# Notes on semimedial semigroups

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*Abstract.* The class of semigroups satisfying semimedial laws is studied. These semigroups are called semimedial semigroups. A connection between semimedial semigroups, trimedial semigroups and exponential semigroups is presented. It is proved that the class of strongly semimedial semigroups coincides with the class of trimedial semigroups and the class of dimedial semigroups is identical with the class of exponential semigroups.

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The present ultrashort note collects a few observations concerning semigroups satisfying some generalized forms of the medial law. Such semigroups were studied in [1] and [11]. Another related class is that of selfdistributive semigroups considered in [2]. Our principal aim in this paper is to characterize semigroups that are semimedial. Throughout this paper we will show under which conditions the class of semimedial semigroups coincides with some special classes of semigroups. In Section 1 we introduce some basic concepts and results needed for Section 2. In Section 2 we show that the class of strongly semimedial semigroups is trimedial and conversely. Finally, it is shown that the dimedial semigroups are exactly the exponential semigroups.

## 1. Preliminaries

A semigroup $S$ is called:

- medial if it satisfies the equation $xyzu = xzyu$;
- trimedial (dimedial, resp.) if every subsemigroup of $S$ generated by at most three (two, resp.) elements is medial;
- left (right or middle, resp.) semimedial if it satisfies the identity $x^2yz = xyxz$ ($zyx^2 = zxyx, xyzx = xzyx$, resp.);
- semimedial if it is both left and right semimedial;
- strongly semimedial if it is semimedial and middle semimedial;
- exponential $(xy)^n = x^n y^n$ for every positive integer $n$;
- left (right, resp.) distributive if it is satisfies $xyz = xyxz$ ($zyx = zxyx$).

**1.1 Proposition.** (i) *Every commutative semigroup is medial.*

(ii) *Every medial semigroup is trimedial.*

(iii) *Every trimedial semigroup is dimedial and strongly semimedial.*
(iv) *Every dimedial semigroup is exponential.*

Proof: It is easy. □

**1.2 Remark.** (i) Trimedial semigroups form an equational class. We will show in Proposition 2.1 that this class coincides with the class of strongly semimedial semigroups and consequently, the equational class of trimedial semigroup is finitely based. If $F$ is the free trimedial semigroup over a four-element set $\{a, b, c, d\}$, then $abcd \neq acbd$ in $F$, and hence $F$ is not medial.

(ii) Dimedial semigroups form an equational class and this class is identical with the class of exponential semigroups (see Proposition 2.4). It is not clear whether if is a finitely based equational class. If $F$ is the free dimedial semigroup over a three-element set $\{a, b, c\}$ then $aabc \neq abac$, $cbaa \neq caba$ and $abca \neq acba$. Consequently, $F$ is neither left nor right nor middle semimedial.

**1.3 Remark.** If $F$ is the free semimedial semigroup over a three-element set $\{a, b, c\}$ then $abca \neq acba$ and it follows that $F$ is not strongly semimedial.

**1.4 Proposition** ([2]). *A left distributive semigroup $S$ is left semimedial if and only if $S$ satisfies the equation $xy^2 = x^2y^2$.*

**1.5 Proposition** ([2]). *The following conditions are equivalent for a left distributive semigroup $S$:*

(i) *$S$ is right semimedial;*
(ii) *$S$ is middle semimedial;*
(iii) *$S$ is medial.*

**1.6 Corollary.** *Let $S$ be an idempotent semigroup. Then:*

(i) *$S$ is left semimedial if and only if $S$ is left distributive;*
(ii) *$S$ is semimedial if and only if $S$ is medial.*

## 2. Main results

**2.1 Proposition.** *A semigroup is trimedial if and only if it is strongly semimedial.*

Proof: Let $W$ denote the free strongly semimedial semigroup of words over a three-element alphabet $A$. Proceeding by induction on the length of words, we are going to show that $uvwd = uwvd$ $(\forall u, v, w, d \in W)$. If $u \notin \mathbf{A}$, then $u = au_1$ for some $a \in A$ and $u_1 \in W$. Now, we have $u_1vwd = u_1wvd$ by induction, and therefore $uvwd = au_1vwd = au_1wvd = uwvd$. Similarly, if $d \notin \mathbf{A}$, and hence we can assume that $u, d \in A$. If $u = d$, then $uvwd = uvwu = uwvu = uwvd$ by the middle semimedial law and consequently, we will assume that $u = a \in A$, and $d = b \in A$, where $a \neq b$.

If $v \notin \mathbf{A}$, then $v = dv_1$ for some $d \in A$, $v_1 \in W$ and we have $uvwd = udv_1wd = udwv_1d = uwdv_1d = uwvd$ by induction. Similarly, if $w \notin \mathbf{A}$.

Finally, if $v = a$, then $uvwd = aawd = awad = uwvd$ by the left semimedial law and, if $v = b$, then $uvwd = ubwb = uwbb = uwvd$ by the right semimedial law. The final choice is $v = c \in A$, where $A = \{a, b, c\}$. Proceeding similarly, we can restrict ourselves to the case $w = c$. Then $uvwd = uccd = uwvd$ trivially. □

**2.2 Proposition.** *Every semimedial semigroup is dimedial.*

PROOF: Similar to that of Proposition 2.1. □

**2.3 Proposition.** *Every semimedial semigroup is exponential.*

**2.4 Proposition.** *A semigroup is dimedial if and only if it is exponential.*

PROOF: Similar to that of Proposition 2.1. □

**2.5 Proposition.** *A semigroup is exponential if and only if it is $p$-exponential ($(xy)^p = x^p y^p$) for every prime $p$.*

PROOF: Let $S$ be a semigroup that is $p$-exponential for every prime $p$. Employing induction, we show that $S$ is $n$-exponential for every positive integer $n$. The case $n \leq 3$ is clear, and hence we can assume that $n \geq 4$. Let $p$ be a prime number dividing $n$. Then $(xy)^n = ((xy)^p))^{n/p} = (x^p y^p)^{n/p} = x^n y^n$. □

**2.6 Remark** ([8], [10])**.** (i) Let $S$ be a 2-exponential semigroup. If $a, b \in S$ are such that $b \in S^1 a S^1$, then $b = xay$ for some $x, y \in S^1$ and we get $b^2 = (xay)^2 = x^2 a^2 y^2$. Thus $b^2 \in S^1 a^2 S^1$.

(ii) Let $S$ be an exponential semigroup. Define a relation $r$ on $S$ by $(a, b) \in r$ if $a^m \in SbS$ and $b^n \in SaS$ for some positive integers $m$ and $n$. It is rather easy to check that $r$ is just the smallest congruence of $S$ such the corresponding factor-semigroup of $S$ is a semilattice.

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