and the lower and upper boundary of $Y_{0,l}(x)$ are images of parts of these lines under F^l, we get that x has positive distance to the lower and upper boundary of $Y_{0,l}(x)$. This gives $d_l^2(x) > 0$. One can consider $D_k^1(x)$ and $d_k^1(x)$ as a kind of horizontal diameter and horizontal radius of $Y_{k,0}(x)$ and $D_l^2(x)$ and $d_l^2(x)$ as a kind of vertical diameter and vertical radius of $Y_{0,l}(x)$.

Lemma 4. *Let F be a regular Lorenz transformation and let $\varepsilon > 0$ be such that $\sup_{y,z\in Y} |u(y) - u(z)| < \varepsilon$ holds for all $Y \in \mathcal{Y}$. If μ is an ergodic F-invariant probability measure with positive entropy, then there is a subset K of $X_{\mathcal{Y}}$ with $\mu(K) = 1$, such that for all $x \in K$ we have*

$$\int u\,d\mu - \varepsilon < \liminf_{k\to\infty} -\frac{1}{k}\log D_k^1(x) \le \limsup_{k\to\infty} -\frac{1}{k}\log D_k^1(x) < \int u\,d\mu + \varepsilon,$$

$$\int u\,d\mu - \varepsilon < \liminf_{k\to\infty} -\frac{1}{k}\log d_k^1(x) \le \limsup_{k\to\infty} -\frac{1}{k}\log d_k^1(x) < \int u\,d\mu + \varepsilon$$

and $$\lim_{l\to\infty} -\frac{1}{l}\log D_l^2(x) = \lim_{l\to\infty} -\frac{1}{l}\log d_l^2(x) = \int v\,d\mu.$$

PROOF: The intervals, whose lengths are $D_k^1(x)$ and $d_k^1(x)$, are mapped by T^k monotonically onto intervals, which have length at least $r_k(\pi(x))$, where r_k is as in Lemma 2. Using the mean value theorem, the ergodic theorem and Lemma 2 with $\eta = \mu_\pi$, which has positive entropy by Lemma 1, one gets the first two assertions. The third assertion is proved similarly, using the maps $x_2 \mapsto g(x_1, x_2)$ which are contractions of vertical lines. Here one needs the assumption that the maps $x_2 \mapsto \log|\partial_2 g(x_1, x_2)|$ are uniformly equicontinuous. The detailed proof can be found in [25] as the proof of Lemma 9. The proof of the third assertion works also, if $\int v\,d\mu = \infty$, since we still have $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} v(F^j(x)) = \int v\,d\mu$ for μ-almost all x. This follows from the ergodic theorem, since v has positive values. $\square$

Lemma 5. *Let F be a regular Lorenz transformation. Suppose that μ is an ergodic F-invariant probability measure with $h_\mu(F) > 0$ satisfying $\int u\,d\mu > 0$ and $\int v\,d\mu > 0$, and that $\sup_{y,z\in Y} |u(y) - u(z)| < \int u\,d\mu$ holds for all $Y \in \mathcal{Y}$. Then there is an F-invariant subset L of $X_{\mathcal{Y}}$ with $\mu(L) = 1$ such that $\mathcal{Y}$ restricted to L is a generating partition for the transformation $F|L$ and such that for every $x \in L$ there are constants c_1, c_2 and c_3 and an integer q with the following properties:*

$$B_r(x) \supseteq Y_{k,l}(x) \text{ if } k \ge 0 \text{ and } l \ge 0 \text{ are such that } c_1 D_l^2(x) < r \text{ and } c_2 D_k^1(x) < r.$$
$$B_r(x) \subseteq Y_{k,l}(x) \text{ if } k \ge 0 \text{ and } l \ge q \text{ are such that } c_3 d_l^2(x) > r \text{ and } d_k^1(x) > r.$$

PROOF: By Lemma 3 there is a set G with $\mu(G) = 1$, such that for $x \in G$ the constant $C(x)$ which bounds the slope of the lower and upper boundary of $Y_{0,n}(x)$ for all $n \ge 1$ is finite. For $x \in G$ set $c_1 = 2$ and $c_2 = 2 + 2C(x)$. By the definitions of $D_k^1(x)$ and $D_l^2(x)$ we get that the distance from x to any other point in $Y_{k,l}(x)$ is bounded by $D_l^2(x) + D_k^1(x)(1 + C(x))$. If now $c_1 D_l^2(x) < r$ and $c_2 D_k^1(x) < r$ hold, then we have also $D_l^2(x) + D_k^1(x)(1 + C(x)) < r$ and hence $B_r(x) \supseteq Y_{k,l}(x)$. Therefore the second assertion is shown for all $x \in G$.

The assumptions of Lemma 4 are fulfilled for some $\varepsilon < \int u\,d\mu$. Let K be the set of μ-measure one found there and let G be as above. Set $H = \bigcap_{j=0}^\infty F^{-j}(G \cap K)$. Then H is an F-invariant subset of $X_{\mathcal{Y}}$, which has μ-measure one. For every $x \in H$ we have now $\lim_{k,l\to\infty} |Y_{k,l}(x)| = 0$ by the second assertion proved above, since $\lim_{k\to\infty} D_k^1(x) = 0$ and $\lim_{l\to\infty} D_l^2(x) = 0$ follow from Lemma 4 because of $\int u\,d\mu - \varepsilon > 0$ and $\int v\,d\mu > 0$. Therefore $\mathcal{Y}$ restricted to H is a generating partition for the transformation $F|H$. This shows the first assertion.

For $x \in X_{\mathcal{Y}}$ we write $Y_{k,l}(x)$ as $Y_{k,0}(x) \cap Y_{0,l}(x)$. Since $\mathcal{Y}$ restricted to H is a generating partition, we can apply Proposition 1 of [25]. It says that there is a set $M \subset H$ with $\mu(M) = 1$ such that for all $x \in M$ there is an open interval I and an integer $\tilde{q}$ with $\pi(Y_{0,l}(x)) = I$ for all $l \ge \tilde{q}$. Set $L = \bigcap_{j=0}^\infty F^{-j}(M)$, which is an

F-invariant subset of $X_{\mathcal{Y}}$ of μ-measure one. If α_l and β_l are the lower and upper boundary of $Y_{0,l}(x)$ we have then $Y_{0,l}(x) = \{(t,s) : t \in I,\, \alpha_l(t) \le s \le \beta_l(t)\}$ for $l \ge \tilde{q}$. By Lemma 3 the slopes of α_l and β_l are bounded by $C(x) < \infty$ on the whole interval I for all $x \in L$. Furthermore, $Y_{k,0}(x)$ is the rectangle $Z_k(\pi(x)) \times [0,1]$.

Fix $x \in L$. Notice that $\lim_{l\to\infty} d_l^2(x) = 0$, since the Lorenz transformation F contracts vertical lines. Therefore we find $q \ge \tilde{q}$ such that the distance of $\pi(x)$ to the nearer endpoint of the open interval I is greater than $d_l^2(x)$ for all $l \ge q$. We set $c_3 = (1+C(x))^{-1/2}$. By the definition of $d_l^2(x)$ and some elementary geometry, we get $B_r(x) \subseteq Y_{0,l}(x)$ if $l \ge q$ and $c_3 d_l^2(x) > r$. By the definition of $d_k^1(x)$, we have $B_r(x) \subseteq Y_{k,0}(x)$, if $d_k^1(x) > r$. Because of $Y_{k,l}(x) = Y_{k,0}(x) \cap Y_{0,l}(x)$, the third assertion is proved for all $x \in L$. $\square$

Lemma 6. *Let F be a regular Lorenz transformation and let μ be an ergodic F-invariant probability measure satisfying $h_\mu(F) > 0$. Then we have $\int u\, d\mu > 0$ and $\int v\, d\mu > 0$. Furthermore, for every $\varepsilon > 0$ there is a finite partition $\mathcal{Z}$ of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, such that $\sup_{y,z\in Y} |u(y) - u(z)| < \varepsilon$ holds for all $Y \in \mathcal{Y}$, where $\mathcal{Y} = \{Z \times [0,1] : Z \in \mathcal{Z}\}$.*

Proof: By the definition of a regular Lorenz transformation the derivative T' has onesided limits and the T-invariant measure μ_π is ergodic, since we assume that μ is ergodic. By Lemma 1 we have $h_{\mu_\pi}(T) = h_\mu(F) > 0$. The assumptions of Theorem 2 in [9] are satisfied, which gives $\int \log|T'|\, d\mu_\pi \ge h_{\mu_\pi}(T)$. This means that $\int u\, d\mu \ge h_\mu(F)$ holds and hence $\int u\, d\mu > 0$ is shown. In the definition of a Lorenz transformation we have $\sup|\partial_2 g| < 1$, which implies $\int v\, d\mu > 0$.

By the definition of a regular Lorenz transformation the function $\log|T'|$ is bounded and has onesided limits. Hence there is a finite partition $\mathcal{Z}$ of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, such that we have $\sup_{s,t\in Z} |\log|T'(s)| - \log|T'(t)|| < \varepsilon$ for all $Z \in \mathcal{Z}$. If $\mathcal{Y} = \{Z \times [0,1] : Z \in \mathcal{Z}\}$, we have then $\sup_{y,z\in Y} |u(y) - u(z)| < \varepsilon$ for all $Y \in \mathcal{Y}$. $\square$

Now we can show

Theorem 2. *Suppose that the weight function w is monotone, which means that $A \subset B$ implies $w(A) \le w(B)$, and that F is a regular Lorenz transformation. Let a be in $\mathbb{R}$ and let μ be an ergodic F-invariant probability measure with positive entropy $h_\mu(F)$. Suppose that for every $\varepsilon > 0$ there is a finite partition $\mathcal{Z}$ of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, such that for any partition $\mathcal{Y} = \{U \times [0,1] : U \in \mathcal{U}\}$ with $\mathcal{U}$ a finite partition of $[0,1]$ into open intervals refining $\mathcal{Z}$ we have for μ-almost all x*

$$\liminf_{k,l\to\infty} \frac{-\log w(Y_{k,l}(x))}{k+l} \ge a - \varepsilon \quad \textit{and} \quad \limsup_{k,l\to\infty} \frac{-\log w(Y_{k,l}(x))}{k+l} \le a + \varepsilon.$$

Then $\int u\, d\mu$ and $\int v\, d\mu$ are > 0 and $\lim_{r\to 0} \frac{\log w(B_r(x))}{\log r} = a\left(\frac{1}{\int u\, d\mu} + \frac{1}{\int v\, d\mu}\right)$ holds for μ-almost all x.

Proof: We have $\int u\,d\mu > 0$ and $\int v\,d\mu > 0$ by Lemma 6. Choose $\varepsilon \in (0, \int u\,d\mu)$ and let $\mathcal{Z}$ be a finite partition of $[0,1]$ having the properties assumed in the theorem. By Lemma 6 there is a finite partition $\mathcal{U}$ of $[0,1]$ into open intervals, which can be assumed to refine $\mathcal{Z}$, such that $\sup_{y,z\in Y}|u(y)-u(z)| < \varepsilon$ holds for all $Y \in \mathcal{Y}$, where $\mathcal{Y} = \{U \times [0,1] : U \in \mathcal{U}\}$. We can apply Lemma 4 and Lemma 5 to the Lorenz transformation with this partition $\mathcal{Y}$. Let K and L be the sets of μ-measure one found in these lemmas. By assumption, there is then a subset M_ε of $K \cap L$ with $\mu(M_\varepsilon) = 1$, such that $\liminf_{k,l\to\infty} \frac{-\log w(Y_{k,l}(x))}{k+l} \ge a - \varepsilon$ and $\limsup_{k,l\to\infty} \frac{-\log w(Y_{k,l}(x))}{k+l} \le a + \varepsilon$ hold for all $x \in M_\varepsilon$.

Fix $x \in M_\varepsilon$ and let the constants c_1, c_2 and c_3 and the integer q be as in Lemma 5. The sequences $(D_k^1(x))_{k\ge 0}$ and $(D_l^2(x))_{l\ge 0}$ converge to zero by Lemma 4, since we have $\int u\,d\mu - \varepsilon > 0$ and $\int v\,d\mu > 0$. Hence for every small $r > 0$ there are k and l satisfying

$$c_2 D_k^1(x) < r \le c_2 D_{k-1}^1(x) \quad\text{and}\quad c_1 D_l^2(x) < r \le c_1 D_{l-1}^2(x).$$

If r tends to zero, then k and l tend to infinity. Hence by the choice of M_ε we get using Lemma 5 and Lemma 4, that there is $r_0(x)$ such that

$$\begin{aligned}\frac{\log w(B_r(x))}{\log r} &\le -\frac{\log w(Y_{k,l}(x))}{k+l}\Big(-\frac{k}{\log r} - \frac{l}{\log r}\Big)\\ &\le -\frac{\log w(Y_{k,l}(x))}{k+l}\Big(-\frac{k}{\log(c_2 D_{k-1}^1(x))} - \frac{l}{\log(c_1 D_{l-1}^2(x))}\Big)\\ &\le (a+2\varepsilon)\Big(\frac{1}{\int u\,d\mu - \varepsilon} + \frac{1}{\int v\,d\mu} + \varepsilon\Big)\end{aligned}$$

holds for $r < r_0(x)$. By Lemma 4 also the sequences $(d_k^1(x))_{k\ge 0}$ and $(d_l^2(x))_{l\ge 0}$ converge to zero. Hence for every small $r > 0$ there are k and l satisfying

$$d_{k+1}^1(x) \le r < d_k^1(x) \quad\text{and}\quad c_3 d_{l+1}^2(x) \le r < c_3 d_l^2(x).$$

If r tends to zero, then k and l tend to infinity. Hence by the choice of M_ε we get using Lemma 5 and Lemma 4, that there is $r_1(x)$ such that

$$\begin{aligned}\frac{\log w(B_r(x))}{\log r} &\ge -\frac{\log w(Y_{k,l}(x))}{k+l}\Big(-\frac{k}{\log r} - \frac{l}{\log r}\Big)\\ &\ge -\frac{\log w(Y_{k,l}(x))}{k+l}\Big(-\frac{k}{\log(d_{k+1}^1(x))} - \frac{l}{\log(c_3 d_{l+1}^2(x))}\Big)\\ &\ge (a-2\varepsilon)\Big(\frac{1}{\int u\,d\mu + \varepsilon} + \frac{1}{\int v\,d\mu} - \varepsilon\Big)\end{aligned}$$

holds for $r < r_1(x)$.

We choose $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and set $M = \bigcap_{n=1}^{\infty} M_{\varepsilon_n}$. Then we have $\mu(M) = 1$ and $\lim_{r\to 0} \frac{\log w(B_r(x))}{\log r} = a(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$ holds for all $x \in M$. □

We get the result which is proved in [25] as a corollary.

Corollary. *Suppose that F is a regular Lorenz transformation and let μ be an ergodic F-invariant probability measure with positive entropy $h_\mu(F)$. Then we have $\lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} = h_\mu(F)(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$ for μ-almost all x.*

Proof: We have $\int u\,d\mu > 0$ by Theorem 2. Let $\mathcal{Z}$ be the partition of $[0,1]$ found for $\varepsilon = \int u\,d\mu$ in Lemma 6. Let $\mathcal{U}$ be a finite partition of $[0,1]$ into open intervals refining $\mathcal{Z}$ and set $\mathcal{Y} = \{U \times [0,1] : U \in \mathcal{U}\}$. Then $\sup_{y,z\in Y} |u(y) - u(z)| < \varepsilon$ holds for all $Y \in \mathcal{Y}$. By Lemma 5 there is then an F-invariant subset L of $X_{\mathcal{Y}}$ with $\mu(L) = 1$ such that $\mathcal{Y}$ restricted to L is a generating partition for the transformation $F|L$. By the Shannon-McMillan-Breiman Theorem applied to $F|L$ we get $\lim_{k,l\to\infty} \frac{-\log \mu(Y_{k,l}(x))}{k+l} = h_\mu(F)$ for μ-almost all $x \in L$. Hence the assumptions of Theorem 2 are fulfilled with $w = \mu$ and $a = h_\mu(F)$ and the corollary follows. □

4. Recurrence dimension

We consider the *recurrence dimension*, which is the extended Hausdorff dimension $\dim_w$ or the extended packing dimension Dim_w with weight function w defined by $w(A) = e^{-s\tau(A)}$, where $s \in \mathbb{R}$ and $\tau(A)$ is the recurrence time of the set A defined by $\tau(A) = \inf\{n \geq 1 : F^n(A) \cap A \neq \emptyset\}$. We show

Theorem 3. *Let the weight function w be defined as above. Let F be a regular Lorenz transformation and let μ be an ergodic F-invariant probability measure with $h_\mu(F) > 0$. Then $\dim_w(\mu) = \mathrm{Dim}_w(\mu) = (h_\mu(F) - s)(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$.*

It suffices to show $\lim_{k,l\to\infty} \frac{\tau(Y_{k,l}(x))}{k+l} = 1$ for μ-almost all $x \in X_{\mathcal{Y}}$, where $\mathcal{Y} = \{U \times [0,1] : U \in \mathcal{U}\}$ and $\mathcal{U}$ is a finite partition of [0,1] into open intervals, which refines the partition found in Lemma 6 for $\varepsilon = \int u\,d\mu$. By Theorem 2 we get then $\lim_{r\to 0} \frac{\log e^{-\tau(B_r(x))}}{\log r} = \frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu}$ and hence $\ell_w(x) = s(\frac{1}{\int u\,d\mu} + \frac{1}{\int v\,d\mu})$ for μ-almost all x. The desired formula follows from the corollary and Theorem 1.

In order to show $\lim_{k,l\to\infty} \frac{\tau(Y_{k,l}(x))}{k+l} = 1$ for μ-almost all $x \in X_{\mathcal{Y}}$, by Lemma 5 we can assume that there is an F-invariant subset L of $X_{\mathcal{Y}}$ with $\mu(L) = 1$ such that $\mathcal{Y}$ restricted to L is a generating partition for $F|L$. It follows from Theorem 4.2 in [2] that $\liminf_{k,l\to\infty} \frac{\tau(Y_{k,l}(x))}{k+l} \geq 1$ holds for μ-almost all $x \in X_{\mathcal{Y}}$. Hence it suffices to show that $\limsup_{k,l\to\infty} \frac{\tau(Y_{k,l}(x))}{k+l} \leq 1$ holds for μ-almost all $x \in X_{\mathcal{Y}}$. To this end we introduce shift spaces.

Let $\mathcal{Z} = \{(c_{i-1}, c_i) : 1 \le i \le N\}$ be a finite partition of [0,1] into open intervals, with respect to which T is piecewise monotone. Define the map ϑ^+ from $R_{\mathcal{Z}}$ to the full shift space $\{1, 2, \dots, N\}^{\mathbb{N}}$ equipped with the usual topology and the shift transformation S by $\vartheta^+(t) = i_0 i_1 i_2 \dots$ such that $T^j(t) \in (c_{i_j - 1}, c_{i_j})$ holds for $j \ge 0$. Then we have $S \circ \vartheta^+ = \vartheta^+ \circ T$ and $\vartheta^+(R_{\mathcal{Z}})$ is S-invariant, but not closed. Set $\mathcal{Y} = \{Z \times [0,1] : Z \in \mathcal{Z}\}$ and define the map ϑ from $X_{\mathcal{Y}}$ to the full shift space $\{1, 2, \dots, N\}^{\mathbb{Z}}$ equipped with the usual topology and the shift transformation S by $\vartheta(x) = \dots i_{-2} i_{-1} i_0 i_1 i_2 \dots$ such that $F^j(x) \in (c_{i_j - 1}, c_{i_j}) \times [0,1]$ holds for $j \in \mathbb{Z}$. Then $S \circ \vartheta = \vartheta \circ F$ and $\vartheta(X_{\mathcal{Y}})$ is S-invariant, but not closed.

Set $C = \{\lim_{t \uparrow c_i} \vartheta^+(t) : 1 \le i \le N\} \cup \{\lim_{t \downarrow c_{i-1}} \vartheta^+(t) : 1 \le i \le N\}$. Let Σ_T^+ be the closure of $\vartheta^+(R_{\mathcal{Z}})$ in $\{1, 2, \dots, N\}^{\mathbb{N}}$ and Σ_F be the closure of $\vartheta(X_{\mathcal{Y}})$ in $\{1, 2, \dots, N\}^{\mathbb{Z}}$. If $i_0 i_1 i_2 \dots \in \Sigma_T^+ \setminus \vartheta^+(R_{\mathcal{Z}})$ then $i_j i_{j+1} i_{j+2} \dots \in C$ for some $j \in \mathbb{N}$. If $\dots i_{-1} i_0 i_1 i_2 \dots \in \Sigma_F \setminus \vartheta(X_{\mathcal{Y}})$ then $i_j i_{j+1} i_{j+2} \dots \in C$ for some $j \in \mathbb{Z}$. Hence both, $\Sigma_T^+ \setminus \vartheta^+(R_{\mathcal{Z}})$ and $\Sigma_F \setminus \vartheta(X_{\mathcal{Y}})$, are countable and contain at most finitely many periodic points.

Let $[i_0 i_1 \dots i_j]^+$ be the set of all elements of Σ_T^+, which begin with the word $i_0 i_1 \dots i_j$ and similarly, let $[i_0 i_1 \dots i_j]$ be the set of all elements of Σ_F, which have the word $i_0 i_1 \dots i_j$ in the places from 0 to j. For $\tilde{x} = \dots i_{-1} i_0 i_1 i_2 \dots \in \Sigma_F$ define $\tilde{Y}_{k,l}(\tilde{x}) = S^l [i_{-l} i_{-l+1} \dots i_k]$. If $x \in X_{\mathcal{Y}}$, then $\vartheta(Y_{k,l}(x)) = \tilde{Y}_{k,l}(\vartheta(x)) \cap \vartheta(X_{\mathcal{Y}})$. This follows easily from the definitions.

Now we can show the following result, which completes the proof of Theorem 3.

Proposition 2. *Let F be a Lorenz transformation and suppose that μ is an ergodic F-invariant probability measure with $h_\mu(F) > 0$. Let $\mathcal{Z}$ be a finite partition of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, and set $\mathcal{Y} = \{Z \times [0,1] : Z \in \mathcal{Z}\}$. Suppose that there is an F-invariant subset L of $X_{\mathcal{Y}}$ with $\mu(L) = 1$ such that $\mathcal{Y}$ restricted to L is a generating partition for $F|L$. Then $\limsup_{k,l \to \infty} \frac{\tau(Y_{k,l}(x))}{k+l} \le 1$ holds for μ-almost all x.*

Proof: We define the Markov diagram for the shift space Σ_T^+, which is a finite or countable oriented graph. Let $\mathcal{W}$ be the set of all words which occur in an element of Σ_T^+. Set $\mathcal{D} = \{S^n([i_0 i_1 \dots i_n]^+) : n \ge 0, i_0 i_1 \dots i_n \in \mathcal{W}\}$. Together with the arrows $S^n([i_0 i_1 \dots i_n]^+) \to S^{n+1}([i_0 i_1 \dots i_n i_{n+1}]^+)$ we get an oriented graph $(\mathcal{D}, \to)$, the socalled Markov diagram of Σ_T^+.

We have $S^n([i_0 i_1 \dots i_n]^+) \subset [i_n]^+$. For each $D \in \mathcal{D}$ define $\xi(D) = i$, if $D \subset [i]^+$. Define further $\Sigma_{\mathcal{D}} = \{\dots D_{-1} D_0 D_1 D_2 \dots : D_j \in \mathcal{D}, D_j \to D_{j+1}$ for $j \in \mathbb{Z}\}$ and $\xi : \Sigma_{\mathcal{D}} \to \{1, 2, \dots, N\}^{\mathbb{Z}}$ by $\xi(\dots D_{-1} D_0 D_1 \dots) = \dots \xi(D_{-1}) \xi(D_0) \xi(D_1) \dots$. For a piecewise increasing T it is shown in [6] that ξ is injective, that $\xi(\Sigma_{\mathcal{D}})$ is a subset of Σ_F and that $\Sigma_F \setminus \xi(\Sigma_{\mathcal{D}})$ is a nullset for each ergodic invariant probability measure with positive entropy. The extension to the general case is done in [7].

Since μ is concentrated on the set $L \subset X_{\mathcal{Y}}$, on which ϑ is injective, there is an ergodic S-invariant probability measure $\tilde{\mu}$ on Σ_F with positive entropy satisfying $\tilde{\mu} = \mu \circ \vartheta^{-1}$. Since $\Sigma_F \setminus \xi(\Sigma_{\mathcal{D}})$ has $\tilde{\mu}$-measure zero, there is an ergodic S-invariant probability measure $\hat{\mu}$ on $\Sigma_{\mathcal{D}}$ with positive entropy satisfying $\tilde{\mu} = \hat{\mu} \circ \xi^{-1}$.

For $\hat{x} = \dots D_{-1}D_0D_1D_2\dots$ let $\hat{Y}_{k,l}(\hat{x})$ be the set of all sequences in $\Sigma_{\mathcal{D}}$, having the word $D_{-l}D_{-l+1}\dots D_k$ in the places from $-l$ to k. Since ξ is defined coordinatewise, we have then $\xi(\hat{Y}_{k,l}(\hat{x})) \subset \tilde{Y}_{k,l}(\xi(\hat{x}))$. Furthermore, for a subset A of a shift space define $\overline{\tau}(A) = \inf\{n \geq 1 : S^n(A) \cap A \text{ contains a periodic point}\}$.

We find $E \in \mathcal{D}$ such that $\hat{\mu}(\hat{U}) > 0$, where $\hat{U}$ is the set of all sequences in $\Sigma_{\mathcal{D}}$ which have the symbol E in place zero. By the ergodic theorem there is then a set $\hat{M} \subset \Sigma_{\mathcal{D}}$ with $\hat{\mu}(\hat{M}) = 1$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} 1_{\hat{U}}(S^j(\hat{x})) = \hat{\mu}(\hat{U})$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} 1_{\hat{U}}(S^{-j}(\hat{x})) = \hat{\mu}(\hat{U})$ hold for all $\hat{x} \in \hat{M}$.

Suppose $\hat{x} = \dots D_{-2}D_{-1}D_0D_1D_2\dots$ is in $\hat{M}$. Let $n_0 = 0 < n_1 < n_2 < \dots$ and $m_0 = 0 < m_1 < m_2 < \dots$ be such that $D_{n_j} = E$ and $D_{-m_j} = E$ hold for $j \geq 1$. Then we have $\lim_{j\to\infty} \frac{j}{n_j} = \hat{\mu}(\hat{U})$ and $\lim_{j\to\infty} \frac{j}{m_j} = \hat{\mu}(\hat{U})$, which implies $\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = 1$ and $\lim_{j\to\infty} \frac{m_{j+1}}{m_j} = 1$. For $k,\, l > 0$ let i and j be such that $n_{j-1} \leq k < n_j$ and $-m_i < -l \leq -m_{i-1}$. Set $\hat{V} = \hat{Y}_{n_j,m_i}(\hat{x})$. Let $\hat{p}$ be the periodic point with period $D_{-m_i}D_{-m_i+1}\dots D_{n_j-1}$, which exists, since we have $D_{n_j-1} \to D_{n_j} = E = D_{-m_i}$. Then $\hat{p} \in \hat{V} \cap S^{m_i+n_j}(\hat{V})$ and hence $\overline{\tau}(\hat{V}) \leq m_i + n_j$. Because of $\hat{V} \subset \hat{Y}_{k,l}(\hat{x})$ we have also $\overline{\tau}(\hat{Y}_{k,l}(\hat{x})) \leq m_i + n_j$ and hence $\limsup_{k,l\to\infty} \frac{\overline{\tau}(\hat{Y}_{k,l}(\hat{x}))}{k+l} \leq \limsup_{i,j\to\infty} \frac{m_i+n_j}{m_{i-1}+n_{j-1}} = 1$.

Now set $\tilde{M} = \xi(\hat{M}) \subset \Sigma_F$. Then $\tilde{\mu}(\tilde{M}) = \hat{\mu}(\xi^{-1}(\tilde{M})) \geq \hat{\mu}(\hat{M}) = 1$. Choose $\tilde{x} \in \tilde{M}$. There is $\hat{x} \in \hat{M}$ with $\xi(\hat{x}) = \tilde{x}$. Because of $\xi(\hat{Y}_{k,l}(\hat{x})) \subset \tilde{Y}_{k,l}(\xi(\hat{x}))$ and since ξ maps periodic points to periodic points and commutes with S, we get $\overline{\tau}(\tilde{Y}_{k,l}(\tilde{x})) \leq \overline{\tau}(\hat{Y}_{k,l}(\hat{x}))$. This implies $\limsup_{k,l\to\infty} \frac{\overline{\tau}(\tilde{Y}_{k,l}(\tilde{x}))}{k+l} \leq 1$.

Finally set $M = \vartheta^{-1}(\tilde{M})$. Then $\mu(M) = \tilde{\mu}(\tilde{M}) = 1$. Choose $x \in M$ and set $\tilde{x} = \vartheta(x) \in \tilde{M}$. Set $n = \overline{\tau}(\tilde{Y}_{k,l}(\tilde{x}))$. Then $\tilde{Y}_{k,l}(\tilde{x}) \cap S^n(\tilde{Y}_{k,l}(\tilde{x}))$ contains a periodic point $\tilde{p}$. Since $\Sigma_F \setminus \vartheta(X_{\mathcal{Y}})$ contains at most finitely many periodic points and since $\tilde{x} \in \vartheta(X_{\mathcal{Y}})$, this periodic point $\tilde{p}$ cannot be in $\Sigma_F \setminus \vartheta(X_{\mathcal{Y}})$, if l and k are large enough. In this case there is a periodic point $p \in X_{\mathcal{Y}}$ with $\vartheta(p) = \tilde{p}$, which is contained in $Y_{k,l}(x) \cap F^n(Y_{k,l}(x))$. In particular, this intersection is nonempty. Therefore, we have shown $\tau(Y_{k,l}(x)) \leq \overline{\tau}(\tilde{Y}_{k,l}(\tilde{x}))$ for all l and k which are large enough. This implies $\limsup_{k,l\to\infty} \frac{\tau(Y_{k,l}(x))}{k+l} \leq 1$. We have shown this for all $x \in M$, which means for μ-almost all x. $\square$

5. Equilibrium states

Another possibility for the weight function w is to choose $w(A) = \varrho(A)^s$ for all Borel sets A, where ϱ is a probability measure and $s \in \mathbb{R}$. In order to get a formula for the extended dimensions we have to show the condition in Theorem 2. This can be done for equilibrium states of certain functions $\varphi : [0,1]^2 \to \mathbb{R}$. An *equilibrium state* of a function φ is an F-invariant probability measure ϱ maximizing $h_\varrho(F) + \int \varphi \, d\varrho$. This maximum is called the *pressure* $p(F,\varphi)$ of φ. If we add a constant to φ then the pressure changes also by this constant, but equilibrium states do not change. Hence we can assume that $p(F,\varphi) = 0$. We say that φ is *piecewise Hölder continuous*, if there are $0 = c_0 < c_1 < \ldots < c_N = 1$ such that $\varphi|(c_{j-1}, c_j) \times [0,1]$ is Hölder continuous for $1 \le j \le N$. Then we have

Theorem 4. *Let F be a regular Lorenz transformation and let μ be an ergodic F-invariant probability measure with positive entropy $h_\mu(F)$. Suppose further that T is topologically transitive. Then every piecewise Hölder continuous function $\varphi : [0,1]^2 \to \mathbb{R}$ with $p(F,\varphi) = 0$ and $\sup \varphi - \inf \varphi < h_{\mathrm{top}}(F)$ has an equilibrium state ϱ such that $\dim_w(\mu) = \mathrm{Dim}_w(\mu) = (h_\mu(F) - s \int \varphi \, d\mu)(\frac{1}{\int u \, d\mu} + \frac{1}{\int v \, d\mu})$ where $s \in \mathbb{R}$ and $w(A) = \varrho(A)^s$ for all Borel sets A.*

Theorem 4 follows from Theorems 1 and 2, if we show, that for every $\varepsilon > 0$ there is a finite partition $\mathcal{Z}$ of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, such that for any partition $\mathcal{Y} = \{U \times [0,1] : U \in \mathcal{U}\}$ with $\mathcal{U}$ a finite partition of $[0,1]$ into open intervals refining $\mathcal{Z}$, we have

$$\liminf_{k,l\to\infty} \frac{-\log \varrho(Y_{k,l}(x))}{k+l} \ge \int \varphi \, d\mu - \varepsilon \quad \text{and} \quad \limsup_{k,l\to\infty} \frac{-\log \varrho(Y_{k,l}(x))}{k+l} \le \int \varphi \, d\mu + \varepsilon$$

for μ-almost all x. In order to show this, we reduce everything to one dimension.

For a function $f : [0,1] \to \mathbb{R}$ and $n \ge 1$ set $S_n f = \sum_{j=0}^{n-1} f \circ T^j$. The idea of the following proof is similar to that of Lemma 1.6 in [4].

Lemma 7. *Let F be a regular Lorenz transformation. Let $\varphi : [0,1]^2 \to \mathbb{R}$ be bounded and piecewise Hölder continuous. Then there are a function $\psi : [0,1] \to \mathbb{R}$, which has bounded p-variation for some $p \ge 1$, and a bounded measurable function $\chi : [0,1]^2 \to \mathbb{R}$, such that $\psi(\pi(x)) = \varphi(x) - \chi(x) + \chi(F(x))$ holds for all x in a set, which has measure one for every ergodic F-invariant probability measure with positive entropy. Moreover $\sup S_n\psi - \inf S_n\psi \le n(\sup \varphi - \inf \varphi) + 4 \sup |\chi|$ holds for all $n \ge 1$.*

Proof: Since $\varphi : [0,1]^2 \to \mathbb{R}$ is piecewise Hölder continuous there are constants $a > 0$ and $\alpha \in (0,1]$ and a partition $\{(c_{j-1}, c_j) : 1 \le j \le N\}$ of $[0,1]$, with respect to which T is piecewise monotone, such that $|\varphi(x) - \varphi(y)| \le a|x-y|^\alpha$ holds, if $\pi(x)$ and $\pi(y)$ are in (c_{j-1}, c_j) for some j. Since φ is bounded, we can assume that $\sup |\varphi| \le a$. If $\pi(x) = c_j$ for some j, we redefine $F(x)$ by $F(x) = (\pi(x), 0)$.

We define $\tilde{\pi} : [0,1]^2 \to [0,1]^2$ by $\tilde{\pi}(x) = (\pi(x), 0)$ and for $x \in [0,1]^2$ we set $\chi(x) = \sum_{j=0}^\infty \varphi(F^j(x)) - \varphi(F^j(\tilde{\pi}(x)))$. Set $q = \sup |\partial_2 g|$. We have $q < 1$ by the definition of a Lorenz transformation and by the above redefinition of F. For all

$j \geq 0$ the two points $F^j(x)$ and $F^j(\tilde{\pi}(x))$ are mapped to the same point by π. This implies $|F^i(x) - F^i(\tilde{\pi}(x))| \leq q^i$ and $|\varphi(F^i(x)) - \varphi(F^i(\tilde{\pi}(x)))| \leq aq^{i\alpha}$ for $i \geq 0$. Therefore χ is a well defined measurable function, which is bounded.

For $t \in [0,1]$ set $\psi(t) = \varphi(t,0) + \sum_{j=1}^{\infty} \varphi(F^j(t,0)) - \varphi(F^{j-1}(T(t),0))$. Because of $\varphi(x) - \chi(x) + \chi(F(x)) = \varphi(\tilde{\pi}(x)) + \sum_{j=1}^{\infty} \varphi(F^j(\tilde{\pi}(x))) - \varphi(F^{j-1}(\tilde{\pi} \circ F(x)))$ we have $\psi(\pi(x)) = \varphi(x) - \chi(x) + \chi(F(x))$ for all $x \in [0,1]^2$. For $n \geq 1$ we get then $S_n\psi(\pi(x)) \leq n \sup \varphi + 2 \sup |\chi|$ and $S_n\psi(\pi(x)) \geq n \inf \varphi - 2 \sup |\chi|$, which implies $\sup S_n\psi - \inf S_n\psi \leq n(\sup \varphi - \inf \varphi) + 4 \sup |\chi|$. If we return to the original F then $\psi(\pi(x)) = \varphi(x) - \chi(x) + \chi(F(x))$ holds for all $x \in [0,1]^2 \setminus \pi^{-1}\{c_0, c_1, \dots, c_N\}$, which is a set of measure one for every ergodic F-invariant probability measure with positive entropy. It remains to show that ψ is of bounded p-variation for some $p \geq 1$.

Choose $L \geq N$ such that $|F(x) - F(y)| \leq L|x - y|$ holds, if $\pi(x)$ and $\pi(y)$ are in (c_{j-1}, c_j) for some j. This is possible, since the partial derivatives of F are bounded. Let $\gamma \in (0,1)$ be the unique solution of $q = L^{1-\frac{1}{\gamma}}$ and fix β satisfying $\frac{1}{N} > \beta > L^{-\frac{1}{\gamma}}$. Set $K_0 = \{c_0, c_1, \dots, c_N\}$ and $K_j = T^{-j}(K_0)$ for $j \geq 1$. Because of $\frac{1}{N} > \beta$ and $\operatorname{card} K_j \leq N^{j+1} + 1$ for $j \geq 0$, we have $\sum_{j=0}^{\infty} \beta^j \operatorname{card} K_j < \infty$. For u and v in $[0,1]$ set $k(u,v) = \sum_{j=0}^{\infty} \beta^j \operatorname{card}(K_j \cap I(u,v))$, where $I(u,v)$ is the closed interval with endpoints u and v. We define a modified distance on the interval $[0,1]$ by $d(u,v) = |u - v| + k(u,v)$.

Fix u and v in $[0,1]$. Let $r \in \mathbb{Z}$ be such that $r < \max(0, -\gamma\frac{\log d(u,v)}{\log L}) \leq r + 1$. Then either $r = -1$ or we have $r < \frac{\log d(u,v)}{\log \beta}$ and hence $\beta^r > d(u,v) \geq k(u,v)$. This implies $K_j \cap I(u,v) = \emptyset$ for $0 \leq j \leq r$. Therefore $K_0 \cap T^j(I(u,v)) = \emptyset$ for $0 \leq j \leq r$, which gives $|\varphi(F^j(u,0)) - \varphi(F^j(v,0))| \leq a(L^j|u - v|)^\alpha$ for $0 \leq j \leq r$ and $|\varphi(F^{j-1}(T(u),0)) - \varphi(F^{j-1}(T(v),0))| \leq a(L^j|u - v|)^\alpha$ for $1 \leq j \leq r$. Since we have $|u - v| \leq d(u,v)$ and $L^r \leq d(u,v)^{-\gamma}$, if $r \geq 0$, and since the empty sum is zero, we get $\sum_{j=0}^{r} |\varphi(F^j(u,0)) - \varphi(F^j(v,0))| \leq \frac{aL^\alpha}{L^\alpha - 1} d(u,v)^{(1-\gamma)\alpha}$ and $\sum_{j=1}^{r} |\varphi(F^{j-1}(T(u),0)) - \varphi(F^{j-1}(T(v),0))| \leq \frac{aL^\alpha}{L^\alpha - 1} d(u,v)^{(1-\gamma)\alpha}$.

If $z \in [0,1]^2$ then $|\varphi(z)| \leq \frac{a}{q}$ and $|\varphi(F^{j-1}(z)) - \varphi(F^{j-1}(\tilde{\pi}(z)))| \leq aq^{(j-1)\alpha}$ for $j \geq 1$. With $z = F(u,0)$ we get $\sum_{j=r+1}^{\infty} |\varphi(F^j(u,0)) - \varphi(F^{j-1}(T(u),0))| \leq \frac{a}{1-q^\alpha} q^{r\alpha}$, where in the case of $r = -1$ the summand for $j = 0$ is only $|\varphi(u,0)|$. The same holds for v instead of u. Because of $q = L^{1-\frac{1}{\gamma}}$ and $r + 1 \geq -\gamma\frac{\log d(u,v)}{\log L}$ we get $q^r = \frac{1}{q} L^{(r+1)(1-\frac{1}{\gamma})} \leq \frac{1}{q} L^{(1-\gamma)\frac{\log d(u,v)}{\log L}} = \frac{1}{q} d(u,v)^{1-\gamma}$.

Putting these inequalities together, we get $|\psi(u) - \psi(v)| \leq bd(u,v)^{(1-\gamma)\alpha}$ with $b = \frac{aL^\alpha}{L^\alpha - 1} + \frac{2}{q^\alpha} \frac{a}{1-q^\alpha}$. This holds for any u and v in $[0,1]$. Set $p = \frac{1}{(1-\gamma)\alpha}$, which is ≥ 1. If we choose now arbitrary points $t_0 < t_1 < \ldots < t_n$ in $[0,1]$ we have

$$\begin{aligned}\sum_{i=1}^{n}|\psi(t_{i-1})-\psi(t_i)|^p &\leq b^p\sum_{i=1}^{n}d(t_{i-1},t_i)\\ &= b^p\sum_{i=1}^{n}|t_{i-1}-t_i| + b^p\sum_{i=1}^{n}k(t_{i-1},t_i)\\ &\leq b^p(1+2\sum_{j=1}^{\infty}\beta^j\operatorname{card}K_j)\end{aligned}$$

which shows that the p-variation of ψ is bounded by $b^p(1+2\sum_{j=1}^{\infty}\beta^j\operatorname{card}K_j)$. □

Now we can use one-dimensional techniques for equilibrium states. For a piecewise monotone transformation $T:[0,1]\to[0,1]$ and a measurable function $\psi:[0,1]\to\mathbb{R}$ we introduce the *transfer operator* P_ψ. It is defined on the vector space of all measurable functions $f:[0,1]\to\mathbb{R}$ by $P_\psi f(t)=\sum_{v\in T^{-1}(t)}e^{\psi(v)}f(v)$. For a measure m let $P_\psi^* m$ be the measure defined by $P_\psi^* m(A)=\int P_\psi 1_A\,dm$. Now we consider equilibrium states for the transformation T and a function $\psi:[0,1]\to\mathbb{R}$. The corresponding pressure is denoted by $p(T,\psi)$.

Proposition 3. *Suppose that $\psi:[0,1]\to\mathbb{R}$ is of bounded p-variation for some $p\geq 1$ and satisfies $\sup S_n\psi-\inf S_n\psi < nh_{\text{top}}(T)$ for some $n\geq 1$. Then there are $\lambda>0$ and a probability measure m on $[0,1]$, which has no atoms and satisfies $P_\psi^* m=\lambda m$. Furthermore we have $p(T,\psi)=\log\lambda$ and there is a bounded m-almost everywhere continuous function $h:[0,1]\to\mathbb{R}^+$ for which $P_\psi h=\lambda h$ holds m-almost everywhere. The measure $\varrho^+=hm$ has positive entropy and is an ergodic equilibrium state for the function ψ.*

Proof: Using a method of [26] it is shown in [13] that a probability measure m on $[0,1]$ and $\lambda>0$ exist satisfying $P_\psi^* m=\lambda m$. In this paper bounded variation is assumed for ψ, but the proof uses only that ψ has onesided limits, which holds also for functions of bounded p-variation. It is also shown there that $\lambda^{-n}\|e^{S_n\psi}\|_\infty < 1$ for some $n\geq 1$. This is done on an extended interval $[0,1]$, one gets by doubling countably many points. For all points x in this extended interval we have $P_\psi 1_{\{x\}}=e^{\psi(x)}1_{\{Tx\}}$ and hence also $m(\{x\})=\lambda^{-1}e^{\psi(x)}m(\{Tx\})$, which implies $m(\{x\})=\lambda^{-k}e^{S_k\psi(x)}m(\{T^kx\})\leq\lambda^{-k}e^{S_k\psi(x)}$ for all $k\geq 1$. Since we have $\lambda^{-n}\|e^{S_n\psi}\|_\infty<1$ for some $n\geq 1$, we get $\lim_{k\to\infty}\lambda^{-k}e^{S_k\psi(x)}=0$ and $m(\{x\})=0$ follows. In particular, the countably many doubled points have measure zero and m can be considered as a measure on the original interval $[0,1]$, which has no atoms.

The existence of a function $h : [0,1] \to \mathbb{R}^+$ satisfying $P_\psi h = \lambda h$ almost everywhere, such that $\varrho^+ = hm$ is an ergodic measure, is then shown in [17] using results from [12]. This function h is contained in a certain Banach space, whose definition implies that it is bounded and m-almost everywhere continuous. It is shown in [12], that ϱ^+ is an equilibrium state of ψ using Theorem 1 of [18], from which applied to $\frac{1}{\lambda}e^\psi \frac{h}{h\circ T}$ we also get $h_{\varrho^+}(T) = \log\lambda - \int \psi\, d\varrho^+ = \log\lambda - \int \frac{1}{n}S_n\psi\, d\varrho^+$ and hence $h_{\varrho^+}(T) > 0$. This also shows $p(T,\psi) = \log\lambda$. $\square$

Let ϱ^+ be an ergodic T-invariant probability measure on $[0,1]$ with positive entropy. We find an F-invariant measure ϱ on $[0,1]^2$, such that its image ϱ_π under π equals ϱ^+, as follows. Let $\mathcal{Z}$ be a partition of $[0,1]$, with respect to which T is piecewise monotone. Then $\varrho^+(R_{\mathcal{Z}}) = 1$ and $\mathcal{Z}$ is a generating partition since, by the assumption that T is topologically transitive, there cannot be a nondegenerate interval I such that $T^n|I$ is monotone for all $n \geq 1$. Set $\mathcal{Y} = \{Z \times [0,1] : Z \in \mathcal{Y}\}$. For $k,\ l \geq 0$ and $x \in X_{\mathcal{Y}}$ we have $F^{-l}(Y_{k,l}(x)) = \pi^{-1}(Z_{k+l}(\pi\circ F^{-l}(x)))$. We set $\varrho(Y_{k,l}(x)) = \varrho^+(Z_{k+l}(\pi\circ F^{-l}(x)))$. This defines a probability measure ϱ on $[0,1]^2$, which is F-invariant, since ϱ^+ is T-invariant, and satisfies $\varrho_\pi = \varrho^+$. By Lemma 1 we have $h_\varrho(F) = h_{\varrho^+}(T)$. For the shift spaces introduced in the last section this construction just means to extend an invariant measure from the onesided shift space Σ_T^+ to its natural extension Σ_F.

Let $\varphi : [0,1]^2 \to \mathbb{R}$ be as in Theorem 4. Let $\psi : [0,1] \to \mathbb{R}$ be the function found in Lemma 7. Since we assume $\sup\varphi - \inf\varphi < h_{\mathrm{top}}(F)$, there is $n \geq 1$ with $\sup S_n\psi - \inf S_n\psi < nh_{\mathrm{top}}(F) = nh_{\mathrm{top}}(T)$ by Lemma 7. Let ϱ^+ be the equilibrium state of ψ found in Proposition 3 and let ϱ be the measure constructed above. We have $\sup\varphi < h_{\mathrm{top}}(F) + \inf\varphi$ and hence there is an F-invariant probability measure ν satisfying $\sup\varphi < h_\nu(F) + \inf\varphi \leq h_\nu(F) + \int\varphi\, d\nu$. Therefore an F-invariant probability measure with zero entropy cannot be an equilibrium state of φ. For every ergodic F-invariant probability measure γ of positive entropy we have $\int\psi\, d\gamma_\pi = \int\psi\circ\pi\, d\gamma = \int\varphi\, d\gamma$ by Lemma 7 and $h_\gamma(F) = h_{\gamma_\pi}(T)$ by Lemma 1. Since ϱ^+ is an equilibrium state of ψ, this implies that ϱ is an equilibrium state of φ and that we have $p(F,\varphi) = p(T,\psi)$. The following lemma completes then the proof of Theorem 4. Notice that we assume $p(F,\varphi) = 0$ in Theorem 4, and hence λ in Proposition 3 equals one.

Lemma 8. *Suppose that μ is an ergodic F-invariant probability measure with positive entropy and that T is topologically transitive. Let ϱ be an F-invariant probability measure with $\varrho_\pi = hm$, where m is a probability measure, which has no atoms and satisfies $P_\psi^* m = m$ for a function $\psi : [0,1] \to \mathbb{R}$ of bounded p-variation, and where $h : [0,1] \to \mathbb{R}^+$ is a bounded m-almost everywhere continuous function, such that $P_\psi h = h$ holds m-almost everywhere. Then for every $\varepsilon > 0$ there is a finite partition $\mathcal{Z}$ of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, such that for any partition $\mathcal{Y} = \{U \times [0,1]$:*

$U \in \mathcal{U}\}$ *with* $\mathcal{U}$ *a finite partition of* $[0,1]$ *into open intervals refining* $\mathcal{Z}$*, we have* $\liminf_{k,l\to\infty} \frac{-\log \varrho(Y_{k,l}(x))}{k+l} \geq \int \psi \circ \pi \, d\mu - \varepsilon$ *and* $\limsup_{k,l\to\infty} \frac{-\log \varrho(Y_{k,l}(x))}{k+l} \leq \int \psi \circ \pi \, d\mu + \varepsilon$ *for* μ*-almost all* x.

Proof: Fix $\varepsilon > 0$. Let $\mathcal{Z}$ be a finite partition of $[0,1]$ into open intervals, with respect to which T is piecewise monotone, such that $\sup_Z \psi - \inf_Z \psi < \varepsilon$ holds for all $Z \in \mathcal{Z}$. This is possible, since $\psi : [0,1] \to \mathbb{R}$ is of bounded p-variation and has therefore onesided limits. Let $\mathcal{U}$ be a finite partition of $[0,1]$ into open intervals refining $\mathcal{Z}$ and set $\mathcal{Y} = \{U \times [0,1] : U \in \mathcal{U}\}$. For $n \geq 1$ and $t \in [0,1]$ let $U_n(t)$ be the unique element of $\mathcal{U}_n = \bigvee_{j=0}^{n} T^{-j}\mathcal{U}$ which contains t. It exists for μ_π-almost all t. For $k,\, l \geq 0$ and $x \in [0,1]^2$ let $Y_{k,l}(x)$ be the unique element of $\mathcal{Y}_{k,l} = \bigvee_{j=-l}^{k} F^{-j}\mathcal{Y}$ which contains x. It exists for μ-almost all x.

If M is a subset of $U_1(t)$ then $P_\psi 1_M \leq e^{\psi(t)+\varepsilon} 1_{T(M)}$, since $|\psi(v) - \psi(t)| < \varepsilon$ for all $v \in U_1(t)$ by the choice of $\mathcal{Z}$. This implies $m(M) \leq e^{\psi(t)+\varepsilon} m(T(M))$. Iterating this estimate, we get $m(U_n(t)) \leq e^{S_n\psi(t)+n\varepsilon} m(T^n U_n(t))$ for all $n \geq 1$. Similarly we get $m(U_n(t)) \geq e^{S_n\psi(t)-n\varepsilon} m(T^n U_n(t))$ for all $n \geq 1$.

By assumption there is a constant d with $h \leq d$. Therefore we get

$$\begin{aligned} \varrho(Y_{k,l}(x)) &= \varrho(F^{-l}(Y_{k,l}(x))) = \varrho(Y_{k+l,0}(F^{-l}(x))) \\ &= \varrho(\pi^{-1} U_{k+l}(\pi \circ F^{-l}(x))) = \varrho_\pi(U_{k+l}(\pi \circ F^{-l}(x))) \\ &\leq d m(U_{k+l}(\pi \circ F^{-l}(x))) \leq d e^{S_{k+l}\psi(\pi \circ F^{-l}(x)) + (k+l)\varepsilon} \end{aligned}$$

since $m(T^{k+l} U_{k+l}(\pi \circ F^{-l}(x))) \leq 1$. Furthermore, we have

$$\begin{aligned} S_{k+l}\psi(\pi \circ F^{-l}(x)) &= \sum_{j=0}^{k+l-1} \psi(T^j \circ \pi \circ F^{-l}(x)) \\ &= \sum_{j=0}^{k+l-1} \psi(\pi \circ F^{-l+j}(x)) = \sum_{j=-l}^{k-1} \psi \circ \pi(F^j(x)). \end{aligned}$$

For μ-almost all $x \in [0,1]^2$ we get $\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \psi \circ \pi(F^j(x)) = \int \psi \circ \pi \, d\mu$ and $\lim_{l\to\infty} \frac{1}{l} \sum_{j=-l}^{-1} \psi \circ \pi(F^j(x)) = \int \psi \circ \pi \, d\mu$ by the ergodic theorem applied to F and F^{-1} and hence also $\lim_{k,l\to\infty} \frac{1}{k+l} \sum_{j=-l}^{k-1} \psi \circ \pi(F^j(x)) = \int \psi \circ \pi \, d\mu$. Together with the above estimate this gives $\limsup_{k,l\to\infty} \frac{-\log \varrho(Y_{k,l}(x))}{k+l} \leq \int \psi \circ \pi \, d\mu + \varepsilon$ for μ-almost all x.

We get $\liminf_{k,l\to\infty} \frac{-\log \varrho(Y_{k,l}(x))}{k+l} \geq \int \psi \circ \pi \, d\mu - \varepsilon$ in the same way, if we show that $\lim_{k,l\to\infty} \frac{1}{k+l} \log m(T^{k+l} U_{k+l}(\pi \circ F^{-l}(x))) = 0$ holds for μ-almost all x and that there is a constant $c > 0$ such that $h(t) \geq c$ holds for m-almost all t.

By assumption, there is $t \in [0,1]$ such that h is continuous in t and $h(t) > 0$. Hence there is an open interval $I \subset [0,1]$ with $\inf_I h > 0$. By Lemma 1 in [10] there is k such that $\bigcup_{j=0}^{k} T^j(I)$ covers $[0,1]$ up to finitely many points. Observe that Lemma 1 in [10] applies to an extended interval with countably many points doubled. For the original interval $[0,1]$ this union covers $[0,1]$ only up to finitely many points. Using that $P_\psi h = h$ holds m-almost everywhere, for a subset J of $[0,1]$ we get that $\operatorname{essinf}_J h > 0$ implies $\operatorname{essinf}_{T(J)} h > 0$, since $P_\psi 1_J \geq 1_{T(J)} \inf e^\psi$. Choosing $c = \inf\{\operatorname{essinf}_{T^j(I)} h : 0 \leq j \leq k\}$ this implies that $h \geq c$ holds m-almost everywhere.

For a subset J of $[0,1]$ we get that $m(J) = 0$ implies $m(T(J)) = 0$, because we have $P_\psi 1_J \geq 1_{T(J)} \inf e^\psi$ and hence also $m(J) \geq m(T(J)) \inf e^\psi$. Again by this result from Lemma 1 in [10] quoted above we would get $m([0,1]) = 0$, if $m(I)$ would be 0 for some open interval I. This shows that m has support $[0,1]$. Furthermore, if Z is an interval, on which T is monotone, $P_\psi^* m = m$ implies that on Z the Radon-Nikodym-derivative $\frac{dm}{dm\circ T}$ equals e^ψ. This implies that $\frac{dm\circ T}{dm}$ equals $e^{-\psi}$, which means that $m(T(J)) = \int_J e^{-\psi}\, dm$ for all intervals J, on which T is monotone. Hence m is a socalled $e^{-\psi}$-conformal measure. It follows from (4) of Lemma 5 in [15] that $\lim_{k\to\infty} \frac{1}{k} \log m(T^k U_k(t)) = 0$ for μ_π-almost all $t \in [0,1]$, since with d_m defined as in [15] we have $d_m(T^k(t), T^k U_k(t)) \leq m(T^k U_k(t)) \leq 1$. This implies $\lim_{k\to\infty} \frac{1}{k} \log m(T^k U_k(\pi(x))) = 0$ for μ-almost all $x \in [0,1]^2$.

In order to show $\lim_{k,l\to\infty} \frac{1}{k+l} \log m(T^{k+l} U_{k+l}(\pi \circ F^{-l}(x))) = 0$ for μ-almost all x, we use the shift spaces Σ_T^+ and Σ_F introduced in Section 4. The subset of Σ_T^+ corresponding to $T^{k+l} U_{k+l}(\pi \circ F^{-l}(x))$ is $S^{k+l}([i_{-l} i_{-l+1} \dots i_k]^+)$ with $\dots i_{-1} i_0 i_1 \dots = \vartheta(x)$. Let C be as in Section 4. For $a = a_0 a_1 \dots \in C$ define $N_a \subset \Sigma_F$ by $N_a = \bigcap_{j=1}^\infty \bigcup_{i=j}^\infty S^{-i}([a_0 a_1 \dots a_{i-1}])$, if a is not periodic, and $N_a = \emptyset$, if a is periodic. Set $K_a = \{\dots i_{-1} i_0 i_1 \dots \in \Sigma_F : i_n i_{n+1} \dots = a \text{ for some } n \in \mathbb{Z}\}$. Let $\tilde\mu$ be defined as in the proof of Proposition 2. Since $\tilde\mu$ is ergodic and has positive entropy, we get $\tilde\mu(K_a) = 0$. It is shown in [6] that $\tilde\mu(N_a) = 0$. In [6] this is shown only for piecewise increasing transformations, the extension to the general case can be done by the method used in [7].

If $a \in C$ is not periodic and $\dots i_{-1} i_0 i_1 \dots \notin N_a \cup K_a$, by the definition of N_a there is l_a such that $i_{-l} i_{-l+1} \dots i_{-1}$ is not an initial segment of a for all $l \geq l_a$. By the definition of K_a we find then an $k_a \geq 0$ such that $i_{-l} i_{-l+1} \dots i_{k_a}$ is not an initial segment of a for all $l \geq 0$. If $a \in C$ is periodic and $\dots i_{-1} i_0 i_1 \dots \notin K_a$, then again there is $k_a \geq 0$ such that $i_{-l} i_{-l+1} \dots i_{k_a}$ is not an initial segment of a for all $l \geq 0$.

Set $L = \bigcup_{a\in C} (N_a \cup K_a)$. Then $\tilde\mu(L) = 0$ and for every $\dots i_{-1} i_0 i_1 \dots \in \Sigma_F \setminus L$ there is k_0 such that $i_{-l} i_{-l+1} \dots i_{k_0}$ is not an initial segment of any $a \in C$ for all $l \geq 0$. It follows then from (2.2) and (1.8) in [6] that $S^{k+l}([i_{-l} i_{-l+1} \dots i_k]^+) = S^k([i_0 i_1 \dots i_k]^+)$ for all $l \geq 0$ and all $k \geq k_0$ (in [6] it is shown only for piecewise increasing transformations, but can be extended to the general case by the method used in [7]). Set $M = \vartheta^{-1}(L)$. Then $\mu(M) = 0$ and for $x \notin M$ there is k_0 with $T^{k+l}U_{k+l}(\pi \circ F^{-l}(x)) = T^k U_k(\pi(x))$ for all $l \geq 0$ and all $k \geq k_0$. Since we have already shown that $\lim_{k\to\infty} \frac{1}{k} \log m(T^k U_k(\pi(x))) = 0$ holds for μ-almost all $x \in [0,1]^2$, this implies $\lim_{k,l\to\infty} \frac{1}{k+l} \log m(T^{k+l}U_{k+l}(\pi \circ F^{-l}(x))) = 0$ for μ-almost all $x \in [0,1]^2$. □

References

[1] Afraimovich V., *Pesins dimension for Poincaré recurrences*, Chaos **7** (1997), 12–20.

[2] Afraimovich V., Chazottes J.-R.,Saussol B., *Pointwise dimension for Poincaré recurrences associated with maps and special flows*, Discrete Contin. Dyn. Syst. A **9** (2003), 263–280.

[3] Billingsley P., *Ergodic Theory and Information*, Krieger, New York, 1978.

[4] Bowen R., *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, 470, Springer, Berlin-Heidelberg-New York, 1975.

[5] Cutler C., *The Hausdorff dimension distribution of finite measures in Euclidian space*, Canad. J. Math. **38** (1986), 1459–1484.

[6] Hofbauer F., *On intrinsic ergodicity of piecewise monotone transformations with positive entropy*, Israel J. Math. **34** (1979), 213–237.

[7] Hofbauer F., *On intrinsic ergodicity of piecewise monotone transformations with positive entropy II*, Israel J. Math. **38** (1981), 107–115.

[8] Hofbauer F., *Piecewise invertible dynamical systems*, Probab. Theory Relat. Fields **72** (1986), 359–386.

[9] Hofbauer F., *An inequality for the Lyapunov exponent of an ergodic invariant measure for a piecewise monotonic map on the interval*, Lyapunov Exponents (Oberwolfach, 1990), L. Arnold, H. Crauel, J.-P. Eckmann, Eds., Lecture Notes in Mathematics, 1486, Springer, Berlin, 1991, pp. 227–231.

[10] Hofbauer F., *Hausdorff dimension and pressure for piecewise monotonic maps of the interval*, J. London Math. Soc. **47** (1993), 142–156.

[11] Hofbauer F., *The recurrence dimension for piecewise monotonic maps of the interval*, Ann. Scuola Norm. Super. Pisa Cl. Sci. **4** (2005), 439–449.

[12] Hofbauer F., Keller G., *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z. **180** (1982), 119–140.

[13] Hofbauer F., Keller G., *Equilibrium states for piecewise monotonic transformations*, Ergodic Theory Dynam. Systems **2** (1982), 23–43.

[14] Hofbauer F., Raith P., *The Hausdorff dimension of an ergodic invariant measure for a piecewise monotonic map of the interval*, Canad. Math. Bull. **35** (1992), 84–98.

[15] Hofbauer F., Raith P., Steinberger T., *Multifractal dimensions for invariant subsets of piecewise monotonic interval maps*, Fund. Math. **176** (2003), 209–232.

[16] Hofbauer F., Urbański M., *Fractal properties of invariant subsets for piecewise monotonic maps of the interval*, Trans. Amer. Math. Soc. **343** (1994), 659–673.

[17] Keller G., *Extended bounded variation and application to piecewise monotonic transformations*, Probab. Theory Relat. Fields **69** (1985), 461–478.

[18] Ledrappier F., *Principe variationnel et systemes dynamiques symboliques*, Probab. Theory Relat. Fields **30** (1974), 185–202.

[19] Mattila P., *Geometry of Sets and Measures in Euclidean space*, Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995.

[20] Olsen L., *A multifractal formalism*, Adv. Math. **116** (1995), 82–196.

[21] Pesin Ya., *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, 1997.
[22] Rychlik M., *Bounded variation and invariant measures*, Studia Math. **76** (1983), 69–80.
[23] Saussol B., Troubetzkoy S., Vaienti S., *Recurrence, dimensions and Lyapunov exponents*, J. Statist. Phys. **106** (2002), 623–634.
[24] Saussol B., Troubetzkoy S., Vaienti S., *Recurrence and Lyapunov exponents*, Moscow Math. J. **3** (2003), 189–203.
[25] Steinberger T., *Local dimension of ergodic measures for two-dimensional Lorenz transformations*, Ergodic Theory Dynam. Systems **20** (2000), 911–923.
[26] Walters P., *Equilibrium states for β-transformations and related transformations*, Math. Z. **159** (1978), 65–88.
[27] Walters P., *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, 79, Springer, New York, 1982.
[28] Young L.-S., *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynam. Systems **2** (1982), 109–124.

Institut für Mathematik, Universität Wien, Nordbergstrasse 15, A 1090 Wien, Austria

E-mail: franz.hofbauer@univie.ac.at

(*Received* October 2, 2007, *revised* January 5, 2009)