Relatively pseudocomplemented directoids

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Abstract. The concept of relative pseudocomplement is introduced in a commutative directoid. It is shown that the operation of relative pseudocomplementation can be characterized by identities and hence the class of these algebras forms a variety. This variety is congruence weakly regular and congruence distributive. A description of congruences via their kernels is presented and the kernels are characterized as the so-called p-ideals.

Keywords: directoid, relative pseudocomplementation, filter, congruence distributivity, congruence weak regularity

Classification: 06A12, 06D15, 08B10

By a *directoid* (a commutative directoid in sense of [3]) we understand a groupoid $\mathcal{D} = (D; \sqcap)$ satisfying the identities

(D1) $x \sqcap x = x$;

(D2) $x \sqcap y = y \sqcap x$;

(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$.

It is known (see [3]) that the relation $\leq$ defined on D by

$$x \leq y \quad \text{if and only if} \quad x \sqcap y = x$$

is an order and for any $x, y \in D$ we have $x \sqcap y \leq x$, $x \sqcap y \leq y$. Also conversely, if $(D; \leq)$ is downward directed ordered set and for any $x, y \in D$ we define $x \sqcap y = y \sqcap x \in L(x, y) = \{z \in D; z \leq x, z \leq y\}$ arbitrarily if x, y are non-comparable and $x \sqcap y = y \sqcap x = x$ if $x \leq y$ then the resulting algebra $(D; \sqcap)$ is a directoid.

The concept of pseudocomplementation was introduced for directoids by the author in [1]. Our aim here is to extend the concept of relative pseudocomplement from semilattices or lattices (see e.g. [2], [4]) to directoids.

If $(S; \wedge)$ is a meet-semilattice and $a, b \in S$, a relative pseudocomplement of a with respect to b is a greatest element x (if it exists) such that

$$a \wedge x \leq b.$$

It is easy to check that this condition is equivalent to

$$a \wedge x = a \wedge b.$$

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However, if a directoid is considered instead of a semilattice, these conditions are not equivalent, see the following

Example 1. Let $\mathcal{D} = (D; \sqcap)$ be a directoid whose diagram is visualized in Figure 1

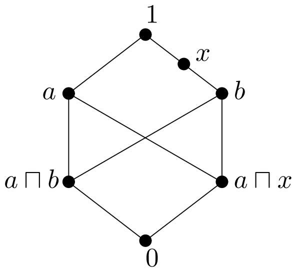

Figure 1

Then clearly, $a \sqcap x \leq b$ but $a \sqcap x \neq a \sqcap b$, in fact the elements $a \sqcap x$, $a \sqcap b$ are non-comparable. $\diamondsuit$

The situation explained in Example 1 is caused by the fact that $a \leq b$ in a directoid $\mathcal{D}$ does not imply $a \sqcap c \leq b \sqcap c$ for any $c \in D$. It holds if and only if $\mathcal{D}$ is a semilattice. To avoid these difficulties, we define

Definition. Let $\mathcal{D} = (D; \sqcap)$ be a directoid, $a, b \in D$. By a *relative pseudocomplement of* a *with respect to* b, $a * b$ in symbol, is meant a greatest element x of D such that $a \sqcap x = a \sqcap b$. A directoid $\mathcal{D}$ is called *relatively pseudocomplemented* if there exists $a * b$ for every $a, b \in D$. We will denote by $\mathcal{D} = (D; \sqcap, *)$ a relatively pseudocomplemented directoid.

In what follows, we will suppose the priority of the operation $*$ and hence we will write e.g. $x \sqcap y * z$ instead of $x \sqcap (y * z)$.

Example 2. Consider the directoid as shown in Figure 2

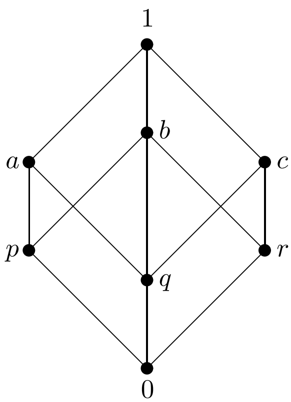

Figure 2

where $a \sqcap b = p$, $a \sqcap c = q$, $b \sqcap c = r$. Then there does not exist a greatest element $x \in D$ with $a \sqcap x \leq b$ since $a \sqcap b = p \leq b$, $a \sqcap c = q \leq b$ but $a \sqcap 1 = a \nleq b$. On the other hand, there is a greatest element $x = b$ with $a \sqcap x = a \sqcap b$ since $a \sqcap c = q \neq p = a \sqcap b$, thus $a * b$ exists and equals to b. $\diamondsuit$

Example 3. An example of relatively pseudocomplemented directoid is depicted in Figure 3

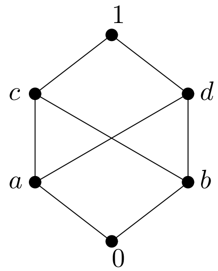

Figure 3

where for non-comparable elements we define $c \sqcap d = a$. Then, relative pseudocomplements are given by the table

$*$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	d	b	1	d	1
d	0	c	b	c	1	1
1	0	a	b	c	d	1

$\diamondsuit$

Theorem 1. *Every relatively pseudocomplemented directoid has a greatest element which is equal to $a * a$ for each $a \in D$.*

Proof: Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid and $a, b \in D$. Let $p = a \sqcap b$. Then $p \leq a$, $p \leq b$ and hence for $p * a$ we have $p \sqcap p * a = p \sqcap a = p$ thus $a \leq p * a$. However $p \leq b$ thus $p \sqcap b = p = p \sqcap a = p \sqcap p * a$ whence $p * a \leq p * b$. Interchanging a and b in the previous reasoning we conclude that $p * a = p * b$. Since $b \leq p * b$, we have that $p * a$ is a common upper bound of both a, b, i.e. $(D; \leq)$ is an upward directed set.

Now, let $a, b \in D$. We have that $a \leq a * a$ and $b \leq b * b$ where $a * a$ is a greatest element with $a \sqcap a * a = a$, i.e. a greatest element over a and $b * b$ is a greatest element over b. Since $(D; \leq)$ is upward directed, it easily yields that $a * a = b * b$, i.e. $a * a$ is a greatest element of $(D; \leq)$. $\square$

For a relatively pseudocomplemented directoid, its greatest element will be denoted by 1.

Lemma 1. *Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid. Then*

(a) $1 * x = x$;
(b) $a \leq b$ *if and only if* $a * b = 1$;
(c) $b \leq a * b$;
(d) $a * b = a * (a \sqcap b)$.

Proof: (a) Since $1 \sqcap x = x$ for each $x \in D$, we get $1 * x = x$ immediately.

(b) Assume $a \leq b$. Then $a \sqcap b = a$ and hence $a \sqcap 1 = a = a \sqcap b$ gets $1 = a * b$. Conversely, if $a * b = 1$ then $a = a \sqcap 1 = a \sqcap b$ whence $a \leq b$.

(c) Since $a * b$ is a greatest element with $a \sqcap a * b = a \sqcap b$, we have $b \leq a * b$.

(d) It follows immediately by the fact that $a \sqcap b = a \sqcap (a \sqcap b)$. □

We are going to show that also conversely, the properties (c), (d) and those of Theorem 1 and of the Definition characterize the operation of relative pseudocomplementation.

Theorem 2. *Let $\mathcal{D} = (D; \sqcap)$ be commutative directoid and $*$ be a binary operation on D. Then $\mathcal{D} = (D; \sqcap, *)$ is relatively pseudocomplemented if and only if it satisfies the following identities*

(S1) $x \sqcap (x * y) = x \sqcap y$;

(S2) $(x * y) \sqcap y = y$;

(S3) $x * y = x * (x \sqcap y)$;

(S4) $x * x = y * y$.

Proof: If $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid then it satisfies (S1)–(S4) directly by the Definition and Theorem 1 and Lemma 1.

Conversely, assume that a directoid $(D; \sqcap)$ with $*$ satisfies (S1)–(S4). By (S2) we have $(y * y) \sqcap y = y$ and, by (S4) we conclude $(x * x) \sqcap y = y$ for each $x, y \in D$ thus $(D; \sqcap)$ has a greatest element $1 = x * x$.

Suppose $a, b \in S$ and $a \sqcap x = a \sqcap b$ for some $x \in D$. By (S3) we obtain

$$a * x = a * (a \sqcap x) = a * (a \sqcap b) = a * b$$

thus, due to (S2), $x \leq a * x = a * b$. By (S1), we have $a \sqcap (a * b) = a \sqcap b$ i.e. $a * b$ is a greatest element x of D satisfying $a \sqcap x = a \sqcap b$ and hence a relative pseudocomplement of a with respect to b. □

Corollary. *The class of all relatively pseudocomplemented directoids is a variety presented by the identities* (D1)–(D3), (S1)–(S4).

Denote by $\mathcal{V}$ the variety of relatively pseudocomplemented directoids. Recall that a variety with a constant 1 is *weakly regular* if for any algebra $\mathcal{A}$ of this variety, every congruence $\theta \in \mathrm{Con}\mathcal{A}$ is determined by its 1-class $[1]_\theta$. By Csákány Theorem (see e.g. [2]), a variety is weakly regular if and only if there exist binary terms $b_1(x,y), \dots, b_n(x,y)$ such that

$$\text{(W)} \qquad b_1(x,y) = \dots = b_n(x,y) = 1 \quad \text{if and only if} \quad x = y.$$

We can state the following

Theorem 3. *The variety $\mathcal{V}$ of relatively pseudocomplemented directoids is weakly regular.*

Proof: Consider the term $b_1(x,y) = (x*y) \sqcap (y*x)$. If $x = y$ then, by Theorem 1, $b_1(x,x) = (x*x) \sqcap (x*x) = 1 \sqcap 1 = 1$. Conversely, let $b_1(x,y) = 1$. Since 1 is the greatest element of this directoid, it yields that $x * y = 1$ and $y * x = 1$. By (b) of Lemma 1 we have $x \leq y$ and $y \leq x$ thus $x = y$. By (W), $\mathcal{V}$ is weakly regular. $\square$

It means that congruences on a relatively pseudocomplemented directoid will be fully described when the congruence kernels are known. In what follows we get this description.

Let $\mathcal{D} = (D; \sqcap)$ be a directoid. A non-void subset $F \subseteq D$ is called a *filter of $\mathcal{D}$* if it satisfies the following conditions:

(i) if $a, b \in F$ then also $a \sqcap b \in F$;
(ii) if $a \in F$ and $a \leq x$ for $x \in D$ then $x \in F$.

For relatively pseudocomplemented directoids, filters can be characterized as the so-called deductive systems, i.e. as subset closed under Modus Ponens, see the following

Lemma 2. *Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid and $F \subseteq D$. The following are equivalent:*

(a) *F is a filter of $\mathcal{D}$;*
(b) *$1 \in F$ and if $x \in F$ and $x * y \in F$ then also $y \in F$.*

Proof: (a)$\Rightarrow$(b): Since $F \neq \emptyset$, there is $a \in F$. By (ii) of the previous definition we conclude $1 \in F$ since $a \leq 1$. Assume $x \in F$ and $x * y \in F$. Then, by (i) and the definition of relative pseudocomplement we have

$$x \sqcap y = x \sqcap x * y \in F,$$

thus $x \sqcap y \leq y$ yields $y \in F$.

(b)$\Rightarrow$(a): Assume $b \in F$ and $b \leq a$. Then $b * a = 1 \in F$ and, due to (b), also $a \in F$.

Assume now that $c, d \in F$. By Lemma 1(c) we have $d \leq c * d$ thus, as already shown, also $c * d \in F$. However $c * (c \sqcap d) = c * d$ by Lemma 1(d). Since $c \in F$, the condition (b) yields $c \sqcap d \in F$ and hence F is a filter of $\mathcal{D}$. $\square$

Lemma 3. *Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid and $\theta \in \mathrm{Con}\mathcal{D}$. Then $[1]_\theta$ is a filter of $\mathcal{D}$.*

Proof: It is evident that $1 \in [1]_\theta$. Further, $[1]_\theta$ is certainly closed under $\sqcap$ since $1 \sqcap 1 = 1$ and θ is a congruence on $\mathcal{D}$. Moreover, if $x \in [1]_\theta$ and $x \leq y$, then $x = x \sqcap y \,\theta\, 1 \sqcap y = y$, so $y \in [1]_\theta$. Hence $[1]_\theta$ is a filter of $\mathcal{D}$. □

Theorem 4. *Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid, $\theta \in \mathrm{Con}\mathcal{D}$. Then*

$$\langle x, y\rangle \in \theta \quad \textit{if and only if} \quad x * y \sqcap y * x \in [1]_\theta.$$

Proof: Let $\langle x, y\rangle \in \theta$. Then $\langle 1, x * y\rangle = \langle x * x, x * y\rangle \in \theta$ and $\langle y * x, 1\rangle = \langle y * x, y * y\rangle \in \theta$. Hence $x * y, y * x \in [1]_\theta$. By Lemma 3, $[1]_\theta$ is a filter thus also $x * y \sqcap y * x \in [1]_\theta$.

Conversely, assume $x * y \sqcap y * x \in [1]_\theta$. Then $\langle x * y \sqcap y * x, 1\rangle \in \theta$ thus also

$$\langle x * y \sqcap y * x, x * y\rangle = \langle (x * y) \sqcap (x * y \sqcap y * x), (x * y) \sqcap 1\rangle \in \theta$$

and, analogously, $\langle x * y \sqcap y * x, y * x\rangle \in \theta$. Using transitivity, we get $\langle x * y, y * x\rangle \in \theta$. This yields

$$\langle x \sqcap y, x\rangle = \langle x \sqcap x * y, x \sqcap y * x\rangle \in \theta \qquad \text{and}$$

$$\langle y, x \sqcap y\rangle = \langle y \sqcap x * y, y \sqcap y * x\rangle \in \theta$$

whence $\langle x, y\rangle \in \theta$. □

To describe congruence kernels, let us introduce the following concept.

A filter F of a relatively pseudocomplemented directoid is called a *p-filter* if it satisfies the following condition

$$\text{if } x * y \in F \text{ and } y * x \in F \text{ then}$$
$$(x * z) * (y * z) \in F \text{ and } (z * x) * (z * y) \in F$$
$$\text{and } (x \sqcap z) * (y \sqcap z) \in F \text{ for each } z \in D.$$

Theorem 5. *Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid. A subset $F \subseteq D$ is a congruence kernel if and only if F is a p-filter. If F is a p-filter of $\mathcal{D}$ then it induces the congruence θ_F given by $\langle x, y\rangle \in \theta_F$ if and only if $x * y \sqcap y * x \in F$.*

Proof: Let $\theta \in \mathrm{Con}\mathcal{D}$ and $F = [1]_\theta$. By Lemma 3, F is a filter of $\mathcal{D}$. Assume $x * y \in F$ and $y * x \in F$. Then also $x * y \sqcap y * x \in F$ and, by Theorem 4, $\langle x, y\rangle \in \theta$. Thus also $\langle x * z, y * z\rangle \in \theta, \langle z * x, z * y\rangle \in \theta$ and $\langle x \sqcap z, y \sqcap z\rangle \in \theta$. By Theorem 4 we easily conclude that F is a p-filter.

Conversely, let F be a p-filter of $\mathcal{D}$. Define $\langle x, y\rangle \in \theta_F$ if and only if $x * y \sqcap y * x \in F$. Since $1 \in F$, the relation θ_F is reflexive. Symmetry of θ_F follows immediately. Assume $\langle x, y\rangle \in \theta_F$ and $z \in D$. Then $x * y \sqcap y * x \in F$ thus also $x * y, y * x \in F$. Since F is a p-filter, we conclude $(x * z) * (y * z), (z * x) * (z * y) \in F$. Moreover, the same memberships with x and y permuted follow as the condition "$x * y \in F$

and $y*x \in F$" is symmetric, so we get also $(y*z)*(x*z) \in F$, $(z*y)*(z*y) \in F$. Thus $(x*z)*(y*z) \sqcap (y*z)*(x*z) \in F$ whence

$$\langle x*z, y*z \rangle \in \theta_F.$$

Analogously it can be shown $\langle z*x, z*y \rangle \in \theta_F$ and $\langle x \sqcap z, y \sqcap z \rangle \in \theta_F$.

Assume now that also $\langle y, z \rangle \in \theta_F$. Then $y*z, z*y \in F$. Since $y*z \in F$ and $(y*z)*(x*z) \in F$, by Lemma 2 also $x*z \in F$. Analogously we check $z*x \in F$ and hence $x*z \sqcap z*x \in F$ giving $\langle x, z \rangle \in \theta_F$. Altogether, θ_F is also transitive and hence an equivalence on D and, due to the previous compatibility conditions, θ_F is a congruence on $\mathcal{D}$. The fact that $F = [1]_{\theta_F}$ follows directly by Theorem 4. $\square$

Theorem 6. *The variety $\mathcal{V}$ of relatively pseudocomplemented directoids is congruence distributive.*

Proof: We need only to find Jónsson terms. For this, take

$$t_0(x,y,z) = x = t_1(x,y,z), \quad t_2(x,y,z) = x \sqcap y*z,$$
$$t_3(x,y,z) = x \sqcap z, \quad t_4(x,y,z) = z \sqcap y*x, \quad t_5(x,y,z) = z.$$

Then clearly $t_0(x,y,x) = t_1(x,y,x) = t_5(x,y,x) = x$ and

$$t_2(x,y,x) = x \sqcap y*x = x, \quad t_3(x,y,x) = x \sqcap x = x,$$
$$t_4(x,y,x) = x \sqcap y*x = x.$$

For i even we have

$$t_0(x,x,y) = x = t_1(x,x,y)$$
$$t_2(x,x,y) = x \sqcap x*y = x \sqcap y = t_3(x,x,y)$$
$$t_4(x,x,y) = y \sqcap x*x = y \sqcap 1 = y = t_5(x,x,y).$$

For i odd we compute

$$t_1(x,y,y) = x = x \sqcap 1 = x \sqcap y*y = t_2(x,y,y)$$
$$t_3(x,y,y) = x \sqcap y = y \sqcap x = y \sqcap y*x = t_4(x,y,y).$$

Hence, $t_0, \ldots, t_5$ are Jónsson terms and thus $\mathcal{V}$ is congruence distributive. $\square$

It is well-known that every relatively pseudocomplemented lattice is distributive. It is not the case of pseudocomplemented directoids (where the second operation can be established) since distributivity of such an algebra yields that it is a lattice, see e.g. [2] or [5]. However, a certain form of distributivity can be considered. If $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid and $M \subseteq D$, denote by $\bigvee M$ the supremum of M provided it exists. An easy transcription of the definition of relative pseudocomplementation shows that $\mathcal{D}$

satisfies the identity

$$x \sqcap \bigvee\{z; x \sqcap z = x \sqcap y\} = x \sqcap y.$$

In what follows we show that relatively pseudocomplemented directoids need not be even distributive ordered sets (see e.g. [6]). For our reasons, let us modify the definition from [6] as follows:

We say that a commutative directoid $\mathcal{D} = (D; \sqcap)$ is *order-distributive* if it satisfies the condition

(D) $$U(a \sqcap b, c) = U(a, c) \sqcap U(b, c) \quad \text{for all} \quad a, b, c \in D,$$

where $U(x, y) = \{z \in D; x \leq z \text{ and } y \leq z\}$ and for subsets $A, B \subseteq D$ we put $A \sqcap B = \{a \sqcap b; a \in A, b \in B\}$ if $A \neq \emptyset \neq B$ and $A \sqcap B = \emptyset$ else. Moreover, denote $U(x) = \{y \in D; x \leq y\}$.

Recall that a meet-semilattice $(S, \wedge)$ is *distributive* if for any $a, b \in S$ and each $c \geq a \wedge b$ there exist $a_1 \geq a$, $b_1 \geq b$ such that $c = a_1 \wedge b_1$.

We can state the following

Theorem 7. *A commutative directoid $\mathcal{D} = (D; \sqcap)$ is order-distributive if and only if it is a distributive meet-semilattice.*

PROOF: Assume that $\mathcal{D} = (D; \sqcap)$ is not a semilattice. Then there are $a, b \in D$ such that either $a \wedge b$ exists but $a \sqcap b < a \wedge b$ or $a \wedge b$ does not exist.

Let $\mathcal{D} = (D; \sqcap)$ be order-distributive.

(a) If $a \wedge b$ exists and $a \sqcap b < a \wedge b = p$, then $U(a \sqcap b, p) = U(p)$ but $p \leq a$, $p \leq b$ yield $U(a, p) = U(a)$, $U(b, p) = U(b)$ and hence

$$a \sqcap b \in U(a) \sqcap U(b) = U(a, p) \sqcap U(b, p), \quad a \sqcap b \notin U(p) = U(a \sqcap b, p)$$

which is a contradiction.

(b) If $a \wedge b$ does not exist then there exists $q \in D$ such that $q \leq a$, $q \leq b$ but $q \parallel a \sqcap b$, see Figure 4.

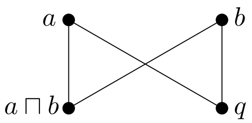

Figure 4

We have $a \sqcap b \in U(a) \sqcap U(b) = U(a, q) \sqcap U(b, q)$ but $a \sqcap b \notin U(a \sqcap b, q)$, a contradiction.

(c) Assume that $\mathcal{D} = (D; \sqcap)$ is a $\wedge$-semilattice, i.e. $a \wedge b$ exist for all $a, b \in D$ and $a \wedge b = a \sqcap b$. Let $a, b \in D$ and $c > a \wedge b$. Then $U(a \wedge b, c) = U(c)$. Since $U(a \wedge b, c) = U(a, c) \wedge U(b, c)$, we have $c \in U(a \wedge b, c)$ and hence there are

$a_1 \in U(a,c)$, $b_1 \in U(b,c)$ such that $c = a_1 \wedge b_1$. Evidently, $a_1 \geq a$, $b_1 \geq b$ thus $\mathcal{D}$ is a distributive semilattice.

Conversely, assume that $\mathcal{D} = (D; \sqcap)$ is a distributive semilattice and $x \in U(a \sqcap b, c)$ for given elements $a, b, c \in D$. Then $x \geq a \sqcap b = a \wedge b$, i.e. there exist $a_1 \geq a$, $b_1 \geq b$ such that $x = a_1 \wedge b_1$. Since $x \geq c$, we have $a_1 \geq a_1 \wedge b_1 \geq c$, $b_1 \geq a_1 \wedge b_1 \geq c$ thus $a_1 \in U(a,c)$, $b_1 \in U(b,c)$ and hence

$$x = a_1 \wedge b_1 \in U(a,c) \wedge U(b,c).$$

Assume $y \in U(a,c) \wedge U(b,c)$. Then there exist $z \in U(a,c)$, $v \in U(b,c)$ such that $y = z \wedge v$. Hence $z \geq a$, $v \geq b$, $z \geq c$, $v \geq c$ thus

$$y = z \wedge v \geq a \wedge b, \quad y \geq c$$

which yield $y \in U(a \wedge b, c)$. We have shown

$$U(a \wedge b, c) = U(a,c) \wedge U(b,c)$$

thus $\mathcal{D}$ is order-distributive. □

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