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Lattices of Scott-closed sets

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Abstract. A dcpo P is continuous if and only if the lattice $C(P)$ of all Scott-closed subsets of P is completely distributive. However, in the case where P is a non-continuous dcpo, little is known about the order structure of $C(P)$. In this paper, we study the order-theoretic properties of $C(P)$ for general dcpo's P. The main results are: (i) every $C(P)$ is C-continuous; (ii) a complete lattice L is isomorphic to $C(P)$ for a complete semilattice P if and only if L is weak-stably C-algebraic; (iii) for any two complete semilattices P and Q, P and Q are isomorphic if and only if $C(P)$ and $C(Q)$ are isomorphic. In addition, we extend the function $P \mapsto C(P)$ to a left adjoint functor from the category **DCPO** of dcpo's to the category **CPAlg** of C-prealgebraic lattices.

Keywords: domain, complete semilattice, Scott-closed set, C-continuous lattice, C-algebraic lattice

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1. Introduction

Since the 1980s, the relation between domains and their complete lattices of Scott-open sets has been extensively studied. A basic but important result is a Characterization Theorem for domains which states that L is a domain if and only if the lattice of Scott-open sets of L (denoted by $\sigma(L)$) is completely distributive (see Theorem II-1.14 of [7]). The subsequent discovery of the connection between domains and logic based on the works of Abramsky [1], Vickers [24] and Smyth [22] provided the mathematical justification for a more general investigation of the domain-theoretic properties of the open set lattice $\mathcal{O}(X)$ for a topological space X.

For closed set lattices of topological spaces, a fundamental result, due to S. Papert, is that a complete lattice L is isomorphic to the lattice of closed subsets of a topological space X if and only if the co-primes of L are join-dense in L (see [18]). For the special case of the Scott topology on a dcpo P, most of what is known about the order structure of the lattice of Scott-closed subsets of P (denoted by $C(P)$ in this paper) is restricted by the assumption that P is continuous (or a domain). More precisely, Hoffmann [10] and Lawson [15] independently proved that a complete lattice L is isomorphic to $\sigma(P)$ for a continuous dcpo P if and only if L is a completely distributive lattice. Since a lattice L is completely distributive if and only if its dual L^{op} is, it follows that the Scott-closed set lattices of continuous dcpo's are, up to isomorphism, exactly the completely distributive lattices. In categorical terms, the function $(P \mapsto C(P))$ can be extended to a functor C (called the *Scott functor*) which establishes an equivalence between **DOM**, the category of continuous dcpo's and **CDL**, the category of completely distributive lattices. The key property of continuous dcpo's used to establish this equivalence is that P can be recovered, up to isomorphism, as the set of all co-primes of the lattice $C(P)$. However, for a *non-continuous* dcpo P, little is known about the order-theoretic properties of the complete lattice $C(P)$.

Besides the mathematical systematics we have considered so far, our investigation of the order-theoretic properties of lattices of Scott-closed subsets is motivated by another seemingly unrelated problem. We explain this by first recalling a definition from [13]. Let $\mathcal{C}$ be a category and $\mathbf{E}$ be a collection of morphisms in $\mathcal{C}$. An object A of $\mathcal{C}$ is said to be $\mathbf{E}$-*projective* if for any $\mathbf{E}$-morphism $r : B \to C$ and any $f : A \to C$ in $\mathcal{C}$ there is at least one morphism $\overline{f} : A \to B$ with $r \circ \overline{f} = f$. In the special case where $\mathbf{E}$ is the collection of all epimorphisms, the $\mathbf{E}$-projective objects are precisely the projective objects of $\mathcal{C}$. Usually the problem of characterizing the $\mathbf{E}$-projectives can be made more manageable if one restricts to a more specific collection $\mathbf{E}$. In the presence of categorical adjunctions, the following restriction is usually considered. Let $F \dashv G$ be an adjunction between the categories $\mathcal{D}$ and $\mathcal{C}$. Consider the collection $\mathbf{E}$ of all $\mathcal{C}$-morphisms $f : A \to B$ such that Gf has a section, i.e., a right inverse in $\mathcal{D}$.

In [26], the second author solved the problem of characterizing the $\mathbf{E}$-projective objects for certain categorical adjunctions. When applied to the adjunction[1] between **SLat**, the category of meet-semilattices (with a top) and **Frm**, the category of frames, his result implies that the $\mathbf{E}$-projective frames are precisely the stably continuous frames. In order to establish this, it is crucial to prove that the frames of ideals are, up to isomorphism, exactly the stably continuous ones.

One natural question is whether one can characterize the $\mathbf{E}$-projective frames for the well-known adjunction between **PreFrm**, the category of preframes and **Frm** (see [3] and [8]). Recall that in this adjunction, the left adjoint is the functor which sends a preframe M to $C(M)$ and the right adjoint is the forgetful functor. Almost inevitably, one ends up with the problem of characterizing the Scott-closed subset lattices of preframes. Additionally, Scott-closed set lattices have also been employed by Mislove to construct the local dcpo-completion of posets (see [17]).

In summary, we do have sufficient motivation for investigating the general order-theoretic properties of the lattices of Scott-closed sets. In this paper, we report the following main results.

(1) All lattices of Scott-closed sets enjoy a property (called C-continuity) analogous to the well studied notion of continuity.
(2) A complete lattice is isomorphic to $C(P)$ for a complete semilattice P iff it is weakly C-stable and C-algebraic.
(3) If P and Q are complete semilattices such that $C(P)$ and $C(Q)$ are isomorphic, then P and Q are isomorphic.

[1] The left adjoint is the ideal functor Id and the right the forgetful functor.

The layout of the paper is as follows: In Section 2, we give some preliminaries on continuous dcpo's and completely distributive lattices. In Section 3, we introduce and study the notion of C-continuous posets. Section 4 is devoted to the study of general properties of the Scott-closed set lattices. In Section 5, we prove that if P is a complete semilattice, then P can be recovered from $C(P)$ as the set of C-compact elements. In the last section, we extend the Scott functor to establish an adjunction between **DCPO**, the category of all dcpo's, and **CAlg**, the category of all C-algebraic lattices. This adjunction restricts to three familiar categorical equivalences, including the classical Lawson-Hoffmann equivalence. In what follows we assume that the reader is familiar with domains, categories and functors. For an introduction to order theory, one may consult [4] and for an excellent treatment of domain theory, one may consult [2] and [7]. For category theory, the book [16] is a comprehensive text.

2. Preliminaries

A partially ordered set will be called a poset. In this paper, we shall use $\sqsubseteq$ to denote the order relation, and $\bigsqcup E$ and $\bigsqcap E$ the supremum and infimum of a subset E, respectively. A nonempty subset D of a poset is said to be *directed* if any two elements in D have an upper bound in D. A poset P is a *dcpo* (short for directed complete poset) if every directed subset of P has a supremum in P. Since the empty set is not directed by definition, a dcpo may fail to have a least element (called the *bottom*). A dcpo with bottom 0 is said to be *pointed*.

For any subset A of a poset P, the subset $\uparrow A$ is defined by

$$\uparrow A = \{x \in P \mid \exists y \in A \,.\, y \sqsubseteq x\}.$$

A subset A of a poset P is said to be *upper* if $A = \uparrow A$. The *lower* subsets are defined dually.

A subset U of a poset P is called *Scott-open* if (i) U is upper, and (ii) for any directed set D in P, $\bigsqcup D \in U$ implies $U \cap D \neq \emptyset$ whenever $\bigsqcup D$ exists. The set of all Scott-open sets of P forms a topology (called the *Scott topology*) on P, denoted by $\sigma(P)$.

The complements of Scott-open sets are the *Scott-closed* sets. We use $C(P)$ to denote the set of all Scott-closed sets of P. Thus a subset $F \subseteq P$ is Scott closed if and only if (i′) F is lower, and (ii′) for any directed subset $D \subseteq F$, if $\bigsqcup D$ exists then $\bigsqcup D \in F$. Both $\sigma(P)$ and $C(P)$ are complete, distributive lattices with respect to the inclusion relation.

Let P be a poset. The *way-below* relation $\ll$ on P is defined by $a \ll b$ for $a, b \in P$ if for any directed set D for which $\bigsqcup D$ exists, $b \sqsubseteq \bigsqcup D$ implies $a \sqsubseteq d$ for some $d \in D$.

A poset P is said to be *continuous* if for any $a \in P$, the set $\twoheaddownarrow a := \{x \in P \mid x \ll a\}$ is directed and $a = \bigsqcup \twoheaddownarrow a$. A continuous dcpo is also called a *domain*. A continuous poset which is also a complete lattice is called a *continuous lattice*.

Note that for any directed set D of a dcpo P, $\Downarrow\bigsqcup D = \bigcup\{\Downarrow d \mid d \in D\}$ and this set is again directed if each of the sets $\Downarrow d$ is directed.

Another binary relation on a complete lattice L often considered is $\lhd$, which is defined as follows: $y \lhd a$ iff for any subset $X \subseteq L$, $a \sqsubseteq \bigsqcup X$ implies $y \sqsubseteq x$ for some $x \in X$.

A complete lattice L is called *completely distributive* iff for any family $\{x_{j,k} \mid j \in J, k \in K(j)\}$ in L the equation

$$\bigsqcap_{j\in J} \bigsqcup_{k\in K(j)} x_{j,k} = \bigsqcup_{f\in F} \bigsqcap_{j\in J} x_{j,f(j)}$$

holds where F is the set of choice functions f choosing for each index j of J some index $f(j)$ in $K(j)$. It is well-known (see [19]) that a complete lattice L is completely distributive if and only if for each $a \in L$, it holds that $a = \bigsqcup\{x \in L \mid x \lhd a\}$.

3. C-continuous posets

Given a poset P, we now define a new auxiliary relation on P which is crucial for us to formulate the properties of lattices of Scott-closed sets.

Definition 3.1. Let P be a poset and $x, y \in P$. We say that x is *beneath* y, denoted by $x \prec y$, if for every nonempty Scott-closed set $C \subseteq P$ for which $\bigsqcup C$ exists, the relation $\bigsqcup C \sqsupseteq y$ always implies that $x \in C$.

The following example reveals that $\prec$ and $\ll$ can be quite different.

Example 3.2. (1) Let $I = [0, 1]$ be the unit interval with the ordinary order $\sqsubseteq$. It is easy to see that $x \prec y$ if and only if $x \sqsubseteq y$. In particular, $x \prec x$ for all $x \in [0, 1]$. However, the only element y of I satisfying the relation $y \ll y$ is 0. Hence $x \prec y$ does not imply $x \ll y$.

(2) Consider the lattice $\mathbf{M_3} = \{0, a, b, c, 1\}$, where the order is defined by $0 < a, b, c < 1$. Since $\mathbf{M_3}$ is a finite lattice, it follows that for any two elements x and y in $\mathbf{M_3}$, $x \ll y$ if and only if $x \sqsubseteq y$. In particular, $a \ll a$ holds. Notice that the set $C = \{0, b, c\}$ is Scott-closed with 1 as its supremum. $\bigsqcup C = 1 \sqsupseteq a$ but $a \notin C$. So, $a \not\prec a$. Hence $a \ll a$ does not imply $a \prec a$.

However, it is trivial that in a complete lattice $x \lhd y$ always implies $x \prec y$.

Now it is routine to verify the following properties of the relation $\prec$.

Proposition 3.3. *Let P be a poset and $u, v, x, y \in P$. Then the following statements hold:*

(i) *$x \prec y$ implies $x \sqsubseteq y$;*
(ii) *$u \sqsubseteq x \prec y \sqsubseteq v$ implies $u \prec v$; and*
(iii) *if P is pointed, then $0 \prec x$ always holds.*

Proposition 3.4. *Let P be a poset and D a directed subset of P such that $\bigsqcup D$ exists. If $d \prec x$ for all $d \in D$, then*

$$\bigsqcup D \prec x.$$

Proof: Let $F \in C(P)$ be non-empty such that $\bigsqcup F$ exists with $\bigsqcup F \sqsupseteq x$. Since $d \prec x$ for all $d \in D$, it follows that $D \subseteq F$. Because F is Scott-closed and D is directed, we have $\bigsqcup D \in F$. Thus $\bigsqcup D \prec x$. □

Notice that neither the relation $\lhd$ nor $\ll$ enjoys such a property.

Propositions 3.3 and 3.4 together imply the following corollary.

Corollary 3.5. *For any element a of a poset P, the set*

$$\{x \in P \mid x \prec a\}$$

is a Scott-closed subset of P.

With the relation $\prec$, we now define a new class of posets.

Definition 3.6. A poset P is said to be *C-continuous* if it satisfies the following approximation axiom: For each $a \in P$,

$$a = \bigsqcup \{x \in P : x \prec a\}.$$

A C-continuous poset which is also a complete lattice is called a *C-continuous lattice*.

Remark 3.7. Notice that one of the requirements of a domain P is that for every $x \in P$ the set $\{p \in P : p \ll x\}$ is directed. In contrast, for any poset Q, the set $\{q \in Q : q \prec a\}$ is automatically Scott-closed for any $a \in Q$ by virtue of Corollary 3.5.

The following proposition has a proof that is similar to that for continuous lattices (see Theorem I-1.10 of [7]).

Proposition 3.8. *For a complete lattice L, the following are equivalent:*

(i) *L is C-continuous;*

(ii) *for each $a \in L$, there is a smallest non-empty Scott-closed set C such that $\bigsqcup C \sqsupseteq a$;*

(iii) *for any collection $\{F_i : i \in I\}$ of Scott-closed subsets of L, the following equation holds:*

$$\bigsqcap_{i \in I} \bigsqcup F_i = \bigsqcup \bigcap_{i \in I} F_i.$$

Example 3.9. (1) Since for any two elements a and b, $a \lhd b$ implies $a \prec b$, it is immediate that every completely distributive lattice is C-continuous.

(2) The finite lattice $\mathbf{M}_3$ in Example 3.2(2) is continuous but not C-continuous.

An example of a C-continuous lattice which is not continuous is given in a later section (see Example 4.10). Meanwhile, the following result implies that every C-continuous lattice is distributive.

Proposition 3.10. *Let L be a C-continuous lattice. Then for any collection $\{F_i : i \in I\}$ of finite subsets of L the following equation holds:*

$$\bigsqcap_{i\in I} \bigsqcup F_i = \bigsqcup_{f\in\Pi_{i\in I} F_i} \bigsqcap_{i\in I} f(i).$$

In particular, every C-continuous lattice is distributive.

Proof: Denote the left hand side (respectively, the right hand side) of the equation by a (respectively, b). It suffices to prove that $a \sqsubseteq b$. Let $a = \bigsqcap_{i\in I} \bigsqcup F_i$ and $x \prec a$. For each $i \in I$, the set $\downarrow F_i$ is a Scott-closed set and $x \prec a \sqsubseteq \bigsqcup F_i$, so there is a $d_i \in F_i$ with $x \sqsubseteq d_i$. Let $f \in \Pi_{i\in I} F_i$ be defined by $f(i) = d_i, i \in I$. Then $x \sqsubseteq \bigsqcap_{i\in I} f(i) \sqsubseteq b$. But L is C-continuous so that $a = \bigsqcup\{x \in L : x \prec a\}$ and thus $a \sqsubseteq b$. □

Theorem 3.11. *Let L be a complete lattice. Then the following are equivalent:*

(i) *L is C-continuous and continuous;*
(ii) *L is completely distributive.*

Proof: It suffices to show that (i) implies (ii). Assume that L is both C-continuous and continuous. Since L is continuous, for each $a \in L$, $a = \bigsqcup\{x \in L : x \ll a\}$. Now for each $x \ll a$, $x = \bigsqcup\{y \in L : y \prec x\}$. It follows that

$$a = \bigsqcup\{y \in L : \exists\, x\, .\, y \prec x \ll a\}.$$

Next, suppose $y \prec x \ll a$, we shall show that $y \lhd a$. Let $X \subseteq L$ with $\bigsqcup X \sqsupseteq a$. Construct the set $E = \{\bigsqcup A : A \text{ is a finite subset of } X\}$. Then E is a directed set and $\bigsqcup E = \bigsqcup X \sqsupseteq a$. Since $x \ll a$, there is a finite subset $A \subseteq X$ such that $x \sqsubseteq \bigsqcup A = \bigsqcup \downarrow A$. Note that the last set $\downarrow A$ is Scott-closed. So it follows from $y \prec x$ that $y \sqsubseteq d$ for some $d \in A \subseteq X$. This implies that $y \lhd a$. Hence L is completely distributive. □

Before proceeding to the next section, let us have one more example.

Example 3.12. Consider the unit interval $[0, 1]$ with its usual Hausdorff topology. Denote by $\mathcal{O}([0, 1])$ the lattice of all such open sets of $[0, 1]$. It is well-known that $\mathcal{O}([0, 1])$ is continuous (since $[0, 1]$ is locally compact) and distributive but not completely distributive. So, by Theorem 3.11, $\mathcal{O}([0, 1])$ cannot be C-continuous.

4. Order properties of lattices of Scott-closed sets

In this section, we reveal some order-theoretic properties of the lattice $(C(P), \subseteq)$ for an arbitrary poset P. In particular, we prove that every lattice of the form $C(P)$ is C-continuous.

The following proposition says that the subset system defined by $C(P)$ for each dcpo P is *union-complete* in the sense of [23].

Proposition 4.1. *Let P be a poset and $\mathcal{C} \in C(C(P))$. Then $\bigsqcup_{C(P)} \mathcal{C} = \bigcup \mathcal{C}$.*

Proof: Note that each member of $\mathcal{C}$ is a Scott-closed subset of P. So to prove the equation, it suffices to show that

$$\bigcup \mathcal{C} \in C(P).$$

Obviously $\bigcup \mathcal{C}$ is a lower subset of P. Now let D be any directed subset of P contained in $\bigcup \mathcal{C}$ such that $\bigsqcup D$ exists in P. We want to prove that $\bigsqcup D \in C$ for some $C \in \mathcal{C}$. First note that $\mathcal{D} = \{\downarrow d : d \in D\}$ is a directed subset of $C(P)$. Moreover, $\mathcal{D} \subseteq \mathcal{C}$ because $\mathcal{C}$ is lower in $C(C(P))$. Since $\mathcal{C}$ is a Scott-closed set of $C(P)$, so $\bigsqcup_{C(P)} \mathcal{D} \in \mathcal{C}$. But $\bigsqcup_{C(P)} \mathcal{D}$ is precisely $\downarrow \bigsqcup D$. Hence $\bigsqcup D \in C$ for some $C \in \mathcal{C}$. □

Definition 4.2. An element x of a poset P is called *C-compact* if $x \prec x$. We use $\kappa(P)$ to denote the set of all the C-compact elements of P.

Recall that an element $r \neq 0$ of a lattice L is called *co-prime* if for any $x, y \in L$, $r \sqsubseteq x \sqcup y$ implies $r \sqsubseteq x$ or $r \sqsubseteq y$.

Proposition 4.3. *Let L be a lattice.*

(i) *If $r \in \kappa(L)$, then r is co-prime.*
(ii) *If L is a completely distributive lattice, then every co-prime element is C-compact.*
(iii) *If L is a complete lattice and $\kappa(L) \neq \emptyset$, then $\kappa(L)$ is a pointed dcpo with respect to the order inherited from L.*

Proof: (i) Suppose r is C-compact and $r \sqsubseteq x \sqcup y$. Let $D = \downarrow \{x, y\}$. Then D is Scott-closed and $\bigsqcup D = x \sqcup y$. Hence $r \in D$, so $r \sqsubseteq x$ or $r \sqsubseteq y$. Thus r is co-prime.

(ii) Let L be a completely distributive lattice and $r \in L$ be co-prime. Note that the set $\beta(r) = \{x \in L : x \lhd r\}$ is a directed set and $\bigsqcup \beta(r) = r$ (cf. [25]). Since $x \lhd r$ implies $x \prec r$, it then follows from Corollary 3.5 that $\bigsqcup \beta(r) \prec r$, i.e., $r \prec r$. Hence r is C-compact.

(iii) Let D be a directed subset in $\kappa(L)$. It suffices to show that $\bigsqcup D \prec \bigsqcup D$. So let $E \in C(L)$ with $\bigsqcup D \sqsubseteq \bigsqcup E$. Thus $d \sqsubseteq \bigsqcup E$ for all $d \in D$. Since $D \subseteq \kappa(L)$, it follows that $d \prec d$ for all $d \in D$ and so $D \subseteq E$. Because E is a Scott-closed subset of L, this implies that $\bigsqcup D \in E$ and so $\bigsqcup D \prec \bigsqcup D$, i.e., $\bigsqcup D \in \kappa(L)$. Also, $0 \prec 0$ implies that $0 \in \kappa(L)$. Hence $\kappa(L)$ is a pointed dcpo with respect to the order inherited from L. □

From the proof of (iii) it is seen that for a complete lattice L, $\kappa(L)$ is a sub-dcpo of L, i.e. it is closed under the supremum of directed sets.

Example 4.4. (1) Let $F([0,1])$ be the lattice of all closed subsets (with respect to the Euclidean metric) of the unit interval $[0,1]$. Then each singleton is co-prime, but not C-compact. For instance, let $A = \{\frac{1}{2}\}$ and $\mathcal{C} = \{\{x\} : x \neq \frac{1}{2}\} \cup \{\emptyset\}$. Then obviously $\mathcal{C}$ is a Scott-closed set of $F([0,1])$ and $\bigsqcup \mathcal{C} = [0,1] \supseteq A$, but $A \notin \mathcal{C}$.

From the above proof, one sees easily that if X is a T_1 space, then for any $x \in X$, $\{x\} \prec_{F(X)} \{x\}$ in the lattice $F(x)$ of all closed subsets of X iff $\{x\}$ is isolated.

(2) Recall that an element a of a dcpo is called *compact* if $a \ll a$. In a finite lattice, every element is compact. Thus it is easy to construct a compact element that is not C-compact.

Proposition 4.5. *Let P be a poset and X be a non-empty Scott-closed subset of P. Then for each $x \in X$, $\downarrow x \prec X$ holds in $C(P)$.*

Proof: Let $x \in X$. Suppose $\mathcal{C} \in C(C(P))$ with $\bigsqcup_{C(P)} \mathcal{C} \supseteq X$. Then, by Proposition 4.1, $X \subseteq \bigcup \mathcal{C}$. Hence there exists $C \in \mathcal{C}$ such that $x \in C$. So $\downarrow x \subseteq C$, and thus $\downarrow x \in \mathcal{C}$. □

Corollary 4.6. *Let P be a poset. Then for each $x \in P$, it holds that $\downarrow x \in \kappa(C(P))$.*

Definition 4.7. A poset P is said to be *C-prealgebraic* if for each $a \in P$,

$$a = \bigsqcup \{x \in \kappa(P) : x \sqsubseteq a\}.$$

A C-prealgebraic poset P is *C-algebraic* if for any $a \in P$,

$$\downarrow \{x \in \kappa(P) : x \sqsubseteq a\} \in C(P).$$

Obviously every C-prealgebraic poset is C-continuous. Again, we call a C-(pre)algebraic poset which is also a complete lattice a *C-(pre)algebraic lattice*.

Proposition 4.8. *For any poset P, the lattice $C(P)$ is C-prealgebraic.*

Proof: This follows from Corollary 4.5 and the fact that $F = \bigsqcup_{C(P)} \{\downarrow x : x \in F\}$ holds for every $F \in C(P)$. □

It is well-known that a poset P is continuous if and only if $C(P)$ is completely distributive ([6]). From Theorem 3.11, we obtain the following:

Corollary 4.9. *For any poset P, the following statements are equivalent:*

(i) *P is a continuous poset;*
(ii) *$C(P)$ is a continuous lattice;*
(iii) *$C(P)$ is a completely distributive lattice.*

Note that in Theorem II-1.14 of [7], an equivalence condition for P to be continuous is that both $C(P)$ and $\sigma(P)$ are continuous. Our result here shows that $\sigma(P)$ being continuous is surplus.

Example 4.10. Take a non-continuous dcpo P. Since P is not continuous, $C(P)$ cannot be completely distributive. But $C(P)$ is C-continuous, so by Corollary 4.9, $C(P)$ cannot be continuous.

5. Scott-closed set lattices of complete semilattices

Recall that a complete semilattice is a dcpo in which every nonempty subset has an infimum. It is well known that a dcpo is a complete semilattice if and only if every subset that is bounded above has a supremum (see [7]).

At this moment of time, we are still unable to give a complete characterization of the Scott-closed set lattice $C(P)$ for an arbitrary dcpo P. However we can do this for a complete semilattice P.

Lemma 5.1. *Let P be a complete semilattice. For any $X \in C(P)$, the set $\mathcal{C}_X := \{F \in C(P) : F \subseteq \downarrow x$ for some $x \in X\}$ is a Scott-closed subset of $C(P)$.*

Proof: Since $\mathcal{C}_X$ is clearly a lower subset of $C(P)$, it remains to show that for any directed subset $\mathcal{E}$ of $\mathcal{C}_X$, $\bigsqcup_{C(P)} \mathcal{E} \in \mathcal{C}_X$. Now for each Scott-closed set $E \in \mathcal{E}$, $E \subseteq \downarrow x$ for some $x \in X$ and, since P is a complete semilattice, $\bigsqcup E$ exists. Clearly the set $\{\bigsqcup E : E \in \mathcal{E}\}$ is a directed subset of P. Let $e = \bigsqcup\{\bigsqcup E : E \in \mathcal{E}\}$. In addition, as X is a lower set in P, it holds that $\bigsqcup E \in X$ for each $E \in \mathcal{E}$. So $e = \bigsqcup\{\bigsqcup E : E \in \mathcal{E}\} \in X$. We claim that $\bigsqcup_{C(P)} \mathcal{E} \subseteq \downarrow e$. From the way e is defined, it is clear that $e \sqsupseteq \bigsqcup Y$ for any $Y \in \mathcal{E}$. This implies that $Y \subseteq \downarrow \bigsqcup Y \subseteq \downarrow e$ for each $Y \in \mathcal{E}$. Consequently $\bigcup \mathcal{E} \subseteq \downarrow e$. But $\bigsqcup_{C(P)} \mathcal{E}$ is the smallest Scott-closed set of $C(P)$ containing $\bigcup \mathcal{E}$, it follows that $\bigsqcup_{C(P)} \mathcal{E} \subseteq \downarrow e$. Since $e \in X$, it follows that $\downarrow e \in \mathcal{C}_X$. Consequently because C_X is a lower set with respect to the inclusion order, it must be that $\bigsqcup_{C(P)} \mathcal{E} \in \mathcal{C}_X$. □

Theorem 5.2. *Let P be a complete semilattice. Then $X \in \kappa(C(P))$ if and only if X is a principal ideal, i.e. $X = \downarrow a$ for some $a \in P$.*

Proof: By Corollary 4.6, it suffices to prove the "only if" part. Suppose $X \in \kappa(C(P))$. By Lemma 5.1, $\mathcal{C}_X = \{F \in C(P) : F \subseteq \downarrow x$ for some $x \in X\}$ is a Scott-closed subset of $C(P)$. It is clear that $\bigsqcup_{C(P)} \mathcal{C}_X = X$. Since $X \prec X$, it follows that $X \in \mathcal{C}_X$. Thus $X \subseteq \downarrow x$ for some $x \in X$. But this means that $X = \downarrow x$. □

Remark 5.3. (1) A closed set F of a topological space X is said to be *irreducible* if it is a co-prime element of the lattice of all closed subsets of X. Recall that a topological space X is called a *sober* space if every irreducible closed set is the closure of a unique singleton set ([7]). For an arbitrary dcpo P, the Scott space $\Sigma P = (P, \sigma(P))$ need not be sober ([12]). Even if P is a complete lattice, ΣP may not be sober ([11]). Indeed, for the complete lattice P constructed by Isbell in [11] there exists an irreducible Scott-closed set F such that $F \neq \downarrow x$ for any $x \in L$. However, by Theorem 5.2, the C-compact elements of $C(L)$ are exactly the principal ideals of L. Hence the above-mentioned F is not C-compact.

(2) We call a topological space X *pre-sober* if for every C-compact closed set C of X, there is a unique element x such that $C = \{x\}^-$, the closure of $\{x\}$. By Proposition 4.3(i), every C-compact element in $C(P)$ is co-prime and thus every sober space is pre-sober. By Theorem 5.2, for any complete lattice L, the Scott space ΣL is pre-sober. So the example constructed by Isbell in [11] yields a pre-sober space which is not sober.

(3) It is then natural to ask: Is the Scott space ΣP of every dcpo pre-sober? The answer is negative. One such counterexample is the one constructed by Johnstone in [12]. We now explain this. Let $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, with the partial order defined by

$$(m, n) \sqsubseteq (m', n') \iff \text{either } m = m' \text{ and } n \sqsubseteq n' \\ \text{or } n' = \infty \textit{ and } n \sqsubseteq m'.$$

Then $(X, \sqsubseteq)$ is a dcpo and there is no $x \in X$ such that $X = \{x\}^- = \downarrow x$. We now show that the set X is C-compact. Suppose that there is a collection $\mathcal{F} = \{F_i : i \in I\}$ of Scott-closed subsets F_i's of X such that $\mathcal{F}$ itself is a Scott-closed subset of $C(X)$, and $\bigsqcup_{C(X)} \mathcal{F} = X$, which means $\bigcup \mathcal{F} = X$ because of Proposition 4.1. For each $k \in \mathbb{N}$, let $E_k = \{(m, n) : m \in \mathbb{N}, n \sqsubseteq k\}$, which is clearly a Scott-closed subset of X. For each k, there exists F_i such that $(k, \infty) \in F_i$, hence $E_k \subseteq \downarrow (k, \infty) \subseteq F_i$, which implies that $E_k \in \mathcal{F}$. Clearly $\{E_k : k \in \mathbb{N}\}$ is a directed set. Now as $\mathcal{F} \in C(C(X))$, we have $\bigsqcup_{C(X)} \{E_k : k \in \mathbb{N}\} \in \mathcal{F}$. But trivially $\bigsqcup_{C(X)} \{E_k : k \in \mathbb{N}\} = X$, so $X \in \mathcal{F}$. Hence X is C-compact.

By Proposition 4.8, Theorem 5.2 and Lemma 5.1, we obtain the following.

Corollary 5.4. *For any complete semilattice P, the lattice $C(P)$ is C-algebraic.*

Since for every completely distributive lattice L there is a continuous dcpo P such that L is isomorphic to $C(P)$ ([15]), we have:

Corollary 5.5. *Every completely distributive lattice is C-prealgebraic.*

Theorem 5.2 reveals that we can recover P, up to isomorphism, from $C(P)$ as the set of all C-compact elements if P is a complete semilattice.

Corollary 5.6. *Let P be a complete semilattice. Then the principal ideal mapping $\downarrow: P \to \kappa(C(P)), x \mapsto \downarrow x$ is an order-isomorphism.*

Corollary 5.7. *For complete semilattices P and Q, the following statements are equivalent:*

(i) $P \cong Q$.
(ii) $\sigma(P) \cong \sigma(Q)$.
(iii) $C(P) \cong C(Q)$.

Remark 5.8. At the moment, we do not know whether the implication (iii) $\Longrightarrow$ (i) in the above corollary holds for any two dcpo's. Of course, if P is a domain, then $\Sigma(P)$ is sober and $C(P)$ is a completely distributive lattice. Thus if $C(Q) \cong C(P)$, then Q must be continuous, thus $\Sigma(Q)$ is sober. Hence we have that P and Q are isomorphic.

At this juncture, the curious reader may wonder if a C-continuous poset exhibits the well-known interpolation property ([7]). More precisely, if P is a C-continuous poset and $x \prec z$ holds in P, is there a $y \in P$ such that $x \prec y \prec z$? Although we do not have the answer to this at the moment, the following counterexample does show that a C-continuous lattice need not exhibit the strong interpolation property ([7]).

Example 5.9. Let $\mathbf{M_3}$ be the standard non-distributive lattice as in Example 3.2. By Proposition 4.8, the lattice $C(P)$ is C-prealgebraic. Note that in $C(P)$, $\downarrow a \prec \downarrow \{a,b,c\}$ but there is no element $C \in C(P)$ for which $C \neq \downarrow a$ and $C \neq \downarrow \{a,b,c\}$ such that $\downarrow a \prec C \prec \downarrow \{a,b,c\}$.

Definition 5.10. Let L be a complete lattice. We say that L is *C-stable* if the following conditions hold:

(i) for any $x \in L$ and for any nonempty set D of L with $x \prec D$ (i.e., $x \prec d$ for all $d \in D$), it holds that $x \prec \bigsqcap_L D$;

(ii) $1_L \prec 1_L$, where 1_L is the top element of L.

If L satisfies only condition (i), we say that L is *weakly C-stable*.

Proposition 5.11. *For any C-algebraic lattice L. the following are equivalent:*

(i) *L is C-stable;*

(ii) *$\kappa(L)$ is a complete lattice with respect to the order inherited from L.*

Proof: (i) $\Longrightarrow$ (ii): Assume L is C-stable. Then $1_L \in \kappa(L)$. It suffices to prove that the infimum of an arbitrary non-empty subset of $\kappa(L)$, say $\{x_i : i \in I\}$, exists. Denote $\bigsqcap_L \{x_i : i \in I\}$ by x. Clearly, by Proposition 3.3, $x \prec x_i$ for each $i \in I$. Thus, $x \prec x$ by the C-stability of L. Hence $x = \bigsqcap_{\kappa(L)} \{x_i : i \in I\}$. So $\kappa(L)$ is a complete lattice with respect to the order inherited from L.

(ii) $\Longrightarrow$ (i): Assume that $\kappa(L)$ is a complete lattice. Since L is C-algebraic, the top element $1_{\kappa(L)}$ of $\kappa(L)$ must coincide with the top element 1_L of L, so $1_L \in \kappa(L)$, i.e. $1_L \prec 1_L$. Now suppose that $x \in L$ and $D \subseteq L$ is a nonempty set such that $x \prec D$. The C-algebraicity of L ensures that for each $d \in D$ there exists $d' \in \kappa(L)$ such that $x \sqsubseteq d' \sqsubseteq d$. Define $D' = \{d' \in \kappa(L) : x \sqsubseteq d' \sqsubseteq d$ for some $d \in D\}$. Since $\kappa(L)$ is complete, the element $e =: \bigsqcap_{\kappa(L)} D' \in \kappa(L)$ exists. If we can prove $x \sqsubseteq e$, then $x \sqsubseteq e \prec e \sqsubseteq \bigsqcap_L D' \sqsubseteq \bigsqcap_L D$ would imply that $x \prec \bigsqcap_L D$. So it remains to verify that indeed $x \sqsubseteq e$.

Now for each $y \in \kappa(L)$ with $y \prec x$, we have $y \sqsubseteq e$. Thus $\bigsqcup_{\kappa(L)} \{y \in \kappa(L) : y \sqsubseteq x\} \sqsubseteq e$. Since L is C-algebraic, $x = \bigsqcup_L \{y \in \kappa(L) : y \sqsubseteq x\}$. Since $\bigsqcup_L \{y \in \kappa(L) : y \sqsubseteq x\} \sqsubseteq \bigsqcup_{\kappa(L)} \{y \in \kappa(L) : y \sqsubseteq x\}$, by the transitivity of $\sqsubseteq$, it follows that $x \sqsubseteq e$ and the proof is complete. $\square$

The following proposition can be proved in a similar way.

Proposition 5.12. *The following statements are equivalent for a C-algebraic lattice L:*

(i) *L is weakly C-stable;*
(ii) *$\kappa(L)$ is a complete semilattice.*

Now we can prove one of the main results in this paper.

Theorem 5.13. *In the following, a (weakly) C-stable and C-algebraic lattice will be called a (weak-) stably C-algebraic lattice.*

(i) *A complete lattice M is isomorphic to the lattice $C(L)$ for a complete lattice L if and only if M is stably C-algebraic.*
(ii) *A complete lattice M is isomorphic to the lattice $C(P)$ for a complete semilattice P if and only if M is weak-stably C-algebraic.*

Proof: Since the proofs for (i) and (ii) are very similar, we prove only (ii).

Assume that P is a complete semilattice and $M \cong C(P)$. By Corollary 5.4, M is C-algebraic. By Corollary 5.6, $\kappa(M)$ is isomorphic to P. Thus $\kappa(M)$ is a complete semilattice. Since M is a C-algebraic lattice, by Proposition 5.12, M is weakly C-stable.

Now assume that M is a weak-stably C-algebraic lattice. Let $P = \kappa(M)$ and $\{x_i : i \in i\} \subseteq P$ be a nonempty subset. Then for all $i \in I$, we have $\bigsqcap_M \{x_i : i \in I\} \sqsubseteq x_i \prec x_i$. As M is weakly-stable, it follows that $\bigsqcap_M \{x_i : i \in I\} \prec \bigsqcap_M \{x_i : i \in I\}$. Consequently $\bigsqcap_M \{x_i : i \in I\} \in P$ and thus $\bigsqcap_M \{x_i : i \in I\} = \bigsqcap_P \{x_i : i \in I\}$. Thus, P is a complete semilattice.

We now show that M is isomorphic to $C(P)$. We claim that the mapping $\zeta : M \to C(P)$, $x \mapsto (\downarrow x) \cap P$, is an order-isomorphism of complete lattices. By Proposition 4.3, P is a dcpo with respect to the order inherited from M. So it follows that $\zeta(x) \in C(P)$ for every $x \in M$. Trivially, ζ is order-preserving. In order to show that ζ is an order-isomorphism, it suffices to show that the mapping $\bigsqcup_M : C(P) \longrightarrow M$ is an inverse of ζ. For each $x \in M$, $\bigsqcup \zeta(x) = \bigsqcup_M (\downarrow x \cap P) = x$ holds because M is C-algebraic. Now for any $C \in C(P)$, let $\bigsqcup_M C = a$. We claim that $(\downarrow a) \cap P = C$. Since $(\downarrow a) \cap P \supseteq C$ is trivial, we only need to show that $(\downarrow a) \cap P \subseteq C$. Let $x \in (\downarrow a) \cap P$, i.e., $x \prec x$ and $x \sqsubseteq a$. This implies that $x \prec a = \bigsqcup_M C$. Let $Q = \downarrow C = \{y \in M : \exists c \in C, y \sqsubseteq c\}$, then $\bigsqcup_M C = \bigsqcup_M Q$. If we can show Q is a Scott-closed set of M, then $x \prec a = \bigsqcup_M C = \bigsqcup_M Q$ will imply that $x \in Q$ and thus $x \sqsubseteq c$ for some $c \in C$, whence $x \in C$. It then follows that $(\downarrow a) \cap P \subseteq C$.

To show that $\downarrow C$ is a Scott-closed set of M, let $D \subseteq \downarrow C$ be a directed subset of M. For each $d \in D$, let $\overline{d} = \bigsqcap_P \{y \in C : d \sqsubseteq y\}$. Note that each $\overline{d} \in C$, and obviously the set $\overline{D} = \{\overline{d} : d \in D\}$ is a directed subset of P. Since C is a Scott-closed set of P, we have $\bigsqcup_P \overline{D} \in C$. However it is clear that $\bigsqcup_M D \sqsubseteq \bigsqcup_M \overline{D} \sqsubseteq \bigsqcup_P \overline{D}$, and hence $\bigsqcup_M D \in \downarrow C$. Since $\downarrow C$ is obviously a lower set, it is thus a Scott-closed set of M. The proof is then complete. □

From the proof of Theorem 5.13 we have the following:

Corollary 5.14. *Let M be a weak-stably C-algebraic lattice.*

(i) *The mapping $\zeta : M \longrightarrow C(P)$, $x \mapsto (\downarrow x) \cap P$ is an order isomorphism, where $P = \kappa(M)$.*

(ii) *The mapping $\bigsqcup_M : C(P) \longrightarrow M$ is an order isomorphism, which is the inverse of ζ.*

6. Some categorical equivalences

After proving Theorem 5.13, it is natural to conjecture that there exists an equivalence between the category of complete semilattices and the category of weak-stably C-algebraic lattices. In this section, we first establish an adjunction between the category of dcpo's and the category of C-prealgebraic lattices. The restriction of this adjunction will give the desired equivalence.

Let **DCPO** be the category whose objects are dcpos and whose morphisms are the Scott-continuous maps (i.e. monotone maps preserving sups of directed sets).

Let L and M be posets. A pair (g, d) of mappings $d : L \longrightarrow M$ and $g : M \longrightarrow L$ is called a *Galois connection* if both d and g are monotone, and for any $x \in L$ and $y \in M$, $d(x) \sqsubseteq y$ holds if and only if $x \sqsubseteq g(y)$ holds. In a Galois connection (g, d), the mapping d is called the *lower adjoint* and g the *upper adjoint*.

Let (g, d) be a Galois connection between complete lattices L and M. It is well-known that if L and M are both complete lattices then a mapping $d : L \longrightarrow M$ is a lower adjoint if and only if it preserves arbitrary suprema ([4]). Likewise, a mapping $g : M \longrightarrow L$ is an upper adjoint if and only if it preserves arbitrary infima. Moreover, if M and L are complete lattices, then

$$\forall\, m \in M\,.\; g(m) = \bigsqcup\{l \in L : d(l) \sqsubseteq m\},$$

$$\forall\, l \in L\,.\; d(l) = \bigsqcap\{m \in M : l \sqsubseteq g(m)\}.$$

A mapping $h : L \longrightarrow M$ between complete lattices is said to preserve the relation $\prec$ if for any $x, y \in L$, $x \prec y$ implies $h(x) \prec h(y)$. Now let **CPAlg** be the category whose objects are the C-prealgebraic lattices and morphisms the lower adjoints which preserve the relation $\prec$.

Lemma 6.1. *Let (g, d) be a Galois connection between two posets M and N, where $d : M \longrightarrow N$ and $g : N \longrightarrow M$. Then for any Scott-closed subset C of N, $\downarrow g(C)$ is a Scott-closed subset of M.*

Proof: Let $D \subseteq \downarrow g(C)$ be directed such that $\bigsqcup D$ exists. Then for each $x \in D$, there exists $c \in C$ such that $x \sqsubseteq g(c)$, and so $d(x) \sqsubseteq c$. Since C is lower in N, $d(x) \in C$, and so $d(D)$ is a directed subset contained in C. Because C is Scott-closed in N and lower adjoints preserve arbitrary sups, it follows that $\bigsqcup d(D) = d(\bigsqcup D) \in C$. Since (g, d) is a Galois connection, we have that $\bigsqcup D \in \downarrow g(C)$ as desired. $\square$

Lemma 6.2. *Let (g, d) be a Galois connection between complete lattices L and M, where $d : L \longrightarrow M$ and $g : M \longrightarrow L$. If g preserves the sups of Scott-closed* *subsets, then d preserves the relation $\prec$. If L is C-continuous, then the converse conclusion is also true.*

Proof: Assume that g preserves the supremum of Scott-closed sets and $x \prec y$ holds in L. Let C be a Scott-closed subset of M satisfying $\bigsqcup C \sqsupseteq d(y)$. It follows that $g(\bigsqcup C) \sqsupseteq g(d(y)) \sqsupseteq y$ since g is the upper adjoint of d. Thus $\bigsqcup \downarrow g(C) = \bigsqcup g(C) = g(\bigsqcup C) \sqsupseteq y$. By Lemma 6.1, $\downarrow g(C)$ is Scott closed. Now it follows from the definition of $x \prec y$ that $x \sqsubseteq g(c)$ holds for some $c \in C$. But d is the lower adjoint so that $d(x) \sqsubseteq c$. Hence $d(x) \in C$ and so $d(x) \prec d(y)$.

Now assume that L is C-continuous and d preserves the relation $\prec$. For any Scott-closed subset C of L, we show that $g(\bigsqcup C) \sqsubseteq \bigsqcup g(C)$ which then implies $g(\bigsqcup C) = \bigsqcup g(C)$. Since L is C-continuous, in order to show $g(\bigsqcup C) \sqsubseteq \bigsqcup g(C)$ it suffices to prove that for every $x \prec g(\bigsqcup C)$, $x \sqsubseteq \bigsqcup g(C)$ holds. For this, let $x \prec g(\bigsqcup C)$, then $d(x) \prec d(g(\bigsqcup C)) \sqsubseteq \bigsqcup C$. Thus as C is Scott-closed, there exists $c \in C$ such that $d(x) \sqsubseteq c$. Hence $x \sqsubseteq g(c) \sqsubseteq \bigsqcup g(C)$ and thus the proof is complete. $\square$

It is well-known that if $f : P \longrightarrow Q$ is a morphism in **DCPO** then f is a (topologically) continuous mapping from the Scott space $\Sigma\, P$ to $\Sigma\, Q$ ([4] and [7]). Hence for each $E \in C(Q)$, $f^{-1}(E) \in C(P)$. Thus the mapping $f^{-1} : C(Q) \longrightarrow C(P)$ is well-defined and preserves arbitrary meets, so it is an upper adjoint. We now show that the lower adjoint of f^{-1} preserves the relation $\prec$.

Lemma 6.3. *Let $f : P \longrightarrow Q$ be a morphism in* **DCPO** *and let $h : C(P) \longrightarrow C(Q)$ be the lower adjoint of f^{-1}. Then h preserves $\prec$.*

Proof: By virtue of Lemma 6.2, it suffices to show that f^{-1} preserves sups of Scott-closed sets of $C(Q)$. For any $\mathcal{C} \in C(C(Q))$, $\bigsqcup_{C(Q)} \mathcal{C} = \bigcup \mathcal{C}$ by Lemma 4.1. Then $f^{-1}(\bigsqcup_{C(Q)} \mathcal{C}) = f^{-1}(\bigcup \mathcal{C})$. Thus $\bigcup f^{-1}(\mathcal{C}) = f^{-1}(\bigcup \mathcal{C}) \in C(P)$, so it follows that

$$\bigsqcup_{C(P)} f^{-1}(\mathcal{C}) = \bigcup f^{-1}(\mathcal{C}) = f^{-1}(\bigsqcup_{C(Q)} \mathcal{C}).$$

Thus f^{-1} preserves sups of Scott-closed subsets. $\square$

It follows from the above lemma that the function $P \longrightarrow C(P)$ can be extended to a functor $C : \mathbf{DCPO} \longrightarrow \mathbf{CPAlg}$, where for each dcpo P, C sends P to the complete lattice $C(P)$ and sends every morphism $f : P \longrightarrow Q$ in **DCPO** to the mapping $C(f) : C(P) \longrightarrow C(Q)$ which is the lower adjoint of f^{-1}.

On the other hand, for each C-prealgebraic A, recall that $\kappa(A)$ is the dcpo of all C-compact elements of A and it is a subset of A closed under the sups of directed sets. If $f : A \longrightarrow B$ is a morphism in **CPAlg** then f restricts to a morphism $\kappa(f) : \kappa(A) \longrightarrow \kappa(B)$ in **DCPO**. Thus we have a functor $\kappa : \mathbf{CPAlg} \longrightarrow \mathbf{DCPO}$.

Theorem 6.4. *The functor κ is right adjoint to the functor C.*

Proof: For each dcpo P, let $\eta_P : P \longrightarrow \kappa(C(P))$ be the mapping defined by $\eta_P(x) = \downarrow x$ for all $x \in P$. It is clear that η_P is a morphism in **DCPO**. Suppose that L is a C-prealgebraic lattice and $h : P \longrightarrow \kappa(L)$ is a morphism in **DCPO**. Define $\overline{h} : C(P) \longrightarrow L$ by

$$\overline{h}(E) = \bigsqcup h(E) \text{ for each } E \in C(P).$$

Then $h(x) = \overline{h}(\eta_P(x)) = \kappa(\overline{h}) \circ \eta_P(x)$ holds for every $x \in P$, so $h = \kappa(\overline{h}) \circ \eta_P$. It remains to prove that $\overline{h}$ is a lower adjoint that preserves the relation $\prec$. Since both $C(P)$ and L are complete lattices, to prove $\overline{h}$ is a lower adjoint, it is enough to show that it preserves sups of arbitrary sets.

For this purpose, let $\mathcal{D} = \{A_i : i \in I\} \in C(P)$. Then, as h is a continuous mapping between the spaces $(P, \sigma(P))$ and $(L, \sigma(L))$, we have

$$h(cl(\bigcup \mathcal{D})) \subseteq cl(h(\bigcup \mathcal{D})).$$

Note that $cl(\bigcup \mathcal{D}) = \bigsqcup_{C(P)} \mathcal{D}$ and $cl(h(\bigcup \mathcal{D})) \subseteq \downarrow (\bigsqcup_{i \in I} \bigsqcup h(A_i)) = \downarrow (\bigsqcup_{i\in I} \overline{h}(A_i))$. Now

$$\begin{aligned} \overline{h}(\textstyle\bigsqcup_{C(P)} \mathcal{D}) &= \textstyle\bigsqcup_L h(cl(\bigcup \mathcal{D})) \\ &\leq \textstyle\bigsqcup_L cl(h(\bigcup \mathcal{D})) \\ &\leq \textstyle\bigsqcup_{i\in I} \overline{h}(A_i). \end{aligned}$$

But clearly $\overline{h}(\bigsqcup_{C(P)} \mathcal{D}) \geq \bigsqcup_{i\in I} \overline{h}(A_i)$, thus

$$\overline{h}(\bigsqcup_{C(P)} \mathcal{D}) = \bigsqcup_{i\in I} \overline{h}(A_i).$$

Next we show that $\overline{h}$ preserves $\prec$. By Lemma 6.2, we only need to prove that the right adjoint $g : L \longrightarrow C(P)$ preserves sups of Scott-closed sets. Let $C \in C(L)$. It suffices to prove that $g(\bigsqcup C) \subseteq \bigsqcup g(C)$. Because g is the upper adjoint of $\overline{h}$, we have the following:

$$\begin{aligned} g(\textstyle\bigsqcup C) &= \textstyle\bigsqcup\{A \in C(P) : \overline{h}(A) \sqsubseteq \bigsqcup C\} \\ &= \textstyle\bigsqcup\{\downarrow x : x \in P, \overline{h}(\downarrow x) \sqsubseteq \bigsqcup C\} \\ &= \textstyle\bigsqcup\{\downarrow x : x \in P, h(x) \sqsubseteq \bigsqcup C\}. \end{aligned}$$

But for each $x \in P, h(x) \in \kappa(L)$, so $h(x) \prec h(x)$. Hence if $h(x) \sqsubseteq \bigsqcup C$ then $h(x) \in C$. Let $c = h(x)$. Then $\overline{h}(\downarrow x) = h(x) = c$ implies $\downarrow x \subseteq g(c)$ because g is right adjoint to h. Therefore $g(\bigsqcup C) \subseteq \bigsqcup g(C)$. □

Let **CL** be the full subcategory of **DCPO** consisting of all complete lattices and let **CSlat** be the full subcategory of **DCPO** consisting of all complete semilattices. Denote by **SCAlg** (respectively, **WSCAlg**) the full subcategory of **CPAlg** consisting of all stably C-algebraic (respectively, weak-stably C-algebraic) lattices.

By Theorem 5.13, for any object P in **CSlat**, $C(P)$ is an object in **WSCAlg**. Thus C restricts to a functor from **CSlat** to **WSCAlg**. For any morphism $f : P \longrightarrow Q$ in **CSlat**, $C(f) : C(P) \longrightarrow C(Q)$ is the lower adjoint of f^{-1}. It follows immediately that $C(f)(\downarrow x) = \downarrow f(x)$ for each $x \in P$. Thus the functor

C is faithful. Now if $s : C(P) \longrightarrow C(Q)$ is a morphism in **WSCAlg**, then s restricts to a morphism $f : P \longrightarrow Q$, where for each $x \in P$, $f(x) \in Q$ such that $\downarrow f(x) = s(\downarrow x)$. It is straightforward to verify that $C(f) = s$. Hence the functor C : **CSlat** $\longrightarrow$ **WSCAlg** is also full. At last, for each object M in **WSCAlg**, let $P = \kappa(M)$, then M is isomorphic to $C(P)$.

Also, P is an object of **CL** iff $C(P)$ is an object of **SCAlg**.

By Theorem IV.4.1(iii) of [16], we have the following:

Proposition 6.5. *The functor* C : **DCPO** $\longrightarrow$ **CPAlg** *restricts to an equivalence between* **CSlat** (*respectively,* **CL**) *and* **WSCAlg** (*respectively,* **SCAlg**).

Let **DOM** denote the full subcategory of **DCPO** consisting of all domains and let **CDL** denote the full subcategory of **CPAlg** consisting of all completely distributive lattices. From a classical result of [7], one has: a dcpo P is continuous if and only if $C(P)$ is completely distributive. And for every completely distributive lattice L, $C(\mathrm{Cospec}(L)) \cong L$ holds, where $\mathrm{Cospec}(L)$ is the set of all co-prime elements of L, which is the same as the set of all the C-compact elements of L by Proposition 4.3.

By a similar argument to the one before we deduce the following:

Corollary 6.6. *The category* **DOM** *is equivalent to the category* **CDL**.

Classically one considers the category **CDL*** of all completely distributive lattices and lower adjoints which preserve the long way-below relation $\lhd$. Similarly, one consider the category **DOM*** of domains and Scott-continuous maps which preserve the way-below relation $\ll$. The Lawson-Hoffmann duality states that the category **DOM*** is equivalent to the category **CDL*** ([15]).

Corollary 6.6 reveals how one can extend the category **CDL*** by including more morphisms so that the new category **CDL** is equivalent to the category **DOM** which has the same objects as **DOM*** but more morphisms.

The following example illustrates how the category **CDL** is indeed "larger" than the category **CDL***.

Example 6.7. Let $L = [0, 1]$ be the usual complete chain of real numbers in the unit interval. Then L is a completely distributive lattice. Let $f : L \longrightarrow L$ be the mapping defined by $f(0) = 0$ and $f(x) = 1$ for $x \neq 0$. Then f is a left adjoint preserving the relation $\prec$ (note that $x \prec y$ holds in L if and only if $x \sqsubseteq y$). But f does not preserve the relation $\lhd$. For instance $1/2 \lhd 1$ but $f(1/2) \not\lhd f(1)$.

7. Concluding remarks and future development

This paper is an improvement of some work [9] which started in 2002. Amazingly, at about the same time, an independent work of Martin Escardó [5] showed that the injective locales over perfect sublocale embeddings coincide with the underlying objects of the algebras of the upper powerlocale monad, and these are characterized as those frames of opens enjoying stably C-continuity. However, the results developed there have a different motivation, and, interestingly, many of these are forced upon us by the fact that the monad under consideration is of Kock-Zöberlein type [14].

The problem of characterizing the lattice $C(P)$ for an arbitrary dcpo P remains open. Thus we still do not know whether the isomorphism of complete lattices $C(P)$ and $C(Q)$ implies that of the dcpo's P and Q. Clearly, further work must be done (i) to achieve a better understanding of the lattices of Scott-closed sets, and (ii) to look for suitable applications of the results in this paper.

One possible research direction, with regards to (i), is to study the order-theoretic properties of Scott-closed set lattices by passing to the Hoare powerdomain. This idea is motivated by a result in [21] which states that the Hoare powerdomain of a dcpo D is isomorphic to the lattice of non-empty Scott-closed subsets of D.

Regarding (ii), for instance, one might be able to apply the technical results herein to characterize the **E**-projective frames for the adjunction between **PreFrm** and **Frm**.

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