Comment.Math.Univ.Carolin. 50,2 (2009) 245–264

On the sign of Colombeau functions and applications to conservation laws

Jiří Jelínek, Dalibor Pražák

Abstract. A generalized concept of sign is introduced in the context of Colombeau algebras. It extends the sign of the point-value in the case of sufficiently regular functions. This concept of generalized sign is then used to characterize the entropy condition for discontinuous solutions of scalar conservation laws.

Keywords: Colombeau algebra, generalized sign, conservation law, entropy condition

Classification: 46F30, 35L67

1. Introduction

Colombeau algebra of generalized functions $\mathscr{G}$ extends the theory of distributions so that not only arbitrary differentiation, but also multiplication of elements of $\mathscr{G}$ is defined. An interesting feature is that the product in $\mathscr{G}$ is not always consistent with the natural pointwise product. A typical example is the Heaviside function h, for which

$$\iota h \cdot \iota h \neq \iota h, \tag{1}$$

ιh being the canonical embedding into $\mathscr{G}$. Intuitively speaking, h is somewhere between 0 and 1 if $x = 0$. Hence, if $h \cdot h - h$ is not zero, the reason is that it is negative at $x = 0$. One of the objectives of this paper is to introduce a generalized concept of sign which is motivated by the above heuristics. The main idea is to detect the sign by multiplying with a class of singular distributions.

Later sections of our paper are devoted to application of the generalized sign to simple conservation law

$$\partial_t u + \partial_x b(u) = 0. \tag{2}$$

Assume $u \in L^\infty_{\mathrm{loc}}$ is given. Applying the canonical embedding, we find its representative $[\iota u] \in \mathscr{G}$, and then evaluate the equation with all the operations (derivative and composition) interpreted in $\mathscr{G}$.

The second author was supported by the research project MŠM 0021620839 financed by MŠMT, and also by the project GAČR 201/08/0315.

As expected, if u is a weak solution (in the usual sense) to (2), then $[\iota u]$ does not satisfy the same equation in $\mathscr{G}$. Certain "error term" m appears on the right-hand side, which is zero only in the weaker sense of association. Examples show that this is intimately related to (1). Here we find another motivation for the concept of generalized sign. It is supposed to serve as a sort of finer criterion, which enables us to detect the admissible (entropy) solution, based on the sign properties of the "infinitesimal" term m.

So far, many authors have studied various PDEs in the context of Colombeau algebras, and hyperbolic problems seem to be of a special interest. See Colombeau's survey paper [1] and the monograph [9] in particular. Concerning the hyperbolic shocks, we refer to [4], [11]. For more recent results, concerning various problems of fluid mechanics, see for example [8], [14] and [15].

In the above works, the Colombeau algebra (or some other nonstandard space) is a priori taken as the underlying functional space of the problem. In some cases, special modifications of Colombeau's original constructions are used ([14], [15]). The (non)existence of solutions is thus studied directly in $\mathscr{G}$.

In the present paper we adopt a somewhat different point of view. Our central interest lies in the concept of entropy solution, which belongs to the classical analysis. Secondly, the generalized sign is always detected via a multiplication with a distribution which arises as a derivative of a certain (possibly discontinuous) function in the ordinary sense. Hence, despite of the use of Colombeau algebras, our approach has several similarities or common links with the classical analysis of hyperbolic problems. Let us mention some of them.

Roughly speaking, to analyze the equation in the context of Colombeau algebras means that the solution is first mollified using a suitable smooth kernel, and the equation is then evaluated on this smoothed function. The resulting object is studied when the kernels converge to a Dirac mass. The key point of the Colombeau analysis is that, as is well-known, the convolution does not commute with nonlinear operations, and hence an additional nontrivial information can be extracted about the weak solution in this way. Here, one is reminded of the classical "commutator estimates", see e.g. [5, Theorem II.1]. Indeed, our Lemma 4 can be seen as version of commutator estimate in the context of Colombeau functions.

One can also see an analogy between our analysis and the so-called kinetic formulation of conservation laws (see [6], [10]). In this approach, one first solves a somewhat artificial kinetic formulation of the given equation, adding a new variable y. Integrating over y then yields the solution of the original equation. It is interesting to note that entropy solutions arise from the solutions of the kinetic equations which contain certain nonnegative terms (measures) to be present on the right-hand side.

In some sense, our approach provides a converse result. We show that a classical solution, when evaluated in a more complicated setting of Colombeau algebras, leaves a certain additional term on the right-hand side, and this term has a correct sign if and only if the original solution is the entropy one.

The content of the paper is the following: in Section 2, we review the basic Colombeau theory. We also introduce the concept of unconditional association, which will be useful in the sequel. In Section 3 we define the generalized sign for functions in $\mathbb{R}$. We show that it has a number of natural properties; among others, we relate the generalized sign to the sign of the value of the distribution.

In Section 4 we introduce the generalized sign for functions in $\mathbb{R}^2$. This is the setting we need for our later applications. Section 5 briefly reviews the basic theory of the equation (2). In particular, we recall the classical concepts of weak and entropy solutions. The same equation is studied in Section 6 from the point of view of Colombeau's algebra. Here we prove the main theorems about the characterization of weak and entropy solutions. Some examples are discussed in Section 7.

2. Basic Colombeau theory

We use the following standard notation: Ω is a domain in $\mathbb{R}^n$, $\mathscr{D}(\Omega)$ or simply $\mathscr{D}$ is the space of infinitely smooth functions with compact support, $\mathscr{D}'(\Omega)$ is the space of the distributions, the duality between those spaces is denoted by $\langle\cdot,\cdot\rangle$. The space of locally integrable and locally bounded functions is denoted L^1_{loc} and L^∞_{loc}, respectively.

The symbols ∂^α (α is a multiindex) or ∂_t, ∂_x, denote the derivative in the classical sense, distributional derivative, and the derivative in the Colombeau space. The meaning is clear from the context. We also use g', $g^{(k)}$ to denote classical derivative of the function g of one real variable.

We follow the standard construction of Colombeau algebra; see [2], [9] for details. We set

$$\mathcal{A}_0 = \left\{\varphi \in \mathscr{D}(\mathbb{R}^n);\ \textstyle\int \varphi(x)\,dx = 1\right\},$$
$$\mathcal{A}_q = \left\{\varphi \in \mathcal{A}_0;\ \int x^\alpha \varphi(x)\,dx = 0,\ \forall\,\alpha,\ 1 \le |\alpha| \le q\right\},$$

where $\alpha \in \mathbb{N}_0^n$ is a multiindex with height $|\alpha|$. The representatives $R \in \mathscr{E}(\Omega)$ are functions

$$R : \mathcal{A}_0 \times \Omega \to \mathbb{R}$$

such that $R(\varphi, \bullet) \in C^\infty(\Omega)$ for any $\varphi \in \mathcal{A}_0$ fixed. Denoting further

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon),$$

we recall that the Colombeau construction is based on two important algebras: the algebra $\mathscr{E}_{\mathrm{M}}(\Omega)$ of moderate representatives R which satisfy

$$\begin{gathered}\big(\forall\, K \Subset \Omega\big)\big(\forall\,\alpha \in \mathbb{N}_0^n\big)\big(\exists\, N,\, q \in \mathbb{N}\big)\big(\forall\,\varphi \in \mathcal{A}_q\big)\\ \big(\exists\,\varepsilon_0,\, c > 0\big)\big(\forall\,\varepsilon \in (0,\varepsilon_0)\big)\Big[\sup_{x\in K}|\partial^\alpha R(\varphi_\varepsilon, x)| \le c\varepsilon^{-N}\Big],\end{gathered}$$

and the ideal $\mathscr{N}(\Omega)$ of negligible representatives R, given by

$$\big(\forall\, K \Subset \Omega\big)\big(\forall\, \alpha \in \mathbb{N}_0^n\big)\big(\forall\, M > 0\big)\big(\exists\, q \in \mathbb{N}\big)\big(\forall\, \varphi \in \mathcal{A}_q\big)$$
$$\big(\exists\, \varepsilon_0,\, c > 0\big)\big(\forall\, \varepsilon \in (0, \varepsilon_0)\big)\Big[\sup_{x\in K} |\partial^\alpha R(\varphi_\varepsilon, x)| \le c\varepsilon^M\Big].$$

Colombeau algebra of generalized functions is defined as a quotient

$$\mathscr{G}(\Omega) = \mathscr{E}_{\mathrm{M}}(\Omega)/\mathscr{N}(\Omega).$$

It is convenient to denote the elements of $\mathscr{G}(\Omega)$ by $[R]$, where $R \in \mathscr{E}_{\mathrm{M}}(\Omega)$ is an arbitrary member of the equivalence class. R is called a representative of the generalized function $[R]$. So $[R'] = [R]$ if and only if $R' - R \in \mathscr{N}(\Omega)$. Sometimes, if needed, we use the notation $[R(\varphi, x)]$ meaning the same as $[R]$.

The operations on $\mathscr{G}(\Omega)$ are defined via the representatives; it is a matter of routine to check that all the definitions below are in fact independent on the particular choice of the representative in view of the properties of $\mathscr{N}(\Omega)$.

For $[R]$, $[S] \in \mathscr{G}(\Omega)$ one defines $[R]\pm[S] = [R\pm S]$, $[R][S] = [RS]$, $\partial^\alpha[R] = [\partial^\alpha R]$ and the derivative of R means the derivative with respect to the second variable, i.e. $\partial_x^\alpha R(\varphi, x)$.

By $C_{\mathrm{M}}^\infty(\mathbb{R})$ we denote the space of infinitely differentiable functions f with moderate growth, i.e.

$$\big(\forall\, k \ge 0\big)\big(\exists\, N, c > 0\big)\big(\forall\, x \in \mathbb{R}\big)\Big[|f^{(k)}(x)| \le c(1 + |x|)^N\Big].$$

The composition of a function $g \in C_{\mathrm{M}}^\infty(\mathbb{R})$ with $[R] \in \mathscr{G}(\Omega)$ is defined by

$$g \circ [R] = [g(R)].$$

Remark that for $[R]$, $[S] \in \mathscr{G}(\Omega)$, $g \in C_{\mathrm{M}}^\infty(\mathbb{R})$, we have

$$\begin{aligned} \partial_x([R][S]) &= \partial_x[R][S] + [R]\partial_x[S],\\ \partial_x(g \circ [R]) &= (g' \circ [R])\partial_x[R]\,; \end{aligned} \tag{3}$$

i.e., the Leibniz rule and the chain rule hold as expected.

The canonical embedding $\iota : \mathscr{D}'(\Omega) \to \mathscr{E}_{\mathrm{M}}(\Omega)$ is defined via the canonical representative ιT, given by[1]

$$\iota T(\varphi, x) = \langle T(y), \varphi(y - x)\rangle.$$

A special case $f \in L^1_{\mathrm{loc}}(\Omega)$ (regular distribution) gives

$$\iota f(\varphi, x) = \int f(x + y)\varphi(y)\, dy.$$

[1]A distribution T is denoted by $T(y)$, when the duality is taken over the explicitly written variable y.

For $x \in K \Subset \Omega$ and φ fixed, $\iota T(\varphi_\varepsilon, x)$, $\iota f(\varphi_\varepsilon, x)$ make sense for ε sufficiently small, which is enough in view of the definition of $\mathscr{N}(\Omega)$. See e.g. [2, §1.2–1.3] where it is shown that for the definition of the generalized function $[R]$ the representative R need not be defined on the whole of $\mathcal{A}_0 \times \Omega$ if only $R(\varphi_\varepsilon, x)$ is defined for (φ_ε, x) needed in the definition of $\mathscr{E}_{\mathrm{M}}$ and $\mathscr{N}$.

Remark that the application $[\iota] : T \mapsto [\iota T]$ of $\mathscr{D}'(\Omega)$ into $\mathscr{G}(\Omega)$ is also injective, i.e. $\iota T \in \mathscr{N}(\Omega)$ only if the distribution T vanishes.

If needed, we use the abusive notation $\iota\big(f(x)\big)$ with the same meaning as ιf or $\iota(f)$. Note that the explicitly written variable x of the function f has nothing to do with the variables of the representative ιf.

Note also that

$$\iota \partial^\alpha T = \partial^\alpha \iota T. \tag{4}$$

We say that representatives $R, S \in \mathscr{E}_{\mathrm{M}}(\Omega)$ are associated (or R is associated to S), if

$$\begin{gathered}\big(\forall\, \omega \in \mathscr{D}(\Omega)\big)\big(\exists\, q \in \mathbb{N}\big)\big(\forall\, \varphi \in \mathcal{A}_q\big)\\ \lim_{\varepsilon \searrow 0} \int \big(R(\varphi_\varepsilon, x) - S(\varphi_\varepsilon, x)\big)\omega(x)\, dx = 0.\end{gathered} \tag{5}$$

We write $R \approx S$. In that case the generalized functions $[R], [S]$ are called associated, too. Evidently this does not depend on the choice of representatives. The association is an equivalence on $\mathscr{E}_{\mathrm{M}}(\Omega)$ and on $\mathscr{G}(\Omega)$. We say that $[R] \in \mathscr{G}(\Omega)$ is associated to a distribution $T \in \mathscr{D}'(\Omega)$ and denote $R \approx T$, if $R \approx \iota T$. So $\iota T \approx T$ for any $T \in \mathscr{D}'(\Omega)$.

For the intention of this paper, we call the association unconditional, if (5) holds for all $\omega \in \mathscr{D}(\Omega)$ and $\varphi \in \mathcal{A}_0$. This relation concerns representatives, but does not concern generalized functions. Evidently $R \in \mathscr{E}_{\mathrm{M}}(\Omega)$ is unconditionally associated to ιT for $T \in \mathscr{D}'(\Omega)$, if and only if

$$\big(\forall\, \omega \in \mathscr{D}(\Omega)\big)\big(\forall\, \varphi \in \mathcal{A}_0\big)\ \lim_{\varepsilon \searrow 0} \int R(\varphi_\varepsilon, x)\omega(x)\, dx = \langle T, \omega\rangle. \tag{6}$$

Equivalently,

$$\big(\forall\, \varphi \in \mathcal{A}_0\big)\ \lim_{\varepsilon \searrow 0} R(\varphi_\varepsilon, \bullet) = T \quad \text{in} \quad \mathscr{D}'(\Omega). \tag{7}$$

We say in that case that the representative R is unconditionally associated to the distribution T. Recall an important property of barrelled spaces. If for some $\varphi \in \mathcal{A}_0$ and for all $\omega \in \mathscr{D}(\Omega)$ the finite left-hand side limit in (6) exists, then the linear form T defined by (6) is automatically continuous on $\mathscr{D}(\Omega)$, i.e. $T \in \mathscr{D}'(\Omega)$. See [13, Théorème XIII, p. 74] or [12, Theorem 6.17, p. 146].

If in addition $\sigma \in C^\infty(\Omega)$, we have even

$$\big(\forall\, \varphi \in \mathcal{A}_0\big)\ \lim_{\varepsilon \searrow 0} \iota\sigma(\varphi_\varepsilon, \bullet) = \sigma \quad \text{in} \quad C^\infty(\Omega). \tag{8}$$

The functions $\iota\sigma(\varphi_\varepsilon, \bullet)\iota T(\varphi_\varepsilon, \bullet)$ have the same limit σT in $\mathscr{D}'(\Omega)$ as the functions $\iota(\sigma T)(\varphi_\varepsilon, \bullet)$, so

$$\iota\sigma \cdot \iota T \approx \iota(\sigma T) \tag{9}$$

and the association is unconditional.

For $\sigma \in C^\infty(\Omega)$, it is well-known that $\iota\sigma(\varphi, x) - \sigma(x) \in \mathscr{N}(\Omega)$, so the function σ independent on φ also is, beside $\iota\sigma$, a representative of $[\iota\sigma]$. Thus the canonical embedding $[\iota]$ into $\mathscr{G}$ preserves the multiplication of smooth functions: $[\iota(\sigma_1\sigma_2)] = [\iota\sigma_1] \cdot [\iota\sigma_2]$, while in other situation (e.g. (9)) we have only association.

Observe that, given $R, S \in \mathscr{E}_{\mathrm{M}}(\Omega)$, $T \in \mathscr{D}'(\Omega)$,

$$\begin{aligned} R \approx S &\implies \partial^\alpha R \approx \partial^\alpha S \\ R \approx T &\implies \partial^\alpha R \approx \partial^\alpha T. \end{aligned} \tag{10}$$

Moreover, if the left-hand associations are unconditional, so are the right-hand ones, too.

Further, we use the following — not commonly used — concepts. We write

$$R \gtrapprox 0, \qquad [R] \gtrapprox 0$$

if R is associated to a non-negative distribution. Recall that $T \in \mathscr{D}'(\Omega)$ is non-negative, if $\langle T, \omega\rangle \geq 0$ for any $\omega \in \mathscr{D}(\Omega)$, $\omega \geq 0$. Note that a linear form T on $\mathscr{D}(\Omega)$ is automatically continuous (is a non-negative measure), if it is non-negative in the above sense.

Finally, we say that $R \in \mathscr{E}_{\mathrm{M}}(\Omega)$ is (unconditionally) locally bounded, if

$$\big(\forall\, K \Subset \Omega\big)\big(\forall\, \varphi \in \mathcal{A}_0\big)\big(\exists\, c > 0\big)\big(\exists\, \varepsilon_0 > 0\big)\big(\forall\, \varepsilon \in (0, \varepsilon_0)\big) \sup_{x \in K} |R(\varphi_\varepsilon, x)| \leq c. \tag{11}$$

It is clear that if $u \in L^\infty_{\mathrm{loc}}$, then its canonical representative ιu is locally bounded in the above sense.

Remark. Note that if $R \in \mathscr{E}_{\mathrm{M}}(\Omega)$ is locally bounded, then the composition $g(R)$ belongs to $\mathscr{E}_{\mathrm{M}}(\Omega)$ even if g is a smooth (but not necessarily moderate) function. This can be proved by a simple modification of [2, Proposition 1.4.2]. Similarly, the resulting generalized function $[g(R)]$ is independent of the choice of the (unconditionally) bounded representative (cf. [2, Theorem 1.4.3].) In the following, we will use this type of composition frequently.

If $g \in C^\infty(\mathbb{R})$, $u \in L^\infty_{\mathrm{loc}}(\Omega)$, then the classical composition, denoted by $g(u)$ or $g \circ u$, belongs to $L^\infty_{\mathrm{loc}}(\Omega)$ and we have

$$\iota(g \circ u) \approx g(\iota u). \tag{12}$$

The association is unconditional. Indeed, evidently the representatives $\iota(g \circ u)(\varphi_\varepsilon, x)$ and $g\big(\iota u(\varphi_\varepsilon, x)\big)$ are locally bounded and tend to $g(u(x))$ ($\varphi \in \mathcal{A}_0$, $\varepsilon \searrow 0$) at the Lebesgue points x of $g \circ u$ and of u, i.e. almost everywhere. So we obtain the assertion easily from the Lebesgue majorization theorem.

We conclude with examples that will be useful also in the following. The Heaviside function $h \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ is defined as $h(x) = 0$ for $x < 0$ and $h(x) = 1$ for $x > 0$. The Dirac distribution $\delta_0 \in \mathscr{D}'(\mathbb{R})$ is given by $\langle \delta_0, \omega \rangle = \omega(0)$, $\forall\, \omega \in \mathscr{D}(\mathbb{R})$. One has $\partial_x h = \delta_0$ in $\mathscr{D}'(\mathbb{R})$. Note that δ_0 is a non-negative distribution.

The corresponding canonical representatives $H := \iota h$, $\Delta_0 := \iota\delta_0$ read

$$
\begin{aligned}
H(\varphi, x) &= \int_{-x}^{\infty} \varphi(y)\, dy, \\
\Delta_0(\varphi, x) &= \varphi(-x).
\end{aligned} \tag{13}
$$

It follows that $\partial_x H = \Delta_0$.

3. Generalized sign in dimension 1

In this section we introduce the generalized sign in $\mathbb{R}$.

Definition 1. We say that $[R] \in \mathscr{G}(\mathbb{R})$ is non-negative in the generalized sense at the point $x = 0$, if for arbitrary non-decreasing $g \in C^\infty(\mathbb{R})$

$$R \cdot \partial_x g(H) \gtrapprox 0. \tag{14}$$

We write $[R](0) \geq 0$.

The relation $[R](0) \leq 0$ is defined in an analogous way.

The symbol $[R](0)$ can be interpreted as the germ of the generalized function $[R]$ at the point $x = 0$. Note that $\partial_x g(H) = g'(H) \cdot \Delta_0$, hence instead of (14) we can require

$$R \cdot \gamma(H) \cdot \Delta_0 \gtrapprox 0, \tag{15}$$

where $\gamma = g' \in C^\infty(\mathbb{R})$ is an arbitrary non-negative function. The intuitive meaning of the definition is clear: to detect the sign at a given point, we multiply by a class of non-negative singularities.

The definition extends to points x other than zero in an obvious way. The property is local, and one easily verifies the linear properties, e.g.

$$[R](0) \geq 0,\ [S](0) \geq 0 \implies [R + S](0) \geq 0.$$

It would be interesting to see whether some nonlinear properties also hold, as for example

$$[R](0) \geq 0,\ [S](0) \geq 0 \implies [RS](0) \geq 0. \tag{16}$$

We will provide at least some partial answers later. Let us proceed with a proposition which is useful in studying further properties of the generalized sign.

Proposition 1. *Let $m \in C^\infty(\mathbb{R})$. Then*

$$m(H) \cdot \Delta_0 \approx \left(\int_0^1 m(s)\, ds \right) \delta_0.$$

Proof: The left-hand side equals $\partial_x\{M(H)\}$, where $M' = m$. Now $M(H) \approx f$ (see (12)), a regular distribution with the value $f(x) = M(0)$ for $x < 0$, $f(x) = M(1)$ for $x > 0$. By (10),

$$\partial_x M(H) \approx \partial_x f = \{M(1) - M(0)\}\delta_0 = \left(\int_0^1 m(s)\,ds\right)\delta_0.$$

$\square$

One intuitively thinks of the Heaviside function as being somewhere between 0 and 1 for $x = 0$; our concept of generalized sign is consistent with that.

Proposition 2. *Let $m \in C^\infty(\mathbb{R})$. Then $[m(H)](0) \geq 0$ if and only if $m(s) \geq 0$ for all $s \in (0,1)$.*

Proof: By the previous proposition

$$\begin{aligned} m(H)\partial_x\{g(H)\} &= m(H)g'(H)\iota\delta_0 \\ &= \{mg'\}(H)\iota\delta_0 \\ &\approx \left(\int_0^1 m(s)g'(s)\,ds\right)\delta_0. \end{aligned}$$

Clearly the integral is non-negative for any $g \in C^\infty(\mathbb{R})$ non-decreasing if and only if $m \geq 0$ on $(0,1)$. $\square$

As a corollary we deduce that $[H](0) \geq 0$, $[H^2 - H](0) \leq 0$. Both signs are strict (in the sense that opposite inequalities do not hold). Note that $H^2 - H$ is (unconditionally) associated to zero.

Let us turn again to the problem whether (16) holds. Proposition 2 gives a positive answer if $R = m(H)$, $S = \tilde{m}(H)$ for certain m, $\tilde{m} \in C^\infty(\mathbb{R})$. Another partial result is given in the following:

Recall that $F \in \mathscr{D}'(\mathbb{R})$ admits the value $k \in \mathbb{R}$ at the point $x = 0$ (in the Lojasiewicz's sense [7]), if for every $\varphi \in \mathcal{A}_0$

$$\lim_{\varepsilon \searrow 0} \langle F, \varphi_\varepsilon \rangle = k.$$

Proposition 3. *Let $F \in \mathscr{D}'(\mathbb{R})$ admit the value $k \in \mathbb{R}$ at the point $x = 0$. Then $[\iota F](0) \geq 0$ if and only if $k \geq 0$.*

Proof: We can write $F = F_0 + k$, where F_0 admits the value 0 at $x = 0$. In view of (15) and Proposition 1, for the constant function k, $[\iota k](0) \geq 0$ if and only if $k \geq 0$. Hence, it suffices to show that

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \iota F_0(\varphi_\varepsilon, x) g'(H(\varphi_\varepsilon, x)) \Delta_0(\varphi_\varepsilon, x)\omega(x)\,dx = 0, \tag{17}$$

where $\omega \in \mathscr{D}(\mathbb{R})$ is fixed. Recall that

$$\iota F_0(\varphi_\varepsilon, x) = \langle F_0(y), \varphi_\varepsilon(y - x)\rangle.$$

Since (cf. (13))

$$H(\varphi_\varepsilon, \varepsilon x) = H(\varphi, x),$$

by substitution $x \to \varepsilon x$ the integral in (17) equals

$$\begin{aligned}&\int_{\mathbb{R}} \langle F_0(y), \varphi_\varepsilon(y - \varepsilon x)\rangle\, g'(H(\varphi, x))\, \varphi(-x)\, \omega(\varepsilon x)\, dx \\ &= \left\langle F_0(\varepsilon y),\ \underbrace{\int_{\mathbb{R}} \varphi(y - x)\, g'(H(\varphi, x))\, \varphi(-x)\, \omega(\varepsilon x)\, dx}_{\chi(y,\varepsilon)} \right\rangle.\end{aligned}$$

Here $F_0(\varepsilon y)$ is defined by $\langle F_0(\varepsilon y), \varphi(y)\rangle = \langle F_0(y), \varphi_\varepsilon(y)\rangle$. In order to prove (17), it is enough to consider a sequence $\varepsilon_n \searrow 0$. By our assumption, $F_0(\varepsilon_n y) \to 0$ weakly in $\mathscr{D}'(\mathbb{R})$. However, for sequences of distributions the weak and strong convergence coincide (see e.g. [13, Théorème XIII, p. 74]). On the other hand, $\chi(\cdot, \varepsilon_n)$ form a bounded set in $\mathscr{D}(\mathbb{R})$, in view of smooth dependence on ε and uniformly bounded supports. Hence $\langle F_0(\varepsilon_n y), \chi(y, \varepsilon_n)\rangle \to 0$ and we are done. $\square$

As a corollary, we obtain that if $x = 0$ is a Lebesgue point of $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ (in particular, if f is continuous at $x = 0$), then $[\iota f](0) \geq 0$ if and only if $f(0) \geq 0$.

4. Generalized sign in dimension 2

In this section we extend the concept of sign to generalized functions that are defined in $\mathbb{R}^2$. We do not, however, speak of the sign at a point, but at a line.

We use the notation that is suitable for our later applications to evolutionary PDEs: the considered domain is $Q = \mathbb{R} \times (0, \infty)$, with the variables denoted by x and t.

We introduce $\delta_c \in \mathscr{D}'(\mathbb{R}^2)$ by

$$\delta_c = \partial_x h(x - c(t)),$$

where $c : \mathbb{R} \to \mathbb{R}$ is a given smooth function. The derivative is computed in $\mathscr{D}'(\mathbb{R}^2)$. The corresponding canonical representatives will be denoted by

$$H_c := \iota\big(h(x - c(t))\big), \tag{18}$$

$$\Delta_c := \iota\delta_c. \tag{19}$$

One has

$$\begin{aligned} H_c(\varphi; x, t) &= \int_{\mathbb{R}^2} h(x + y - c(t + s))\varphi(y, s)\, dyds \\ &= \int\limits_{x+y>c(t+s)} \varphi(y, s)\, dyds, \end{aligned} \tag{20}$$

$$\Delta_c(\varphi; x, t) = \partial_x H_c(\varphi; x, t) = \int_{\mathbb{R}} \varphi(c(t + s) - x, s)\, ds.$$

Definition 2. We say that $[R] \in \mathscr{G}(\Omega)$ $(\Omega \subseteq \mathbb{R}^2$, open) is non-negative (resp. non-positive) in the generalized sense at the line $\{x = c(t)\}$, if for arbitrary $g \in C^\infty(\mathbb{R})$ non-decreasing, the product

$$R \cdot \partial_x (g \circ H_c)$$

is associated to a non-negative (resp. non-positive) distribution on Ω. We write $[R]\,\{x = c(t)\} \geq 0$ (resp. $[R]\,\{x = c(t)\} \leq 0$). We use this definition namely for $\Omega = \mathbb{R}^2$ or $\Omega = Q$.

Observe that $\partial_x(g \circ H_c) = (g' \circ H_c) \cdot \Delta_c$. Thus the sign is again detected by multiplying with a certain class of positive singularities.

Basic properties of the sign in $\mathbb{R}^2$ are analogous to the results in $\mathbb{R}$. We will prove only those that will be needed in the sequel. In analogy to Proposition 1, we establish:

Proposition 4. *Let $m \in C^\infty(\mathbb{R})$. Then*

$$m\big(H_c\big) \cdot \Delta_c \approx \left(\int_0^1 m(s)\, ds \right) \delta_c.$$

Proof: We have

$$m\big(H_c\big) \cdot \Delta_c = \partial_x\big(M(H_c)\big),$$

where $M \in C^\infty(\mathbb{R})$ is primitive to m. Now $M(H_c)$ is associated to a regular distribution (cf. (12)) equal to the function $M(h(x - c(t)))$. One finds easily that its ∂_x (distributional derivative) is $(M(1) - M(0))\delta_c$. The conclusion follows by (10). □

An immediate corollary is the following proposition.

Proposition 5. *Let $m \in C^\infty(\mathbb{R})$. Then $m(H_c)\,\{x = c(t)\} \geq 0$, if and only if $m(s) \geq 0$ for $\forall\, s \in [0, 1]$.*

Proof: Completely analogous to the proof of Proposition 2. □

The rest of this section is devoted to proofs of several auxiliary results of technical nature. In particular, Lemma 3 below asserts that the generalized sign in $\mathbb{R}^2$ can be detected by a more general class of functions depending on t. This result will be needed in our later applications.

Lemma 1. *Let $c(t)$, $\sigma(t) \in C^\infty(\mathbb{R})$. Then $\forall\, \varphi \in \mathcal{A}_0$*

(i) *the representative*

$$\iota\big(\sigma(t) h(x - c(t))\big)(\varphi_\varepsilon; x, t) - \sigma(t) H_c(\varphi_\varepsilon; x, t)$$

tends to 0 *locally uniformly w.r. to x, t as $\varepsilon \searrow 0$;*

(ii) *the representative*

$$\partial_x \iota\big(\sigma(t) h(x - c(t))\big)(\varphi_\varepsilon; x, t) - \sigma(t) \Delta_c(\varphi_\varepsilon; x, t)$$

is locally bounded;

(iii)

$$\int |\Delta_c(\varphi_\varepsilon; x,t)|\, dx$$

is bounded independently on $t > 0$ and on sufficiently small $\varepsilon > 0$.

Proof: One has

$$\iota\big(\sigma(t)h(x-c(t))\big)(\varphi_\varepsilon; x,t) = \int\limits_{x+y>c(t+s)} \sigma(t+s)\, \varphi_\varepsilon(y,s)\, dyds. \tag{21}$$

Concerning (i), we thus need to estimate (cf. also (20))

$$\int\limits_{x+y>c(t+s)} \big[\sigma(t+s) - \sigma(t)\big] \varphi_\varepsilon(y,s)\, dyds. \tag{22}$$

Assume that $|x|$, $|t| < k$ and let further supp $\varphi \subset [-k,k]^2$ and $|\varphi| \le k$. As the integrand is zero for $|s| > k\varepsilon$, we have the estimate $|\sigma(t+s) - \sigma(t)| \le k'\varepsilon$, and (22) is estimated as

$$k'\varepsilon \int_{\mathbb{R}^2} |\varphi(y,s)|\, dyds = k''\varepsilon.$$

To handle (ii), we first deduce from (21) that

$$\partial_x \iota\big(\sigma(t)h(x-c(t))\big)(\varphi_\varepsilon; x,t) = \int_{s\in\mathbb{R}} \sigma(t+s)\varphi_\varepsilon(c(t+s)-x, s)\, ds,$$

and proceeding as above we get (see (20) again)

$$\begin{aligned} \Big| \int_{s\in\mathbb{R}} \big[\sigma(t+s) - \sigma(t)\big] \varphi_\varepsilon(c(t+s)-x,s)\, ds \Big| & \\ \le k'\varepsilon \int\limits_{|s|\le k\varepsilon} \frac{1}{\varepsilon^2} \big|\varphi\big(\tfrac{c(t+s)-x}{\varepsilon}, \tfrac{s}{\varepsilon}\big)\big|\, ds & \\ \le k''\,. & \end{aligned}$$

Concerning (iii), we proceed similarly as in (ii):

$$\int |\Delta_c(\varphi_\varepsilon; x,t)|\, dx \le \int |\varphi_\varepsilon(c(t+s)-x,s)|\, dsdx$$

$$= \int \frac{1}{\varepsilon^2} \big|\varphi\big(\tfrac{c(t+s)-x}{\varepsilon}, \tfrac{s}{\varepsilon}\big)\big|\, dxds = \int\limits_{\substack{|\mathrm{x}|\le \mathrm{k}\varepsilon \\ |s|\le k\varepsilon}} \frac{1}{\varepsilon^2} \big|\varphi\big(\tfrac{x}{\varepsilon}, \tfrac{s}{\varepsilon}\big)\big|\, dxds \le 4k^2 \max|\varphi|.$$

□

Lemma 2. *Let f be a non-negative, continuous function on $K \Subset \mathbb{R}^2$; let $\eta > 0$ be given. Then there exist non-negative functions γ_n, $\psi_n \in \mathscr{D}(\mathbb{R})$, $n = 1, \dots, N$, such that*

$$\Big| f(y,t) - \sum_{n=1}^{N} \gamma_n(y)\psi_n(t) \Big| \le \eta, \qquad \forall\, (y,t) \in K.$$

Proof: We can assume that $K \subset [\frac{1}{4}, \frac{3}{4}]^2$ and f is non-negative and continuous on $[0,1]^2$. The key step are the Bernstein polynomials

$$\begin{aligned} B_{N'}(y,t) &= \sum_{j,k=1}^{N'} f\left(\tfrac{k}{N'}, \tfrac{j}{N'}\right) \binom{N'}{k}\binom{N'}{j} y^k (1-y)^{N'-k}\, t^j (1-t)^{N'-j} \\ &= \sum_{j,k=1}^{N'} \gamma_{j,k}(y)\psi_{j,k}(t), \end{aligned}$$

where e.g.

$$\begin{aligned} \gamma_{j,k}(y) &:= f\left(\tfrac{k}{N'}, \tfrac{j}{N'}\right) \binom{N'}{k} y^k (1-y)^{N'-k}, \\ \psi_{j,k}(t) &:= \binom{N'}{j} t^j (1-t)^{N'-j}. \end{aligned}$$

As is well-known, they approximate f uniformly, and obviously preserve non-negativity on $[0,1]$. We then modify $\gamma_{j,k}$, $\psi_{j,k}$ outside $[\frac{1}{4}, \frac{3}{4}]$ as needed and replace the double-indices j, k with n running from 1 to $N := {N'}^2$. □

Lemma 3. *Suppose the representative $R \in \mathscr{E}_{\mathrm{M}}(Q)$ is locally bounded (defined by (11)), $\sigma, u \in C^\infty(\mathbb{R})$, $\sigma > 0$. Then the following are equivalent:*

(i) *for any non-negative $\Gamma \in C^\infty(\mathbb{R}^2)$, the representative*

$$R(\varphi; x, t) \cdot \Gamma\big(H_c(\varphi; x, t), t\big) \cdot \Delta_c(\varphi; x, t)$$

is unconditionally associated to a distribution, resp. to a non-negative distribution;

(ii) *for any non-negative $\gamma \in C^\infty(\mathbb{R})$, the representative*

$$R(\varphi; x, t) \cdot \gamma\big(\sigma(t) H_c(\varphi; x, t) + u(t)\big) \cdot \Delta_c(\varphi; x, t)$$

is unconditionally associated to a distribution, resp. to a non-negative distribution.

Proof: Implication (i) $\Longrightarrow$ (ii) is obvious.

Conversely, let (ii) hold; we want to show that

$$\int_Q R(\varphi_\varepsilon; x, t) \Gamma\big(H_c(\varphi_\varepsilon; x, t), t\big)\, \Delta_c(\varphi_\varepsilon; x, t)\, \omega(x, t)\, dx dt \tag{23}$$

has a finite (resp. non-negative finite) limit as $\varepsilon \searrow 0$, where $\omega \in \mathscr{D}(Q)$, $\omega \geq 0$, $\varphi \in \mathcal{A}_0$ are fixed. It is enough to consider $(x,t) \in \operatorname{supp}\omega$. As H_c is locally bounded, there is a compact $K \Subset \mathbb{R}^2$ such that, denoting

$$y = \sigma(t) H_c(\varphi_\varepsilon; x, t) + u(t),$$

we have $(y,t) \in K$ for all $\varepsilon > 0$ small enough provided $(x,t) \in \operatorname{supp}\omega$. By Lemma 2, we can write

$$\Gamma\big(\tfrac{y-u(t)}{\sigma(t)}, t\big) = \sum_{n=1}^{N} \gamma_n(y)\psi_n(t) + z(y,t),$$

where $|z(y,t)| \leq \eta$ for $(y,t) \in K$. Hence (23) is equal to

$$\begin{aligned}\sum_{n=1}^{N} \int_Q R(\varphi_\varepsilon; x,t)\, \gamma_n\big(\sigma(t) H_c(\varphi_\varepsilon; x,t) + u(t)\big)\psi_n(t) \\ \cdot \Delta_c(\varphi_\varepsilon; x,t)\, \omega(x,t)\, dxdt \\ + \int_Q R(\varphi_\varepsilon; x,t)\, z\big(\sigma(t) H_c(\varphi_\varepsilon; x,t) + u(t)\ \ t\big) \cdot \Delta_c(\varphi_\varepsilon; x,t)\, \omega(x,t)\, dxdt.\end{aligned}$$

Lemma 1(iii) yields that the last integral is $\mathcal{O}(\eta)$. The sum has a finite (resp. non-negative finite) limit as $\varepsilon \searrow 0$ by (ii), hence the same holds for (23), as it is independent of (arbitrarily small) η. □

5. Scalar conservation law

We consider a simple conservation law

$$\partial_t u + \partial_x b(u) = 0. \tag{24}$$

Here $u = u(x,t) : Q \to \mathbb{R}$, is the unknown function, $Q = \mathbb{R} \times (0,\infty)$. The non-linearity $b \in C^\infty(\mathbb{R})$ is given.

Below we review the basic theory. These results are nowadays classical and can be found in many books, e.g. [3].

It is well-known that the solutions to (24) need not be (globally) smooth or even continuous; this fact is in agreement with the underlying physics. One introduces the concept of weak solution.

Definition 3. *A function $u \in L^\infty_{loc}(Q)$ is called weak solution to* (24) *if*

$$\int_Q \Big(u(x,t)\partial_t\omega(x,t) + b(u(x,t))\partial_x\omega(x,t)\Big)\, dxdt = 0 \tag{25}$$

for all $\omega \in \mathscr{D}(Q)$. This means that u fulfils (24) *in $\mathscr{D}'(Q)$.*

Proposition 6. *Given $u \in L^\infty_{\mathrm{loc}}(Q)$, we set*

$$\ell(\iota u) = \partial_t\big(\iota u\big) + \partial_x\big(b(\iota u)\big). \tag{26}$$

In other words, $\ell(\iota u) \in \mathscr{E}_{\mathrm{M}}(Q)$ is the left-hand side of (24) evaluated in $\mathscr{E}_{\mathrm{M}}$.

If $\ell(u)$ is the left-hand side of (24) evaluated in $\mathscr{D}'$, then $\ell(\iota u)$ is unconditionally associated to the distribution $\ell(u)$.

Consequently, u is a weak solution if and only if $[\ell(\iota u)] \approx 0$. In that case, the association is unconditional.

Proof: Given $\omega \in \mathscr{D}(Q)$, $\varphi \in \mathcal{A}_0$, one has (see (26))

$$\int_Q \ell(\iota u)(\varphi_\varepsilon; x,t)\, \omega(x,t)\, dxdt =$$
$$- \int_Q \Big(\iota u(\varphi_\varepsilon; x,t)\partial_t\omega(x,t) + b(\iota u(\varphi_\varepsilon; x,t))\partial_x\omega(x,t)\Big)\, dxdt.$$

Now $\iota u(\varphi_\varepsilon; x,t) \to u(x,t)$ locally boundedly almost everywhere in Q as $\varepsilon \searrow 0$, hence the last integral converges to $\langle \ell(u), \omega\rangle$. □

Proposition 7. *For $u \in L^\infty_{loc}(Q)$, $b \in C^\infty(\mathbb{R})$ (the non-linearity of (24)), denote*

$$M = b(\iota u) - \iota b(u). \tag{27}$$

Then $M(\varphi_\varepsilon, x)$ is locally bounded (defined by (11)), tends to 0 for almost all x ($\varphi \in \mathcal{A}_0$, $\varepsilon \searrow 0$), is unconditionally associated to 0, and we have the characterization:
u is a weak solution to (24), if and only if

$$\ell(\iota u) = \partial_x M. \tag{28}$$

Proof: For the properties of M, see (12) with its proof. Using (26), (4), (3), we get

$$(28) \Leftrightarrow \partial_t \iota u + \partial_x b(\iota u) = \partial_x b(\iota u) - \iota\partial_x b(u) \Leftrightarrow \iota\left(\partial_t u + \partial_x b(u)\right) = 0.$$

As $[\iota]$ is injective, this means that the distribution $\partial_t u + \partial_x b(u)$ is equal to 0 in $\mathscr{D}'(Q)$, so that u is a weak solution. □

One observes, however, that there exist multiple weak solutions with the same initial condition: functions $u_1(x,t) = h(x-t)$, and

$$u_2(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{2t} & 0 < x < 2t \\ 1 & x > 2t \end{cases}$$

are weak solutions to

$$\partial_t u + \partial_x u^2 = 0 \tag{29}$$

with the same initial condition $u_i(x,0) = h(x)$, $i = 1, 2$.

Apparently, the concept of weak solution is too weak. One has the intuition that some information about u is lost in the term $b(u)$, in situations where the composition is interpreted pointwise near the points of discontinuity. This intuition seems to be also behind the concept of entropy solution.

Definition 4. Functions $\eta, \psi \in C^\infty(\mathbb{R})$ are called entropy/entropy flux pair for (24), if (i) η is convex and (ii) $\psi'(s) = b'(s)\eta'(s)$ for $\forall\, s \in \mathbb{R}$. We say that $u \in L^\infty_{\mathrm{loc}}(Q)$ is entropy solution to (24) if for all entropy/entropy flux pairs η, ψ one has

$$\int_Q \Big(\eta(u(x,t))\partial_t\omega(x,t) + \psi(u(x,t))\partial_x\omega(x,t)\Big)\,dxdt \geq 0 \tag{30}$$

for all non-negative $\omega \in \mathscr{D}(Q)$. This means that for all entropy/entropy flux pairs η, ψ, the distribution $\partial_t\eta(u) + \partial_x\psi(u)$ is a non-positive measure.

Behind this definition one finds the formal calculation:

$$\begin{aligned} \partial_t u + \partial_x b(u) &= 0 \\ \partial_t u + b'(u)\partial_x u &= 0 \qquad / \cdot \eta'(u) \\ \partial_t \eta(u) + \partial_x\psi(u) &= 0 \end{aligned}$$

This of course cannot be justified if u is only a weak solution. It turns out that entropy solution is a stronger concept than weak solution.

In the example above, u_1 is not an entropy solution; while u_2 is — in virtue of being sufficiently regular.

The importance of the concept of entropy solution is highlighted in the celebrated uniqueness result of Kružkov. Note that the time derivative of weak solutions lies in $L^\infty_{loc}(0,T;(W^{1,1}_{loc})')$, hence a suitable continuous (w.r. to time) representative can be defined (see e.g. [3, Theorem 4.1.1]). In particular, one can speak of value $u(t,\cdot)$ for every $t \geq 0$.

Theorem 1. *Let u, $\tilde{u} \in L^\infty_{\mathrm{loc}}(Q)$ be entropy solutions to* (24)*. Then*

$$\int_{-R}^{R} |u(x,t) - \tilde{u}(x,t)|\,dx \leq \int_{-R-K}^{R+K} |u(x,0) - \tilde{u}(x,0)|\,dx,$$

where $K > 0$ depends on t, L^∞-norm of u, $\tilde{u}$, and $b(\cdot)$. In particular, the entropy solution is uniquely determined by the initial condition.

Proof: See e.g. [3, Theorem 5.2.1]. □

6. Applications of the generalized sign

In this section we want to look at the equation (24) in the context of Colombeau theory. If u is a weak solution, one cannot expect that $[\ell(\iota u)] = 0$, i.e., the equation does not hold with the strict equality in $\mathscr{G}$. Our main objective is to characterize the weak and entropy solutions in terms of the properties of $[\ell(\iota u)]$. In particular, we aim to characterize the entropy solution in terms of its (generalized) sign properties.

We start with a simple observation.

Proposition 8. *Let $u \in L^\infty_{loc}(Q)$, $\eta \in C^\infty(\mathbb{R})$. Then $\ell(\iota u) \cdot \eta'(\iota u)$ is unconditionally associated to the distribution $\partial_t \eta(u) + \partial_x \psi(u)$, where ψ is a primitive to $b'\eta'$. Consequently, $u \in L^\infty_{loc}(Q)$ is an entropy solution to* (24) *if and only if for arbitrary non-decreasing $g \in C^\infty(\mathbb{R})$*

$$\ell(\iota u) \cdot g(\iota u) \lessapprox 0 \qquad \text{on } Q. \tag{31}$$

Proof: By (26), (3), (4) and (12),

$$\begin{aligned}\ell(\iota u) \cdot \eta'(\iota u) &= \big(\partial_t \iota u + b'(\iota u)\partial_x \iota u\big) \cdot \eta'(\iota u) \\ &= \partial_t \eta(\iota u) + \partial_x \psi(\iota u) \approx \partial_t \eta(u) + \partial_x \psi(u)\end{aligned}$$

and by (12) the association is unconditional. As η is convex in the case $g = \eta'$ is non-decreasing, the conclusion follows from Definition 4. □

Let us now consider solutions $u \in L^\infty_{\mathrm{loc}}(Q)$ with the special structure:

$$\begin{gathered} u(x,t) = \sigma(t) h(x - c(t)) + u_0(x,t), \\ \text{where } \sigma \neq 0,\ \sigma, c, u_0 \text{ are smooth.} \end{gathered} \tag{32}$$

In other words, the solution admits a jump discontinuity along the curve $x = c(t)$.

It can be shown (see [16, Theorem 5.9.6]) that the function of bounded variation is, roughly speaking, locally of such structure. Since the space BV is a natural setting for our problem (see [3]), the assumption (32) is in fact less restrictive than it might seem at the first sight.

Now, after the following lemma, we can formulate our main theorem. It claims that in the case of solutions (32), the entropy condition is equivalent to a certain sign condition of the "error" term M.

Lemma 4. *Let a weak solution u to* (24) *have the form* (32). *If M is defined by* (27), *then for arbitrary $\eta, \psi \in C^\infty(\mathbb{R})$ with $\psi' = b'\eta'$* (*Definition* 4(ii)), *the representatives*

$$M \cdot \partial_x\big(\eta'(\iota u)\big) \quad \text{and} \quad M(\varphi; x, t) \cdot \partial_x \eta'\big(\sigma(t) H_c(\varphi; x, t) + u_0(c(t), t)\big)$$

are unconditionally associated and both are unconditionally associated to the distribution $-\partial_t \eta(u) - \partial_x \psi(u) \in \mathscr{D}'(Q)$. In addition, the representative $M \cdot \partial_x\big(\eta' \circ H_c\big)$ is unconditionally associated to a distribution on Q.

Proof: By Proposition 7, $\ell(\iota u) = \partial_x M$. By Proposition 8,

$$\partial_x M \cdot \eta'(\iota u) = \ell(\iota u) \cdot \eta'(\iota u) \approx \partial_t \eta(u) + \partial_x \psi(u)$$

and the association is unconditional.
We have

$$\partial_x\big(M\cdot\eta'(\iota u)\big)=\partial_x M\cdot\eta'(\iota u)+M\cdot\partial_x\eta'(\iota u).$$

As M and $\eta'(\iota u)$ are locally bounded and the representatives $M(\varphi_\varepsilon;x,t)$ tend to 0 ($\forall\,\varphi\in\mathcal{A}_0$, $\varepsilon\searrow 0$) almost everywhere (Proposition 7), one easily deduces by the Lebesgue majorization theorem that $M\cdot\eta'(\iota u)$ is unconditionally associated to 0. So is $\partial_x\big(M\cdot\eta'(\iota u)\big)$ (see (10)) and we deduce

$$M\cdot\partial_x\big(\eta'(\iota u)\big)\approx-\partial_t\eta(u)-\partial_x\psi(u)\tag{33}$$

and the association is unconditional. For u of the form (32), the left-hand side of the association (33) reads

$$M\cdot\eta''(\iota u)\cdot\partial_x\iota u=M\cdot\eta''\big(\iota(\sigma(t)h(x-c(t)))+\iota u_0\big)\cdot\big(\partial_x\iota\big(\sigma(t)h(x-c(t))\big)+\partial_x\iota u_0\big).$$

As above, we deduce by the Lebesgue majorization theorem that
$M\cdot\eta''(\iota u)\cdot\partial_x\iota u_0$ is unconditionally associated to 0. Hence the left-hand side of (33) is unconditionally associated to

$$M\cdot\eta''\big(\iota\big(\sigma(t)h(x-c(t))\big)+\iota u_0\big)\cdot\partial_x\iota\big(\sigma(t)h(x-c(t))\big).$$

By the same token, using Lemma 1(ii), this is unconditionally associated to

$$M(\varphi;x,t)\cdot\eta''\big(\iota\big(\sigma(t)h(x-c(t))\big)(\varphi;x,t)+\iota u_0(\varphi;x,t)\big)\cdot\sigma(t)\Delta_c(\varphi;x,t).$$

By Lemma 1(i) and (iii), this is unconditionally associated to

$$M(\varphi;x,t)\cdot\eta''\big(\sigma(t)H_c(\varphi;x,t)+\iota u_0(\varphi;x,t)\big)\cdot\sigma(t)\Delta_c(\varphi;x,t)$$

and similarly also to (cf. (8))

$$M(\varphi;x,t)\cdot\eta''\big(\sigma(t)H_c(\varphi;x,t)+u_0(x,t)\big)\cdot\sigma(t)\Delta_c(\varphi;x,t).\tag{34}$$

Thanks to (33), we see that this representative is unconditionally associated to the distribution $-\partial_t\eta(u)-\partial_x\psi(u)$. Now we prove that $u_0(x,t)$ can be replaced in the last expression with $u_0(c(t),t)$. Indeed, the unconditional association of (34) is defined using the limit

$$\lim_{\varepsilon\searrow 0}\int M(\varphi_\varepsilon;x,t)\cdot\eta''\big(\sigma(t)H_c(\varphi_\varepsilon;x,t)+u_0(x,t)\big)\\ \cdot\sigma(t)\Delta_c(\varphi_\varepsilon;x,t)\,\omega(x,t)\,dxdt.\tag{35}$$

First, thanks to the integral expression (20) of Δ_c, this is equal to

$$\lim_{\varepsilon\searrow 0}\int M(\varphi_\varepsilon;x,t)\cdot\eta''\big(\sigma(t)H_c(\varphi_\varepsilon;x,t)+u_0(x,t)\big)\\ \cdot\sigma(t)\varphi_\varepsilon(c(t+s)-x,s)\,\omega(x,t)\,dxdsdt$$

and then, substituting x, we obtain

$$\lim_{\varepsilon\searrow 0}\int M(\varphi_\varepsilon; x+c(t+s),t) \cdot \eta''\big(\sigma(t)H_c(\varphi_\varepsilon; x+c(t+s),t)+u_0(x+c(t+s),t)\big) \cdot \sigma(t)\varphi_\varepsilon(-x,s)\,\omega(x+c(t+s),t)\,dxdsdt.$$

If e.g. supp $\varphi(x,s)$ is contained in $|x|\le k$, $|s|\le k$, and supp $\omega(x,t)$ is contained in $|x|\le k$, $|t|\le k$, we can restrict the integration on $|x|\le k\varepsilon$, $|s|\le k\varepsilon$, $|t|\le k$. Replacing $u_0(x,t)$ with $u_0(c(t),t)$ in (34) only results in replacing the term $u_0(x+c(t+s),t)$ in the last integral with $u_0(c(t),t)$. As M and H_c are locally bounded, u_0 and η'' locally Lipschitz and $|\varphi_\varepsilon|\le\frac{1}{\varepsilon^2}\max|\varphi|$, this replacement has no effect on the limit (35). So we obtain that the representative

$$\begin{aligned} &M(\varphi;x,t)\cdot\partial_x\eta'\big(\sigma(t)H_c(\varphi;x,t)+u_0(c(t),t)\big)\\ &= M(\varphi;x,t)\cdot\eta''\big(\sigma(t)H_c(\varphi;x,t)+u_0(c(t),t)\big)\cdot\sigma(t)\Delta_c(\varphi;x,t)\end{aligned} \tag{36}$$

is unconditionally associated to (34) and by (33) also to the distribution $-\partial_t\eta(u)-\partial_x\psi(u)$. Finally, we apply Lemma 3 for $R=M$, $\gamma=g'$, $g=\eta'$, $u(t)=u_0(c(t),t)$. As $\eta''\in C^\infty(\mathbb{R})$ is an arbitrary nonnegative function and the statement (i) of Lemma 3 depends neither on $\sigma(t)$ nor on $u_0(c(t),t)$, we can choose $\sigma(t)=1$ and $u(t)=u_0(c(t),t)=0$ in the equivalent statement (ii). We get $M\cdot(\eta''\circ H_c)\cdot\Delta_c = M\cdot\partial_x(\eta'\circ H_c)$ is unconditionally associated to a distribution and the lemma is proved. $\square$

Theorem 2. *Let a weak solution u to* (24) *have the form* (32)*. Then the following are equivalent.*

(1) *u is an entropy solution to* (24).
(2) *For the representative M defined by* (27),

$$[\sigma(t)M(\varphi;x,t)]\,\{x=c(t)\}\ge 0. \tag{37}$$

To put it loosely, the generalized sign of $[M]$ on $x=c(t)$ is (*non-strictly*) *the same as the sign of the jump $\sigma(t)$.*

Proof: By Definition 4, the assertion (1) of the theorem is equivalent to: for arbitrary convex $\eta\in C^\infty(\mathbb{R})$ and $\psi'=b'\eta'$, $\partial_t\eta(u)+\partial_x\psi(u)$ is a non-positive measure. Consequently, by Lemma 4, the assertion (1) of the theorem is equivalent to: the representative

$$M(\varphi;x,t)\cdot\partial_x\eta'\big(\sigma(t)H_c(\varphi;x,t)+u_0(c(t),t)\big)$$

is unconditionally associated to a non-negative measure. The last expression is equal to (see (20))

$$M(\varphi;x,t)\cdot\eta''\big(\sigma(t)H_c(\varphi;x,t)+u_0(c(t),t)\big)\cdot\sigma(t)\Delta_c(\varphi;x,t).$$

For an arbitrary convex function $\eta \in C^\infty(\mathbb{R})$, $\eta'' \in C^\infty(\mathbb{R})$ is an arbitrary non-negative function and we can use the equivalence of assertions (i) and (ii) of Lemma 3 (R means M, $u(t)$ means $u_0(c(t),t)$). The assertion (i) is independent on σ and u, so we can choose $\sigma = 1$ and $u = 0$ in the assertion (ii), too.

Thus, we obtain that the assertion (1) of the theorem is equivalent to: For arbitrary non-negative $\gamma \in C^\infty(\mathbb{R})$, the representative $M\gamma(H_c)\Delta_c = M\partial_x(g \circ H_c)$ (where $g' = \gamma$) is unconditionally associated to a non-negative distribution.
It is sufficient to say "associated" instead of "unconditionally associated", because by the previous lemma the association is automatically unconditional. The theorem follows by Definition 2. □

7. Examples

1. Consider the equation

$$\partial_t u + \partial_x u^2 = 0$$

and set $u = h(x-t)$. This has the special form (32) with $\sigma(t) = 1$. Denoting $\iota u = H_{x-t}$, we have

$$\ell(\iota u) = \partial_t H_{x-t} + \partial_x \big(H_{x-t}\big)^2 = \partial_x M,$$

where, as one easily verifies, $\partial_t H_{x-t} = -\partial_x H_{x-t}$, and thus we can take

$$M = \big(H_{x-t}\big)^2 - H_{x-t}.$$

Obviously $M \approx 0$ (unconditionally), and by Theorem 7 we see that u is a weak solution. On the other hand, by Proposition 5, $[M](x=t) \geq 0$ does not hold. Thus u is not an entropy solution.

2. Consider the equation

$$\partial_t u + \partial_x \frac{u^4}{2} = 0$$

and $u = h(-2x+t) = 1 - h(x - t/2)$. One has $\iota u = 1 - H_{x-t/2}$. Hence

$$\ell(\iota u) = \partial_t(1 - H_{x-t/2}) + \partial_x \tfrac{1}{2}\big(1 - H_{x-t/2}\big)^4 = \partial_x M,$$

where

$$M = \tfrac{1}{2}\big[\big(1 - H_{x-t/2}\big)^4 + H_{x-t/2} - 1\big].$$

Clearly $M \approx 0$, hence u is a weak solution.

One can write $M = m(H_{x-t/2})$, where $m(s) = ((1-s)^4 + s - 1)/2$, which is negative for $s \in (0,1)$. Hence, the generalized sign of M at $x = t/2$ agrees with the sign of the jump $\sigma(t) = -1$. Thus u satisfies the entropy condition.

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Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, CZ-186 75 Prague 8, Czech Republic

Email: jelinek@karlin.mff.cuni.cz
prazak@karlin.mff.cuni.cz

(*Received* May 20, 2008, *revised* January 20, 2009)