The theorems of Stewart and Steiner in the Poincaré disc model of hyperbolic geometry

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Abstract. In [Comput. Math. Appl. **41** (2001), 135–147], A.A. Ungar employs the Möbius gyrovector spaces for the introduction of the hyperbolic trigonometry. This Ungar's work plays a major role in translating some theorems from Euclidean geometry to corresponding theorems in hyperbolic geometry. In this paper we explore the theorems of Stewart and Steiner in the Poincaré disc model of hyperbolic geometry.

Keywords: Möbius transformation, hyperbolic geometry, gyrogroups, gyrovector spaces and hyperbolic trigonometry

Classification: 51B10, 51M10, 30F45, 20N05

1. Introduction

Hyperbolic geometry is a subset of a large class of geometries called non-Euclidean geometries, however hyperbolic geometry is similar to Euclidean geometry in many aspects. It has concepts of distance and angle, and there are many theorems common to both.

There are finite and infinite models in hyperbolic geometry. Poincaré disc model, Weierstrass model, Klein model, Gans model are well known in literature. Moreover, there are some isomorphisms between these models of hyperbolic geometry. For instance, the Weierstrass model is isomorphic to the Klein, Poincaré, and Gans models, see [10].

Throughout of this study, we only deal with Poincaré disc model of hyperbolic geometry.

This paper is inspired by the beautiful paper [5] by A.A. Ungar on hyperbolic trigonometry. A.A. Ungar showed that the hyperbolic sine and the hyperbolic cosine rules are valid in the Poincaré ball model of hyperbolic geometry in a form analogous to their Euclidean counterparts. In [11], Demirel and Soytürk proved that the hyperbolic Carnot theorem and its reverse hold true in the Poincaré disc model of hyperbolic geometry. In this paper we shall apply hyperbolic trigonometry to the study of the hyperbolic Stewart theorem and the hyperbolic Steiner's theorem in the Poincaré ball model of hyperbolic geometry.

2. Möbius transforms of the disc

In complex analysis Möbius transformations are well known. The most general Möbius transformation of the complex open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex $z-$plane

$$z \rightarrow e^{i\theta}\frac{z_0+z}{1+\overline{z_0}z} = e^{i\theta}\left(z_0 \oplus z\right)$$

defines the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as Möbius *left gyrotranslation*

$$z \rightarrow z_0 \oplus z = \frac{z_0+z}{1+\overline{z_0}z}$$

followed by rotation. Here θ is a real number, $z_0 \in \mathbb{D}$, and $\overline{z_0}$ is the complex conjugate of z_0. Möbius subtraction "$\ominus$" is given by $a \ominus z = a \oplus (-z)$. Clearly $z \ominus z = 0$ and $\ominus z = -z$. Möbius addition $\oplus$ is a binary operation in the disc $\mathbb{D}$, however it is neither commutative nor associative. The Möbius addition $\oplus$ gives rise to the groupoid $(\mathbb{D}, \oplus)$ studied by A.A. Ungar in several books and articles including [1], [2], [3], [8]. Möbius addition is analogous to the common vector addition $+$ in the Euclidean plane geometry. Since the Möbius addition $\oplus$ is neither commutative nor associative, the groupoid $(\mathbb{D}, \oplus)$ is not a group but it has a group-like structure that we present below.

The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration,

$$\mathrm{gyr} : \mathbb{D} \times \mathbb{D} \rightarrow \mathrm{Aut}(\mathbb{D}, \oplus)$$

given by the equation

$$\mathrm{gyr}[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1+a\overline{b}}{1+\overline{a}b}\,, \tag{1}$$

where $\mathrm{Aut}(\mathbb{D}, \oplus)$ is the automorphism group of the groupoid $(\mathbb{D}, \oplus)$. Therefore, the *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyration in (1),

$$a \oplus b = \mathrm{gyr}[a,b]\,(b \oplus a)\,. \tag{2}$$

Coincidentally, the gyration $\mathrm{gyr}[a,b]$ that repairs the breakdown of the commutative law of $\oplus$ in (2), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left* and *right gyroassociative laws*

$$\begin{aligned} a \oplus (b \oplus c) &= (a \oplus b) \oplus \mathrm{gyr}[a,b]c \\ (a \oplus b) \oplus c &= a \oplus (b \oplus \mathrm{gyr}[b,a]c) \end{aligned}$$

for all $a, b, c \in \mathbb{D}$.

Definition 1. A groupoid $(\mathbb{G}, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms

(G1) $0 \oplus a = 0$, left identity property
(G2) $\ominus a \oplus a = 0$, left inverse property
(G3) $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c$, left gyroassociative law

(G4) $\mathrm{gyr}[a,b] \in \mathrm{Aut}(\mathbb{G},\oplus)$, gyroautomorphism
(G5) $\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]$, left loop property

for all $a, b, c \in \mathbb{G}$.

Additionally, if the binary operation "$\oplus$" obeys the gyrocommutative law

(G6) $a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$, gyrocommutative law

for all $a, b, c \in \mathbb{G}$, then $(\mathbb{G},\oplus)$ is called a gyrocommutative gyrogroup.

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid $(\mathbb{D},\oplus)$ is a gyrocommutative gyrogroup.

The axioms in Definition 1 imply the right identity property, the right inverse property, the right gyyroassociative law and the right loop property. We refer readers to [1] and [2] for more details about gyrogroups.

Now define the secondary binary operation $\boxplus$ in $\mathbb{G}$ by

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b.$$

The primary and secondary operations of $\mathbb{G}$ are collectively called the dual operations of gyrogroups.

Let a, b be two elements of a gyrogroup $(\mathbb{G},\oplus)$. Then the unique solution of the equation

$$a \oplus x = b$$

for the unknown x is

$$x = \ominus a \oplus b$$

and the unique solution of the equation

$$x \oplus a = b$$

for the unknown x is

$$x = b \boxminus a.$$

For further details see [1], [2].

3. Möbius gyrogroups: from the disc to the ball

Let us identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way:

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2.$$

Then the equations

$$\begin{array}{rcl} \mathbf{u} \cdot \mathbf{v} & = & \mathrm{Re}(\overline{u}v) \\ \|\mathbf{u}\| & = & |u| \end{array} \tag{3}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc $\mathbb{R}^2_1 = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < 1\}$ of $\mathbb{R}^2$. In fact we get from (3) that

$$
\begin{aligned}
u \oplus v &= \frac{u+v}{1+\bar{u}v} \\
&= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\
&= \frac{\left(1+\bar{u}v+u\bar{v}+|v|^2\right)u + \left(1-|u|^2\right)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\
&= \frac{\left(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2\right)\mathbf{u} + \left(1-\|\mathbf{u}\|^2\right)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \mathbf{u} \oplus \mathbf{v}
\end{aligned} \tag{4}
$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_1$.

4. Möbius addition in the ball

Let $\mathbb{V}$ be any inner-product space and

$$\mathbb{V}_s = \{v \in \mathbb{V} : \|v\| < s\}$$

be the open ball of $\mathbb{V}$ with radius $s > 0$. Möbius addition in $\mathbb{V}_s$ is motivated by (4). It is given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{\left(1+(2/s^2)\,\mathbf{u}\cdot\mathbf{v}+(1/s^2)\,\|\mathbf{v}\|^2\right)\mathbf{u} + \left(1-(1/s^2)\,\|\mathbf{u}\|^2\right)\mathbf{v}}{1+(2/s^2)\,\mathbf{u}\cdot\mathbf{v}+(1/s^4)\,\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2}, \tag{5}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$ and where, ambiguously, $+$ denotes both addition of real numbers on the real line and addition of vectors in $\mathbb{V}$.

Without loss of generality, we may assume that $s = 1$ in (5). However we prefer to keep s as a free positive parameter in order to exhibit the results in the limit as $s \to \infty$, when the ball $\mathbb{V}_s$ expands the whole of its real inner product space $\mathbb{V}$, and Möbius addition $\oplus$ reduces to vector addition $+$ in $\mathbb{V}$, i.e.,

$$\lim_{s\to\infty} \mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$$

and

$$\lim_{s\to\infty} \mathbb{V}_s = \mathbb{V}.$$

Möbius scalar multiplication is given by the equation

$$\begin{aligned} r \otimes \mathbf{v} &= s\frac{(1+\|\mathbf{v}\|/s)^r - (1-\|\mathbf{v}\|/s)^r}{(1+\|\mathbf{v}\|/s)^r + (1-\|\mathbf{v}\|/s)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s\tanh\left(r\tanh^{-1}\|\mathbf{v}\|/s\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned}$$

where $r \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq 0$ and $r \otimes 0 = 0$.

Möbius scalar multiplication possesses the following properties:

- $n \otimes \mathbf{v} = v \oplus v \oplus \cdots \oplus v$, n-terms
- $(r_1 + r_2) \otimes \mathbf{v} = r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}$ scalar distribute law
- $(r_1 r_2) \otimes \mathbf{v} = r_1 \otimes (r_2 \otimes \mathbf{v})$ scalar associative law
- $r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) = r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v})$ monodistribute law
- $\|r \otimes \mathbf{v}\| = |r| \otimes \|\mathbf{v}\|$ homogeneity property
- $\frac{\|r|\otimes\mathbf{v}}{\|r\otimes\mathbf{v}\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ scaling property
- $\mathrm{gyr}[\mathbf{a}, \mathbf{b}](r \otimes \mathbf{v}) = r \otimes \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v}$ gyroautomorphism property
- $1 \otimes \mathbf{v} = \mathbf{v}$ multiplicative unit property

Definition 2 (Möbius gyrovector spaces)**.** Let $(\mathbb{V}_s, \oplus)$ be a Möbius gyrogroup equipped with scalar multiplication $\otimes$. The triple $(\mathbb{V}_s, \oplus, \otimes)$ is called a Möbius gyrovector space.

5. Möbius geodesics and angles

As it is well known from Euclidean geometry, the straight line passing though two given points A and B of a vector space $\mathbb{R}^n$ can be represented by the expression

$$A + (-A + B)\,t,$$

$t \in \mathbb{R}$. Obviously it passes through A when $t = 0$, and through B when $t = 1$.

In full analogy with Euclidean geometry, the unique Möbius geodesic passing though two given points $\mathbf{A}$ and $\mathbf{B}$ of a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is represented by the parametric gyrovector equation

$$L_{\mathbf{AB}} = \mathbf{A} \oplus (\ominus\mathbf{A} \oplus \mathbf{B}) \otimes t$$

with parameter $t \in \mathbb{R}$. It passes through $\mathbf{A}$ when $t = 0$, and through $\mathbf{B}$ when $t = 1$. The gyroline $L_{\mathbf{AB}}$ turns out to be a circular arc that intersects the boundary of the disc $\mathbb{V}_s$ orthogonally. The gyromidpoint $M_{\mathbf{AB}}$ of the points $\mathbf{A}$ and $\mathbf{B}$ corresponds to the parameter $t = 1/2$ of the gyroline $L_{\mathbf{AB}}$, see [4],

$$M_{\mathbf{AB}} = \mathbf{A} \oplus (\ominus\mathbf{A} \oplus \mathbf{B}) \otimes \frac{1}{2}.$$

The measure of a Möbius angle between two intersecting geodesic rays equals the measure of the Euclidean angle between corresponding intersecting tangent lines, as shown in Figure 1 below.

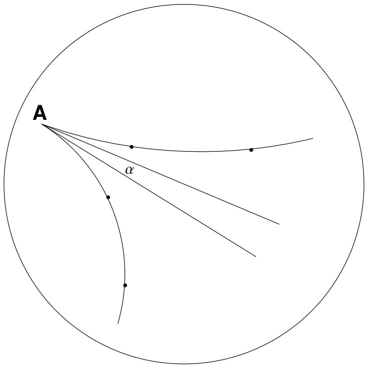

Figure 1. The unique 2-dimensional geodesics that pass through two given points and the hyperbolic angle between two intersecting geodesics rays in a Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. For the non-zero gyrovectors $\ominus\mathbf{A} \oplus \mathbf{B}$ and $\ominus\mathbf{A} \oplus \mathbf{C}$ or, equivalently, $\ominus\mathbf{A} \oplus \mathbf{E}$ and $\ominus\mathbf{A} \oplus \mathbf{D}$, the measure of the gyroangle α is given by the equation $\cos\alpha = \frac{\ominus\mathbf{A}\oplus\mathbf{B}}{\|\ominus\mathbf{A}\oplus\mathbf{B}\|} \cdot \frac{\ominus\mathbf{A}\oplus\mathbf{C}}{\|\ominus\mathbf{A}\oplus\mathbf{C}\|}$ or, equivalently, by the equation $\cos\alpha = \frac{\ominus\mathbf{A}\oplus\mathbf{E}}{\|\ominus\mathbf{A}\oplus\mathbf{E}\|} \cdot \frac{\ominus\mathbf{A}\oplus\mathbf{D}}{\|\ominus\mathbf{A}\oplus\mathbf{D}\|}$.

The hyperbolic angle is invariant under left gyrotranslations and rotations, see [3].

Definition 3. The hyperbolic distance function in $\mathbb{R}^n_s$ is given by the equation

$$d(\mathbf{A}, \mathbf{B}) = \|\mathbf{A} \ominus \mathbf{B}\| \quad \text{for} \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^n_s.$$

6. Gyrotriangles and gyrotrigonometry in Möbius gyrovector spaces

Definition 4 ([2])**.** A gyrotriangle $\Delta\mathbf{ABC}$ in a gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is a gyrovector space object formed by the three points $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$, called the vertices of the gyrotriangle, and the gyrovectors $\ominus\mathbf{A} \oplus \mathbf{B}$, $\ominus\mathbf{B} \oplus \mathbf{C}$ and $\ominus\mathbf{C} \oplus \mathbf{A}$, called the sides of the gyrotriangle. These are respectively the sides opposite to the vertices $\mathbf{C}, \mathbf{A}$ and $\mathbf{B}$. The gyrotriangle sides generate the three gyrotriangle gyroangles α, β and γ at the respective vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, as shown in Figure 2 below.

Definition 5 ([2])**.** In any gyrotriangle, gyroangle sum is always smaller than π. The difference between this sum and π i.e. $\delta = \pi - (\alpha + \beta + \gamma)$ is called the defect of the gyrotriangle.

In hyperbolic geometry, gyrotriangle gyroangles determine uniquely its side gyrolenghts as follows:

Theorem 6 ([2])**.** *Let $\Delta\mathbf{ABC}$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, corresponding gyroangles α, β, γ with $0 < \alpha + \beta + \gamma < \pi$, and side gyrolenghts $\| \ominus \mathbf{B} \oplus \mathbf{C}\|$, $\| \ominus \mathbf{C} \oplus \mathbf{A}\|$, $\| \ominus \mathbf{A} \oplus \mathbf{B}\|$.*

The side gyrolengths of the gyrotriangle $\Delta\mathbf{ABC}$ are determined by its gyroangles according to the AAA to SSS conversion equations

$$\begin{aligned}
\left(\frac{\|\ominus\mathbf{B}\oplus\mathbf{C}\|}{s}\right)^2 &= \frac{\cos\alpha+\cos(\beta+\gamma)}{\cos\alpha+\cos(\beta-\gamma)}\\
\left(\frac{\|\ominus\mathbf{C}\oplus\mathbf{A}\|}{s}\right)^2 &= \frac{\cos\beta+\cos(\alpha+\gamma)}{\cos\beta+\cos(\alpha-\gamma)}\\
\left(\frac{\|\ominus\mathbf{A}\oplus\mathbf{B}\|}{s}\right)^2 &= \frac{\cos\gamma+\cos(\alpha+\beta)}{\cos\gamma+\cos(\alpha-\beta)}.
\end{aligned}$$

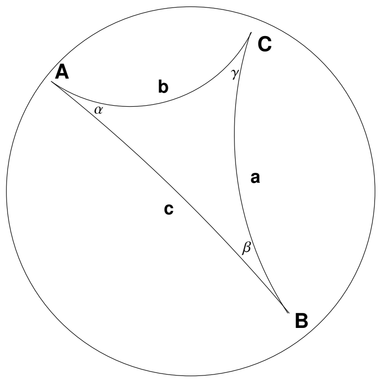

Figure 2. A gyrotriangle in a Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$.

The hyperbolic law of cosine and the hyperbolic law of sine can be recast in a form fully analogous to the form of their Euclidean counterparts. Let us use the notation

$$\|\mathbf{a}\|_M = \gamma_{\mathbf{a}}^2\|\mathbf{a}\|$$

where $\gamma_{\mathbf{a}}$ is the gamma factor

$$\gamma_{\mathbf{a}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{a}\|^2}{s^2}}},$$

so that, conversely

$$\frac{\|\mathbf{a}\|}{s} = \frac{2\left(\|\mathbf{a}\|_M/s\right)}{1+\sqrt{1+4\left(\|\mathbf{a}\|_M/s\right)^2}}.$$

Theorem 7 ([5]). *Let $\Delta\mathbf{ABC}$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$ and sides $\mathbf{a} = \ominus\mathbf{B}\oplus\mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C}\oplus\mathbf{A}$ and $\mathbf{c} = \ominus\mathbf{A}\oplus\mathbf{B}$ with hyperbolic angles α, β and γ at the vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. Then* *we have the hyperbolic law of sine,*

$$\frac{\|\mathbf{a}\|_M}{\sin\alpha} = \frac{\|\mathbf{b}\|_M}{\sin\beta} = \frac{\|\mathbf{c}\|_M}{\sin\gamma}. \tag{6}$$

Theorem 8 ([5])**.** *Let $\Delta\mathbf{ABC}$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$ and sides $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ and $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$ with hyperbolic angles α, β and γ at the vertices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Then we have the hyperbolic law of cosine,*

$$\frac{1}{s}c^2 = \frac{1}{s}a^2 \oplus \frac{1}{s}b^2 \ominus \frac{1}{s}\frac{2ab\cos\gamma}{\left(1+\frac{a^2}{s^2}\right)\left(1+\frac{b^2}{s^2}\right) - \frac{2}{s^2}ab\cos\gamma}, \tag{7}$$

where $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$.

Theorem 9 ([5])**.** *Let $\Delta\mathbf{ABC}$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$ and sides $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ and $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$ with hyperbolic angles α, β and γ at the vertices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. If $\gamma = \pi/2$ then we have the hyperbolic Pythagorean identity,*

$$\frac{1}{s}c^2 = \frac{1}{s}a^2 \oplus \frac{1}{s}b^2$$

where $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$.

In Euclidean geometry, the Pythagorean theorem is well known and it has many proofs in literature. For example, in [13], there are more than 75 different proofs and all of them are intelligible. Some of them are concerned with squares, i.e., in the proof some squares are used. In [4], A.A. Ungar gave the notion of gyrosquares and proved that the hyperbolic square is richer in structure than its Euclidean counterpart.

Problem 10. *Is it possible to prove hyperbolic Pythagorean theorem by using a gyrosquare?*

In the Euclidean geometry, the converse of the Pythagorean theorem does exist. Naturally, one may wonder whether the converse of the Pythagorean theorem in hyperbolic geometry exists. Indeed, the converse of the theorem does exist as we show in the following theorem.

Theorem 11. *Let $\Delta\mathbf{ABC}$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_s$ and sides $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ and $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$ with hyperbolic angles α, β and γ at the vertices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. If the (nonzero) three side lengths of a triangle $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ satisfy the relation*

$$\frac{1}{s}c^2 = \frac{1}{s}a^2 \oplus \frac{1}{s}b^2$$

where $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, then the gyrotriangle $\Delta\mathbf{ABC}$ is a right gyrotriangle.

PROOF: The proof of this theorem is not different from its Euclidean counterpart, see [12]. $\square$

In Euclidean geometry, the Carnot theorem is an direct application of Pythagoras. In [11], Demirel and Soytürk proved that the Carnot theorem and its (partial) reverse holds true in the Poincaré disc model of hyperbolic geometry. These theorems are presented below.

Theorem 12 ([11]). *Let $\Delta\mathbf{ABC}$ be a hyperbolic triangle in the Poincaré disc, whose vertices are the points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of the disc and whose sides (directed counterclockwise) are $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ and $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$. Let points $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ be located on the sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of hyperbolic triangle $\Delta\mathbf{ABC}$ respectively. If the perpendiculars to the sides of the hyperbolic triangle at points $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ are concurrent, then the following holds:*

$$\begin{aligned}(8)\quad &\left|\ominus\mathbf{A} \oplus \mathbf{C}'\right|^2 \ominus \left|\ominus\mathbf{B} \oplus \mathbf{C}'\right|^2 \oplus \left|\ominus\mathbf{B} \oplus \mathbf{A}'\right|^2 \ominus \left|\ominus\mathbf{C} \oplus \mathbf{A}'\right|^2 \\ &\qquad\qquad\qquad \oplus \left|\ominus\mathbf{C} \oplus \mathbf{B}'\right|^2 \ominus \left|\ominus\mathbf{A} \oplus \mathbf{B}'\right|^2 = 0.\end{aligned}$$

Theorem 13 ([11]). *Let $\Delta\mathbf{ABC}$ be a hyperbolic triangle in the Poincaré disc, whose vertices are the points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of the disc and whose sides (directed counterclockwise) are $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ and $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$. Let points $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ be located on the sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ of hyperbolic triangle $\Delta\mathbf{ABC}$ respectively. If (8) holds and two of the three perpendiculars to the sides of the hyperbolic triangle at points $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$ are concurrent, then the three perpendiculars are concurrent.*

7. The theorems of Stewart and Steiner in the Poincaré disc model of hyperbolic geometry

In Euclidean geometry, Stewart's theorem yields a relation between the lengths of the sides of a triangle and the length of segment from a vertex to a point on the opposite side.

Theorem 14 (Stewart's theorem). *Let ΔABC be a triangle and a, b, c be the sides of ΔABC. Let p be a segment from A to a point on a dividing a itself in m and n. Then*

$$p^2 = \frac{b^2 m + c^2 n}{a} - mn.$$

Theorem 15 (Steiner's theorem). *Let ΔABC be a triangle and a, b, c be the sides of ΔABC, and let M and N be points on a such that $\angle BAM = \angle NAC$. Then*

$$\frac{BM}{MC}\frac{BN}{NC} = \frac{c^2}{b^2}.$$

The theorems of Stewart and Steiner are well known and fundamental in Euclidean geometry and these are just direct applications of cosine law and sine law, respectively. Since the laws of sine and cosine are valid in the Poincaré disc model of hyperbolic geometry (6) and (7), we present and prove their counterparts in hyperbolic geometry below:

Theorem 16. *Let* $\Delta\mathbf{ABC}$ *be a gyrotriangle in a Möbius gyrovector space* $(\mathbb{R}^2_1, \oplus, \otimes)$ *with vertices* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^2_1$ *and sides* $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ *and* $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$. *Let* $\mathbf{X}$ *be a point on* $\mathbf{a}$ *such that* $\mathbf{m} := \ominus\mathbf{X} \oplus \mathbf{B}$, $\mathbf{n} := \ominus\mathbf{C} \oplus \mathbf{X}$ *and* $\mathbf{p} := \ominus\mathbf{X} \oplus \mathbf{A}$, *as shown in Figure* 3. *Then*

$$a\left(p^2 + \left(mn \oplus c^2 b^2\right)\right) = p^2\left(b^2 n \oplus mc^2\right) + \left(nc^2 \oplus mb^2\right),$$

or equivalently

$$p^2 = \frac{\left(c^2 n \oplus mb^2\right) - a\left(mn \oplus c^2 b^2\right)}{a - \left(b^2 n \oplus mc^2\right)},$$

where $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, $p = \|\mathbf{p}\|$, $m = \|\mathbf{m}\|$, *and* $n = \|\mathbf{n}\|$.

Proof: Let us take $\angle\mathbf{AXB} := \theta$ and $\angle\mathbf{AXC} := \psi$. Cosine law on $\Delta\mathbf{AXB}$ and $\Delta\mathbf{AXC}$ yields

$$\cos\theta = \frac{\left(p^2 \oplus m^2 \ominus c^2\right)\left(1 + p^2\right)\left(1 + m^2\right)}{\left(1 + \left(p^2 \oplus m^2 \ominus c^2\right)\right) 2pm}$$

and

$$\cos\psi = \frac{\left(p^2 \oplus n^2 \ominus b^2\right)\left(1 + p^2\right)\left(1 + n^2\right)}{\left(1 + \left(p^2 \oplus n^2 \ominus b^2\right)\right) 2pn}$$

respectively, see [5]. Since $\theta + \psi = \pi$, we get

$$\frac{\left(p^2 \oplus m^2 \ominus c^2\right)\left(1 + m^2\right)}{\left(1 + \left(p^2 \oplus m^2 \ominus c^2\right)\right) m} = -\frac{\left(p^2 \oplus n^2 \ominus b^2\right)\left(1 + n^2\right)}{\left(1 + \left(p^2 \oplus n^2 \ominus b^2\right)\right) n}.$$

A simple calculation shows that

$$(m \oplus n)\left(1 + \frac{1}{p^2}\left(mn \oplus c^2 b^2\right)\right) = \left(\left(b^2 n \oplus mc^2\right) + \frac{1}{p^2}\left(c^2 n \oplus mb^2\right)\right)$$

i.e.,

$$a\left(p^2 + \left(mn \oplus c^2 b^2\right)\right) = p^2\left(b^2 n \oplus mc^2\right) + \left(c^2 n \oplus mb^2\right)$$

holds and this implies that

$$p^2 = \frac{\left(c^2 n \oplus mb^2\right) - a\left(mn \oplus c^2 b^2\right)}{a - \left(b^2 n \oplus mc^2\right)}$$

is valid.

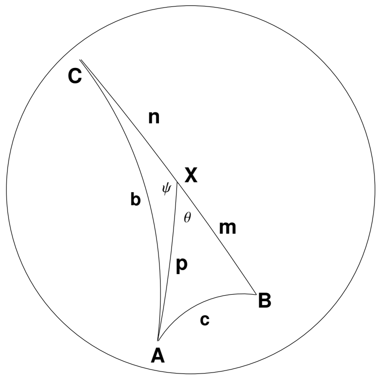

Figure 3. The theorem of Stewart in the Poincaré disc model of hyperbolic geometry. The gyrovectors $\mathbf{m}$ and $\mathbf{n}$, defined in Theorem 16, are illustrated, satisfying the gyrotriangle equality $\|\mathbf{m}\| \oplus \|\mathbf{n}\| = \|\mathbf{a}\|$.

□

Theorem 17. *Let* $\Delta\mathbf{ABC}$ *be a gyrotriangle in a Möbius gyrovector space* $(\mathbb{R}_1^2, \oplus, \otimes)$ *with vertices* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}_1^2$ *and sides* $\mathbf{a} = \ominus\mathbf{B} \oplus \mathbf{C}$, $\mathbf{b} = \ominus\mathbf{C} \oplus \mathbf{A}$ *and* $\mathbf{c} = \ominus\mathbf{A} \oplus \mathbf{B}$. *Let* $\mathbf{M}$, $\mathbf{N}$ *be points on a such that* $\angle\mathbf{BAM} = \angle\mathbf{NAC}$. *Then*

$$\frac{\|\ominus\mathbf{B} \oplus \mathbf{M}\|_M}{\|\ominus\mathbf{M} \oplus \mathbf{C}\|_M} \frac{\|\ominus\mathbf{B} \oplus \mathbf{N}\|_M}{\|\ominus\mathbf{N} \oplus \mathbf{C}\|_M} = \frac{\| \ominus \mathbf{A} \oplus \mathbf{B}\|_M^2}{\| \ominus \mathbf{C} \oplus \mathbf{A}\|_M^2}.$$

Proof: Let us take $\angle\mathbf{BAM} = \angle\mathbf{NAC} = \alpha$, $\angle\mathbf{MAN} = \delta$, $\angle\mathbf{ACB} = \gamma$ and $\angle\mathbf{ABC} = \beta$. From hyperbolic sine law, we get

$$\frac{\|\ominus\mathbf{A} \oplus \mathbf{B}\|_M}{\sin\gamma} = \frac{\|\ominus\mathbf{A} \oplus \mathbf{C}\|_M}{\sin\beta}, \tag{9}$$

$$\frac{\|\ominus\mathbf{N} \oplus \mathbf{B}\|_M}{\sin(\alpha+\delta)} = \frac{\|\ominus\mathbf{N} \oplus \mathbf{A}\|_M}{\sin\beta}, \tag{10}$$

$$\frac{\|\ominus\mathbf{M} \oplus \mathbf{C}\|_M}{\sin(\alpha+\delta)} = \frac{\|\ominus\mathbf{M} \oplus \mathbf{A}\|_M}{\sin\gamma}, \tag{11}$$

$$\frac{\|\ominus\mathbf{M} \oplus \mathbf{B}\|_M}{\sin\alpha} = \frac{\|\ominus\mathbf{M} \oplus \mathbf{A}\|_M}{\sin\beta}, \tag{12}$$

$$\frac{\|\ominus\mathbf{N} \oplus \mathbf{C}\|_M}{\sin\alpha} = \frac{\|\ominus\mathbf{N} \oplus \mathbf{A}\|_M}{\sin\gamma}. \tag{13}$$

Dividing (10) by (11), and dividing (12) by (13), we get

$$\frac{\|\ominus\mathbf{M}\oplus\mathbf{A}\|_M}{\|\ominus\mathbf{N}\oplus\mathbf{A}\|_M}\frac{\|\ominus\mathbf{N}\oplus\mathbf{B}\|_M}{\|\ominus\mathbf{M}\oplus\mathbf{C}\|_M} = \frac{\sin\gamma}{\sin\beta} \tag{14}$$

and

$$\frac{\|\ominus\mathbf{N}\oplus\mathbf{A}\|_M}{\|\ominus\mathbf{M}\oplus\mathbf{A}\|_M}\frac{\|\ominus\mathbf{M}\oplus\mathbf{B}\|_M}{\|\ominus\mathbf{N}\oplus\mathbf{C}\|_M} = \frac{\sin\gamma}{\sin\beta}, \tag{15}$$

respectively. Multiplying (14) by (15), we have

$$\frac{\|\ominus\mathbf{N}\oplus\mathbf{B}\|_M}{\|\ominus\mathbf{M}\oplus\mathbf{C}\|_M}\frac{\|\ominus\mathbf{M}\oplus\mathbf{B}\|_M}{\|\ominus\mathbf{N}\oplus\mathbf{C}\|_M} = \frac{\sin^2\gamma}{\sin^2\beta}, \tag{16}$$

and from (9), we get

$$\frac{\sin^2\gamma}{\sin^2\beta} = \frac{\|\ominus\mathbf{A}\oplus\mathbf{B}\|_M^2}{\|\ominus\mathbf{C}\oplus\mathbf{A}\|_M^2}. \tag{17}$$

Thus, the left-hand side of (16) and the right-hand side of (17) are equal.

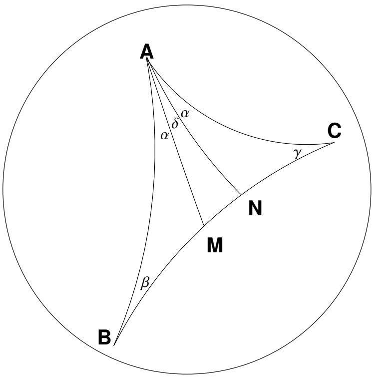

Figure 4. The theorem of Steiner in the Poincaré disc model of hyperbolic geometry.

□

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