Central subsets of Urysohn universal spaces

Piotr Niemiec

Abstract. A subset A of a metric space (X, d) is central iff for every Katětov map $f : X \to \mathbb{R}$ upper bounded by the diameter of X and any finite subset B of X there is $x \in X$ such that $f(a) = d(x, a)$ for each $a \in A \cup B$. Central subsets of the Urysohn universal space $\mathbb{U}$ (see introduction) are studied. It is proved that a metric space X is isometrically embeddable into $\mathbb{U}$ as a central set iff X has the collinearity property. The Katětov maps of the real line are characterized.

Keywords: Urysohn's universal space, ultrahomogeneous spaces, extensions of isometries

Classification: 54E50, 54D65

In [15] Urysohn introduced his universal separable metric space which turned out to be uniquely determined (up to isometry) by the three conditions: completeness, ultrahomogeneity and universality. (Ultrahomogeneity of a metric space X means that any isometry between its two finite subsets is extendable to an isometry of the whole space onto itself; while universality of X means that every separable metric space is isometrically embeddable in X.) About thirty years later, Huhunaišvili [7] has proved that every isometry between compact subsets of the Urysohn universal space admits a bijective isometric extension defined on the whole space. This is probably the most important result on this space which has been obtained after the paper of Urysohn and before Katětov's [10] one, where the author presented a very useful method of constructing the Urysohn space. Since that time, the literature concerning Urysohn's universal space $\mathbb{U}$ is still growing up and we mention here only a part of it: Uspenskij has shown in [16] that the group of isometries of $\mathbb{U}$ is a universal Polish group and in [17] that $\mathbb{U}$ is homeomorphic to a separable Hilbert space; Holmes [5] has proved that the space $\mathbb{U}$ generates a unique (up to linear isometry preserving the points of $\mathbb{U}$) Banach space (see also [6] or [13] for short proofs); Cameron and Vershik [2] have shown that $\mathbb{U}$ can be endowed with the structure of a monothetic group; Melleray [12] has obtained the converse theorem to Huhunaišvili's one: if any isometry between two arbitrary (isometric) copies of a metric space X admits a bijective isometric extension defined on $\mathbb{U}$, then the completion of X is compact.

In the present paper we will study central subsets (defined in the Abstract) of $\mathbb{U}$ and its 'spherical' geometry.

Section 1 deals with Katětov maps and hulls. It is proved that the Katětov hull of a metric space is always hyperconvex and Katětov maps defined on the real line are characterized.

Section 2 is devoted to central subsets of Urysohn spaces and common spheres (that is, intersections of spheres) with centres in them. We give an alternative proof of the above mentioned Huhunaišvili theorem. Several results on common spheres isometric to Urysohn spaces are presented. It is also shown that every central subset of a metric space X is central in the completion of X. As a consequence of this, we prove that a metric space is isometrically embeddable in the Urysohn universal space as a central set iff it has the so-called *collinearity property* (see Theorem 1.2 for the definition of this).

Terminology and notation. The sets of all nonnegative real numbers is denoted by $\mathbb{R}_+$. The identity map on a set X is denoted by id_X. For two numbers $p, q \in [-\infty, +\infty]$, $p \wedge q$ and $p \vee q$ stand, respectively, for the minimum and maximum of them. Similarly, if f and g are two real functions with a common domain or if one of them is a real function and the other is an element of $[-\infty, +\infty]$, $f \wedge g$ and $f \vee g$ are the minimum and maximum functions of them.

The open and the closed ball with centre at a and of radius r in a metric space (X, d) are denoted by $B_X(a, r)$ and $\bar{B}_X(a, r)$, respectively. The sphere $\{x \in X\colon\ d(a, x) = r\} = \bar{B}_X(a, r) \setminus B_X(a, r)$ is denoted by $S_X(a, r)$. Additionally, let $B_X(a, 0) = \emptyset$ and $\bar{B}_X(a, 0) = S_X(a, 0) = \{a\}$. We say that the space X is *precompact* if its completion is compact or, equivalently, if X is totally bounded; and X is *Heine-Borel* if every closed ball in X is compact.

A map $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *nonexpansive* [λ*-isometric*] if for every $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$ [$d_Y(f(x_1), f(x_2)) = \lambda d_X(x_1, x_2)$]. A λ*-isometry* is a λ-isometric bijection. The spaces X and Y are said to be Λ*-isometric* iff there is a λ-isometry of X onto Y for some positive λ.

The *Hausdorff distance* between two nonempty subsets A and B of a metric space (X, d) is denoted by $\mathrm{dist}_d(A, B)$ ($\in [0, +\infty]$). The function dist_d is a metric on the space $\mathcal{D}_b(X)$ of all nonempty, bounded and closed subsets of X.

For metric spaces (X, d) and (Y, ϱ), we shall write $(X, d) \subset (Y, \varrho)$ if $X \subset Y$ and $\varrho\big|_{X \times X} = d$.

1. Katětov maps

From now to the end of the section (X, d) is a nonempty metric space.

1.1 Definition. A function $f\colon X \to \mathbb{R}$ is a *Katětov map* if $|f(a) - f(b)| \leq d(a, b) \leq f(a) + f(b)$ for any $a, b \in X$. If additionally $f(X) \subset [0, \operatorname{diam} X]$, we call it an *inner* Katětov map.

The *diameter* of a Katětov map f is the number

$$\delta(f) = \delta_X(f) = 2 \inf f(X) \in \mathbb{R}_+,$$

while the *bound* of it is defined by $l(f) = l_X(f) = \frac{1}{2} \inf_{x,y \in X}(f(x) + f(y) - d(x, y)) \in \mathbb{R}_+$. The set of all [inner] Katětov maps on X is denoted by $E(X)$ [$E^i(X)$]. Additionally, for a number $r \in [0, +\infty]$, let $E_r(X)$ be the set of all

Katětov maps f such that $f(X) \subset [0,r]$. Observe that $E_\infty(X) = E(X)$ and $E_{\operatorname{diam} X}(X) = E^i(X)$.

We call the set $E(X)$ the *Katětov hull* of the metric space (X,d). The Katětov hull and all its subsets are considered with the metric induced from the 'supremum' norm, denoted by $\|\cdot\|$. (Katětov maps may be unbounded, their difference however is always bounded.)

Katětov maps are precisely the functions that arise in a natural way from one-point metric extensions of the given space.

For an element x of the space X, put $e_x \colon X \ni y \mapsto d(x,y) \in \mathbb{R}_+$ and $\mathfrak{e}(X) = \{e_x \colon\ x \in X\}$. Easily $\mathfrak{e}(X) \subset E^i(X)$. The reader will easily check that if $f \in E(X)$, then $\delta_X(f) = 0$ iff $f \in \mathfrak{e}(X)$, provided X is complete. Further, one shows that the *Kuratowski map* $\mathfrak{e} \colon X \ni x \mapsto e_x \in E(X)$ is isometric and $\|f - e_x\| = f(x)$ for each $x \in X$ and $f \in E(X)$.

Basic properties of Katětov maps and hulls the reader can find in [10], [3] or [13]. Here we recall only the most important ones. For a nonempty subset A of X and a Katětov map $f \colon A \to \mathbb{R}$, let $\widehat{f} \colon X \to \mathbb{R}$ be defined by $\widehat{f}(x) = \inf_{a \in A}(f(a) + d(x,a))$. Then $\widehat{f}$ is a Katětov map on X which extends f and the map $E(A) \ni f \mapsto \widehat{f} \in E(X)$ is isometric. Having this, the reader shall easily check that if $r \geq \operatorname{diam} X$, then the map $E_r(A) \ni f \mapsto \widehat{f} \wedge r \in E_r(X)$ is isometric. The map $\widehat{f}$ is called the *Katětov extension* of f.

In the sequel we shall be working with metric spaces with separable Katětov hulls. Therefore it seems to be worthwhile to mention the following result, due to Melleray [12]:

1.2 Theorem. *The Katětov hull, $E(X)$, of a metric space (X,d) is separable iff X has the collinearity property, i.e. if there is no infinite subset A of X such that*

$$\inf\{d(x,y) + d(y,z) - d(x,z) : x,y,z \text{ are distinct points of } A\} > 0.$$

The completion of a metric space with the collinearity property is Heine-Borel.

In fact Melleray obtained the above condition by combining his criterion involving *inline subsequences* with its equivalence with the collinearity property, proved by Kalton [9].

Theorem 1.2 immediately implies that the Katětov hull of each subset of the real line is separable. In the next result, which will be applied in the next section, we shall characterize Katětov maps on $\mathbb{R}$. In order to do this, we have to recall the classical theorem of real analysis. Namely, if $f \colon \mathbb{R}_+ \to \mathbb{R}$ is a nonexpansive map, then there exists a measurable function $g \colon \mathbb{R}_+ \to \mathbb{R}$ such that $|g| \leq 1$ and $f(x) = f(0) + \int_0^x g(t)\,\mathrm{d}t$. Below, $\operatorname{sgn} \colon \mathbb{R} \to \{-1,0,1\}$ denotes the *signum function*.

1.3 Theorem. (i) *A function $f \colon \mathbb{R}_+ \to \mathbb{R}$ is a Katětov map if and only if f is of the form*

$$f(x) = \alpha + x + \int_0^{+\infty} u(t) \operatorname{sgn}(t-x)\,\mathrm{d}t \qquad (x \in \mathbb{R}_+), \tag{1.1}$$

where α is a nonnegative constant and $u\colon \mathbb{R}_+ \to [0,1]$ is a Lebesgue integrable function. Moreover, $l_A(f) = \alpha$ for every subset A of $\mathbb{R}_+$ such that $0 \in A$ and $\sup A = +\infty$, and u, as an element of $L^1(\mathbb{R}_+)$, is uniquely determined by (1.1).

(ii) *If $a \in \mathbb{R}$ is fixed, then a function $f\colon \mathbb{R} \to \mathbb{R}$ is a Katětov map if and only if f is of the form*

$$f(x) = \gamma + |x-a| + \int_{-\infty}^{+\infty} w(t)\,\mathrm{sgn}(t-a)\,\mathrm{sgn}(t-x)\,\mathrm{d}t, \tag{1.2}$$

where $\gamma \geq 0$ and $w\colon \mathbb{R} \to [0,1]$ is a Lebesgue integrable function. Moreover, $l_C(f) = \gamma$ for every nonempty subset C of $\mathbb{R}$ such that $\sup C = -\inf C = +\infty$ and w, as an element of $L^1(\mathbb{R})$, is uniquely determined by (1.2).

Proof: (i). Suppose that f is a Katětov map. By the note preceding the statement of the theorem, there exists a measurable function $g\colon \mathbb{R}_+ \to [-1,1]$ such that $f(x) = f(0) + \int_0^x g(t)\,\mathrm{d}t$. Put $u = \frac{1}{2}(1-g)\colon \mathbb{R}_+ \to [0,1]$. Let $h \in \mathbb{R}_+$. Since f is a Katětov map, so $h \leq f(0) + f(h) = 2f(0) + \int_0^h g(t)\,\mathrm{d}t$ and therefore $\int_0^h (1-g(t))\,\mathrm{d}t = h - \int_0^h g(t)\,\mathrm{d}t \leq 2f(0)$. This implies, thanks to the nonnegativity of the function $1-g$, that $\int_0^{+\infty}(1-g(t))\,\mathrm{d}t \leq 2f(0)$. Thus u is Lebesgue integrable and $\int_0^{+\infty} u(t)\,\mathrm{d}t \leq f(0)$. Put $\alpha = f(0) - \int_0^{+\infty} u(t)\,\mathrm{d}t \geq 0$. Finally, we obtain

$$\begin{aligned} f(x) &= f(0) + \int_0^x g(t)\,\mathrm{d}t = \alpha + \int_0^{+\infty} u(t)\,\mathrm{d}t + \int_0^x (1-2u(t))\,\mathrm{d}t \\ &= \alpha + x + \int_x^{+\infty} u(t)\,\mathrm{d}t - \int_0^x u(t)\,\mathrm{d}t = \alpha + x + \int_0^{+\infty} u(t)\,\mathrm{sgn}(t-x)\,\mathrm{d}t. \end{aligned}$$

Now suppose that f is given by the formula (1.1). Then, for $x, y \in \mathbb{R}_+$ such that $x \leq y$, we have:

$$\begin{aligned} |f(y) - f(x)| &= \Big| y - x + \int_0^{+\infty} u(t)(\mathrm{sgn}(t-y) - \mathrm{sgn}(t-x))\,\mathrm{d}t \Big| \\ &= \Big| \int_x^y (1-2u(t))\,\mathrm{d}t \Big| \leq \int_x^y |1-2u(t)|\,\mathrm{d}t \leq \int_x^y 1\,\mathrm{d}t = |y-x| \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}(f(x) + f(y) - |y-x|) &= \alpha + x + \frac{1}{2}\int_0^{+\infty} u(t)(\mathrm{sgn}(t-x) + \mathrm{sgn}(t-y))\,\mathrm{d}t \\ &= \alpha + \int_0^x (1-u(t))\,\mathrm{d}t + \int_y^{+\infty} u(t)\,\mathrm{d}t \geq \alpha \geq 0, \end{aligned}$$

which yields that $f \in E(\mathbb{R}_+)$. What is more, if $x = 0 \in A$ and $y \in A$ tends to $+\infty$, then the expression which follows the last equality sign in the foregoing calculations tends to α and therefore $l_A(f) \le \alpha$. On the other hand, the above argument shows that also $f - \alpha \in E(\mathbb{R}_+)$, so $\alpha = l_A(f)$.

Now, if u' is a Lebesgue integrable function such that

$$f(x) = \alpha' + x + \int_0^{+\infty} u'(t)\,\mathrm{sgn}(t-x)\,\mathrm{d}t$$

for each $x \ge 0$ and some constant $\alpha' \ge 0$, then $\alpha' = l(f)$ and $0 = \int_0^{+\infty}(u(t) - u'(t))\,\mathrm{sgn}(t-x)\,\mathrm{d}t = -2\int_0^x (u(t)-u'(t))\,\mathrm{d}t + \int_0^{+\infty}(u(t)-u'(t))\,\mathrm{d}t$ for every $x \in \mathbb{R}_+$. Putting $x = 0$, we obtain $\int_0^{+\infty}(u(t)-u'(t))\,\mathrm{d}t = 0$ and thus $\int_0^x (u(t)-u'(t))\,\mathrm{d}t = 0$ for any $x \in \mathbb{R}_+$, which implies that $u - u' = 0$ in $L^1(\mathbb{R}_+)$.

(ii). First suppose that $a = 0$. Since the set $\mathbb{R}_- = (-\infty, 0]$ is isometric to $\mathbb{R}_+$, so, simply changing the variable ($x \rightsquigarrow -x$) and thanks to (i), we conclude that there exist nonnegative constants γ_- and γ_+ and Lebesgue integrable functions w_- and w_+ defined on the intervals $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively, with values in $[0,1]$ and such that $f(x) = \gamma_- - x - \int_{-\infty}^0 w_-(t)\,\mathrm{sgn}(t-x)\,\mathrm{d}t$ for $x \le 0$ and $f(x) = \gamma_+ + x + \int_0^{+\infty} w_+(t)\,\mathrm{sgn}(t-x)\,\mathrm{d}t$ for $x \ge 0$. Let $w\colon \mathbb{R} \to [0,1]$ be the union of the functions w_- and w_+ (the value at 0 has no matter). Thus w is Lebesgue integrable and

$$f(\pm x) = \gamma_\pm + |x| \pm \int_{\mathbb{R}_\pm} w(t)\,\mathrm{sgn}(t \mp x)\,\mathrm{d}t \qquad (x \in \mathbb{R}_+). \tag{1.3}$$

Furthermore, if $x < 0 < y$, then

$$\begin{aligned} y - x \le f(x) + f(y) &= \gamma_- + \gamma_+ + y - x \\ &\quad + \int_{-\infty}^x w(t)\,\mathrm{d}t - \int_x^y w(t)\,\mathrm{d}t + \int_y^{+\infty} w(t)\,\mathrm{d}t \end{aligned}$$

and therefore

$$\int_x^y w(t)\,\mathrm{d}t \le \gamma_- + \gamma_+ + \int_{-\infty}^x w(t)\,\mathrm{d}t + \int_y^{+\infty} w(t)\,\mathrm{d}t.$$

Now letting $x \to -\infty$ and $y \to +\infty$, we obtain $\int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t \le \gamma_- + \gamma_+$. But, thanks to (1.3), $\gamma_- + \gamma_+ = 2f(0) - \int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t$ and hence $\int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t \le f(0)$. Put $\gamma = f(0) - \int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t$ (≥ 0). Observe that $\gamma_\pm = f(0) - \int_{\mathbb{R}_\pm} w(t)\,\mathrm{d}t = \gamma + \int_{\mathbb{R}_\mp} w(t)\,\mathrm{d}t$. Now, by (1.3), we conclude that for $x \le 0$,

$$\begin{aligned} f(x) = \gamma + \int_0^{+\infty} w(t)\,\mathrm{d}t + |x| - \int_{-\infty}^0 w(t)\,\mathrm{sgn}(t-x)\,\mathrm{d}t \\ = \gamma + |x| + \int_{-\infty}^{+\infty} w(t)\,\mathrm{sgn}(t)\,\mathrm{sgn}(t-x)\,\mathrm{d}t \end{aligned}$$

and for $x \geq 0$,

$$f(x) = \gamma + \int_{-\infty}^{0} w(t)\,\mathrm{d}t + |x| + \int_{0}^{+\infty} w(t)\operatorname{sgn}(t-x)\,\mathrm{d}t$$
$$= \gamma + |x| + \int_{-\infty}^{+\infty} w(t)\operatorname{sgn}(t)\operatorname{sgn}(t-x)\,\mathrm{d}t,$$

which finishes the proof in the case of $a = 0$.

Now if a is arbitrary and $f \in E(\mathbb{R})$, then also $f_a \in E(\mathbb{R})$, where $f_a(x) = f(a+x)$. Applying the foregoing part of the proof for f_a and changing the variable, we obtain the required formula (1.2).

For the proof of the converse statement, suppose that f is of the form (1.2). Let $x, y \in \mathbb{R}$ be such that $x \leq y$. Observe that $|y-a| - |x-a| = \int_x^y \operatorname{sgn}(t-a)\,\mathrm{d}t$ and hence

$$|f(y) - f(x)| = \Big|\int_x^y (1 - 2w(t))\operatorname{sgn}(t-a)\,\mathrm{d}t\Big| \leq \int_x^y |1 - 2w(t)|\,\mathrm{d}t \leq y - x$$

and

$$\begin{aligned}\frac{1}{2}(f(x) + f(y) - |y-x|) &= \gamma + \frac{1}{2}(|x-a| + |y-a| - (y-x)) \\ &\quad + \frac{1}{2}\int_{-\infty}^{+\infty} w(t)\operatorname{sgn}(t-a)(\operatorname{sgn}(t-x) + \operatorname{sgn}(t-y))\,\mathrm{d}t \\ &= \gamma + (x \vee a - y \wedge a) + \int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t - \int_a^x w(t)\operatorname{sgn}(t-a)\,\mathrm{d}t \\ &\quad + \int_y^a w(t)\operatorname{sgn}(t-a)\,\mathrm{d}t.\end{aligned} \tag{1.4}$$

Now if $y \leq a$, then, continuing (1.4), we obtain

$$\frac{1}{2}(f(x) + f(y) - |y-x|) = \gamma + \int_y^a (1 - w(t))\,\mathrm{d}t + \int_{-\infty}^{x} w(t)\,\mathrm{d}t + \int_a^{+\infty} w(t)\,\mathrm{d}t,$$

which is no less than γ. Similarly, if $x \geq a$, then (1.4) gives

$$\frac{1}{2}(f(x) + f(y) - |y-x|) = \gamma + \int_a^x (1 - w(t))\,\mathrm{d}t + \int_{-\infty}^{a} w(t)\,\mathrm{d}t + \int_y^{+\infty} w(t)\,\mathrm{d}t,$$

which is also no less than γ. Finally, if $x \leq a \leq y$, the calculations in (1.4) can be continued as follows

$$\frac{1}{2}(f(x) + f(y) - |y-x|) = \gamma + \int_{-\infty}^{x} w(t)\,\mathrm{d}t + \int_y^{+\infty} w(t)\,\mathrm{d}t \geq \gamma \geq 0. \tag{1.5}$$

Thus $f \in E(\mathbb{R})$ and as in the proof of (i), also $f - \gamma \in E(\mathbb{R})$, so $\gamma \leq l_C(f)$. On the other hand, if $x, y \in C$ and $x \to -\infty$ and $y \to +\infty$, then the expression which follows the last equality sign in (1.5) tends to γ and therefore $l_C(f) = \gamma$. To justify that w is unique, use analogous argument as that in the proof of (i). $\square$

Before we end the section, we shall establish an important property of Katětov hulls, namely the hyperconvexity of them. The fundamental theorem of Aronszajn and Panitchpakdi [1] states that a nonempty metric space (M, ϱ) is hyperconvex iff it is *injective*, i.e. if every nonexpansive map defined on a subset of an arbitrary metric space Y with values in M is extendable to a nonexpansive map defined on the whole space Y (and with values in M as well). They have also shown that M is hyperconvex if $\bigcap_{x \in M} \bar{B}_M(x, f(x)) \neq \emptyset$ for each $f \in E(M)$. For definition and more on hyperconvex spaces the reader is referred to [11].

1.4 Theorem (cf. [8]). *If $r \in (0, +\infty]$ is such that $r \geq \operatorname{diam} X$, then the space $E_r(X)$ is hyperconvex.*

Proof: Firstly we show that $Y = E(X)$ is hyperconvex. Let $F \in E(Y)$. It is enough to show that $\bigcap_{h \in Y} \bar{B}_Y(h, F(h)) \neq \emptyset$. Put $f\colon X \ni x \mapsto F(e_x) \in \mathbb{R}$. It is easily seen that $f \in E(X)$. Furthermore, if $g \in E(X)$, then $|f(x) - g(x)| = \Big|F(e_x) - \|g - e_x\|\Big| \leq F(g)$ for any $x \in X$ and therefore $\|f - g\| \leq F(g)$, which means that $f \in \bar{B}_Y(g, F(g))$.

Now consider a map $\Phi\colon E(X) \ni f \mapsto f \wedge r \in E_r(X)$. It is easy to check that it is well defined. What is more, Φ is nonexpansive and $\Phi(f) = f$ for $f \in E_r(X)$. So, since $E(X)$ is hyperconvex, so is $E_r(X)$. $\square$

2. Central sets

For a subset A of a metric space X and a function $f\colon B \to \mathbb{R}_+$ with $B \supset A$, the symbol $S_X(A, f)$ stands for the intersection $\bigcap_{a \in A} S_X(a, f(a))$, provided A is nonempty (and $S_X(\emptyset, f) = X$), and is called the *common sphere* with centres in A and radii of f. Recall that if the common sphere $S_X(A, f)$ is nonempty, then $f\big|_A$ is a Katětov map upper bounded by $\operatorname{diam} X$.

2.1 Definition. Let X be a nonempty metric space, $n \in \mathbb{N}$ and let $f \in E^i(X)$. A subset A of X is said to be *n-central for f*, if $S_X(A \cup B, F) \neq \emptyset$ for every subset B of X with $\operatorname{card} B \leq n$ and any $F \in E^i(X)$ such that $F\big|_A = f\big|_A$. A is *central for f* if it is n-central for f for each n. Let $\mathcal{O}^n(f) = \mathcal{O}^n_X(f)$ denote the family of all n-central subsets of X for f and let $\mathcal{O}_X(f)$ be the collection of central subsets for f.

The set A is *n-central* if it is n-central for any $f \in E^i(X)$. Similarly, A is *central* if $A \in \mathcal{O}_X(f)$ for each $f \in E^i(X)$. The family of all central [n-central] subsets of X is denoted by $\mathcal{O}(X)$ [$\mathcal{O}_n(X)$]. Note that $A \in \mathcal{O}(X)$ iff $S_X(A \cup B) \neq \emptyset$ for each finite $B \subset X$ and $f \in E^i(X)$.

The reader will easily check the following facts for a metric space (X, d) of positive diameter and its two arbitrary subsets A and B: (a) if $B \subset A$ and

$A \in \mathcal{O}(X)$, then $B \in \mathcal{O}(X)$; (b) if $A \in \mathcal{O}(X)$, then $\bar{A} \in \mathcal{O}(X)$; (c) if $A \in \mathcal{O}(X)$ and B is finite, then $A \cup B \in \mathcal{O}(X)$; (d) if $\varphi\colon X \to X$ is an isometry and $A \in \mathcal{O}(X)$, then $\varphi(A) \in \mathcal{O}(X)$; (e) $X \notin \mathcal{O}(X)$.

Recall that a metric space X is *finitely injective* iff the following condition is fulfilled: whenever B is a metric space of finite cardinality and of diameter no greater than $\operatorname{diam} X$, and A is a subset of B, then every isometric map of A into X admits an isometric extension defined on the whole space B (and with values in X). (In the literature finitely injective spaces satisfy our condition and in addition are unbounded.) It is easy to check that the space X is finitely injective iff $\mathcal{O}(X)$ is nonempty, iff $\mathcal{O}(X)$ contains all finite subsets of X.

Now it is a good time to put

2.2 Definition. An *Urysohn space* is a separable complete metric space X such that every separable metric space of diameter no greater than $\operatorname{diam} X$ is isometrically embeddable in X and each isometry between finite subsets of X is extendable to an isometry of X onto itself. An Urysohn space is nontrivial if it has more than one point.

For an arbitrarily fixed number $r \in [0, +\infty]$ there is a unique (up to isometry) Urysohn space of diameter r. We shall denote it by $\mathbb{U}_r$, and $\mathbb{U}$ will stand for the unbounded Urysohn space.

The fundamental result on Urysohn spaces is the following result due to Urysohn [15] (cf. [13, Section 3], [14, Lemma 5.1.17] or [18, Proposition 3.10] and references therein).

2.3 Theorem. *The completion of a finitely injective metric space is finitely injective. A metric space is Urysohn iff it is separable, complete and finitely injective.*

In the sequel we shall prove the strengthened version of the above result (see Corollary 2.16).

2.4 Lemma. *Let (X, d) be a nonempty metric space and let $f \in E^i(X)$. If A and B are two nonempty members of $\mathcal{O}^1_X(f)$, then*

$$\operatorname{dist}_d(S_X(A, f), S_X(B, f)) \le 2 \operatorname{dist}_d(A, B).$$

Proof: Since $A, B \in \mathcal{O}^1_X(f)$, the sets $S_X(A, f)$ and $S_X(B, f)$ are nonempty. It suffices to show that $\operatorname{dist}_d(x, S_X(B, f)) \le 2 \operatorname{dist}_d(A, B)$ for any $x \in S_X(A, f)$. If $x \in S_X(A, f)$, then $d(x, a) = f(a)$ for each $a \in A$. Let $F \in E(B \cup \{x\})$ be an extension of $f\big|_B$ with $F(x) = \sup_{b \in B} |f(b) - d(x, b)|$. Then $F(x) \le f(x)$. We infer from this that $S_X(B \cup \{x\}, F) \neq \emptyset$. Take any element c of the latter common sphere. Then easily $c \in S_X(B, f)$ and $d(x, c) = F(x)$. So, it is enough to check that $F(x) \le 2 \operatorname{dist}_d(A, B)$. For arbitrary $b \in B$ and $a \in A$, we have

$$d(b, x) - f(b) \le d(b, a) + d(a, x) - f(b) = d(a, b) + f(a) - f(b) \le 2d(a, b) \tag{2.1}$$

and

$$(2.2)\quad f(b) - d(b,x) \leq f(b) - d(a,x) + d(a,b) = f(b) - f(a) + d(a,b) \leq 2d(a,b).$$

Now, by (2.1), $d(b,x) - f(b) \leq \inf_{a\in A} 2d(a,b) = 2\operatorname{dist}_d(b,A) \leq 2\operatorname{dist}_d(A,B)$. Similarly, by (2.2), $f(b) - d(b,x) \leq 2\operatorname{dist}_d(A,B)$. $\square$

2.5 Theorem. *Let (X,d) be a complete metric space. If $(A_n)_n$ is a sequence of nonempty members of $\mathcal{O}(X)$ such that*

$$(2.3)\qquad \operatorname{dist}_d(A_n, A) \to 0 \ (n \to +\infty)$$

for some nonempty subset A of X, then $A \in \mathcal{O}(X)$ as well.

Proof: Let B be a finite subset of X and let f be an inner Katětov map on X. Since $A_n \in \mathcal{O}(X)$, therefore $A_n \cup B \in \mathcal{O}(X) \subset \mathcal{O}^1_X(f)$ for each n. Moreover, $\operatorname{dist}_d(A_n \cup B, A_m \cup B) \leq \operatorname{dist}_d(A_n, A_m)$. But this, combined with Lemma 2.4, yields $\operatorname{dist}_d(S_X(A_n \cup B, f), S_X(A_m \cup B, f)) \leq 2\operatorname{dist}_d(A_n, A_m)$, which implies that the sequence $(S_X(A_n \cup B, f))_n$ is a fundamental sequence in the space $\mathcal{D}_b(X)$ of all nonempty, bounded and closed subsets of X. Since X is complete, so is the space $\mathcal{D}_b(X)$ with respect to the Hausdorff distance (see [4]). Therefore there exists a nonempty set V such that

$$(2.4)\qquad \operatorname{dist}_d(S_X(A_n \cup B, f), V) \to 0 \ (n \to +\infty).$$

Take any $v \in V$. We shall show that $v \in S_X(A \cup B, f)$. By (2.4), there exists a sequence $(v_n)_n$ such that $v_n \in S_X(A_n \cup B, f)$ for every n and $d(v_n, v) \to 0$ $(n \to +\infty)$. Let $a \in A \cup B$. If $a \in B$, then $d(v_n, a) = f(a)$ and since $v_n \to v$ $(n \to +\infty)$, so $f(a) = d(v,a)$, which means that $v \in S_X(a, f(a))$. Now assume that $a \in A$. As before, thanks to (2.3), there exists a sequence $(a_n)_n$ such that $a_n \in A_n$ for each n and $d(a_n, a) \to 0$ $(n \to +\infty)$. Now for any n, we have $f(a_n) = d(v_n, a_n)$ and hence, by the continuity of f, $f(a) = d(v,a)$, which yields that $v \in S_X(a, f(a))$. $\square$

Having the above theorem, we immediately get

2.6 Corollary. *If K is a precompact subset of a complete metric space X and $A \in \mathcal{O}(X)$, then $A \cup K \in \mathcal{O}(X)$.*

Note that Huhunaišvili's theorem [7] follows from Corollary 2.6.

Further properties of central sets and common spheres in Urysohn spaces are collected in the next theorem. A part of them is known (we shall comment this after the proof).

2.7 Theorem. *Let $r \in (0,+\infty]$, d be the metric of $\mathbb{U}_r$ and let A be a nonempty member of $\mathcal{O}(\mathbb{U}_r)$.*

(U0) *$E(A)$ is separable (it is enough to require that $A \in \mathcal{O}_0(\mathbb{U}_r)$).*

(U1) *If* $f\colon A \to \mathbb{R}_+$ *is any function (not necessarily a Katětov map), then the set* $Z = \mathbb{U}_r \setminus \bigcup_{a\in A} B_{\mathbb{U}_r}(a, f(a))$ *is isometric to* $\mathbb{U}_r$*, provided* Z *is nonempty. What is more,* Z *is nonempty if and only if:*
- $f(a) \leq r$ *for each* $a \in A$*, provided* $r < +\infty$*,*
- *there exists* $x \in \mathbb{U}_r$ *for which the map* $f - e_x\big|_A$ *is bounded, provided* $r = +\infty$*.*

(U2) *If* $f \in E^i(\mathbb{U}_r)$*, then the set* $T = S_{\mathbb{U}_r}(A, f)$ *is isometric to* $\mathbb{U}_s$ *with* $s = r \wedge \delta_A(f)$*.*

(U3) *If* A *is bounded and* $s \in \mathbb{R}_+$ *is such a number that* $\frac{1}{2}\operatorname{diam} A \leq s \leq r$*, then the set* $\Delta(A, s) = \{x \in \mathbb{U}_r\colon\ e_x\big|_A = \text{const} \geq s\}$ *is isometric to* $\mathbb{U}_r$*.*

(U4) *The map* $E_r(A) \ni f \mapsto S_{\mathbb{U}_r}(A, f) \in \mathcal{D}_b(\mathbb{U}_r)$ *is isometric. The family* $\{S_{\mathbb{U}_r}(A, f)\}_{f\in E_r(A)}$ *is a cover of* $\mathbb{U}_r$ *and consists of pairwise disjoint subsets, and there is an isometric map* $\psi\colon E_r(A) \to \mathbb{U}_r$ *such that* $\psi(f) \in S_{\mathbb{U}_r}(A, f)$ *for each* $f \in E_r(A)$*.*

(U5) *There exists a family of isometric maps* $(\varphi_t\colon \mathbb{U}_r \to \mathbb{U}_r)_{t\in I}$*, where* $I = [0, r] \cap \mathbb{R}$*, such that* $\varphi_0 = \mathrm{id}_{\mathbb{U}_r}$ *and*

$$d(\varphi_t(x), \varphi_s(x)) = |t - s| \quad \textit{and} \quad \operatorname{dist}_d(\varphi_t(x), A) \geq t \tag{2.5}$$

for each $x \in \mathbb{U}_r$ *and* $t, s \in I$*.*

(U6) *If* Y *is a separable metric space of diameter no greater than* r *and* B *its subset, then every isometric map* φ *of* B *into* A *is extendable to an isometric map of* Y *into* $\mathbb{U}_r$*. Every isometry between* A *and another central subset of* $\mathbb{U}_r$ *is extendable to an isometry of the whole space* $\mathbb{U}_r$*.*

(U7) *There is a hyperconvex subset* R *of* $\mathbb{U}_r$ *such that* $A \subset R$*. In particular, if* Y *is a metric space and* B *is a subset of* Y*, then every nonexpansive map of* B *into* A *is extendable to a nonexpansive map of* Y *into* $\mathbb{U}_r$*.*

Proof: The points (U0) and (U6) have standard proofs (therefore we omit them), while (U7) follows from (U4) and Theorem 1.4.

(U1): The set Z is clearly closed. We shall show that it is finitely injective and that $\operatorname{diam} Z = r$, provided $Z \neq \emptyset$. Clearly, $\operatorname{diam} Z \leq r$. Let B be a finite nonempty subset of Z and let $g \in E_r(B)$. Put $G = \widehat{g} \wedge r$. Then $G \in E^i(\mathbb{U}_r)$ and $S_{\mathbb{U}_r}(A \cup B, G) \neq \emptyset$, since $A \in \mathcal{O}(\mathbb{U}_r)$. It suffices to check that $G(a) \geq f(a)$ for $a \in A$, because then $S_{\mathbb{U}_r}(A \cup B, G) \subset S_{\mathbb{U}_r}(B, g) \cap Z$. If $a \in A$, then $f(a) \leq r$ and we only need to show that $\inf_{b\in B}(g(b) + d(b, a)) \geq f(a)$. But if $b \in B$, then $b \in Z$ and therefore $b \notin B_{\mathbb{U}_r}(a, f(a))$, which finally gives $g(b) + d(b, a) \geq d(b, a) \geq f(a)$. The remainder is simple.

(U2): The nonemptiness of T simply follows from the fact that A is central. Also T is clearly closed. If $a, b \in T = S_{\mathbb{U}_r}(A, f)$ and $x \in A$, then $d(a, b) \leq d(a, x) + d(b, x) = 2f(x)$, which yields that $\operatorname{diam} T \leq \delta_A(f)$. Now let B be a finite subset of T and let $g \in E(B)$ be such a Katětov map that g is upper bounded by $r \wedge \delta_A(f)$ on B. We shall show that

$$f \cup g \in E(A \cup B). \tag{2.6}$$

Take $a \in A$ and $b \in B$. Since $b \in T$, hence $f(a) - g(b) = d(a,b) - g(b) \leq d(a,b)$. On the other hand, $g(b) - f(a) \leq \delta_A(f) - f(a) \leq f(a) = d(a,b)$ and therefore $|f(a) - g(b)| \leq d(a,b)$. What is more, $d(a,b) = f(a) \leq f(a) + g(b)$, which finishes the proof of (2.6). Now since A is central, therefore $S_{\mathbb{U}_r}(A \cup B, h) \neq \emptyset$, where $h \in E(A \cup B)$ is the union of f and g. If c belongs to the latter common sphere, then $c \in S_{\mathbb{U}_r}(A, f) = T$ and $c \in S_{\mathbb{U}_r}(B, g)$ and thus $S_{\mathbb{U}_r}(B, g) \cap T$ is nonempty.

(U3): It is clear that the set $\Delta(A, s)$ is closed and that $\operatorname{diam} \Delta(A, s) \leq r$. It is nonempty, since it contains $S_{\mathbb{U}_r}(A, f)$, where $f \equiv s$ on A. Let B be a finite nonempty subset of $\Delta(A, s)$ and let $g \in E(B)$ be such a Katětov map that g is upper bounded by r on B. Put $G = \widehat{g} \wedge r \in E^i(\mathbb{U}_r)$. Take $y \in S_{\mathbb{U}_r}(A \cup B, G)$. Then $y \in S_{\mathbb{U}_r}(B, g)$ (because $G\big|_B = g$) and $d(x, y) = G(x)$ for $x \in A$. If $x_1, x_2 \in A$, then, by the definition of $\Delta(A, s)$ $(\supset B)$, $\widehat{g}(x_1) = \inf_{b \in B}(d(b, x_1) + g(b)) = \inf_{b \in B}(d(b, x_2) + g(b)) = \widehat{g}(x_2)$ and therefore $e_y(x_1) = G(x_1) = G(x_2) = e_y(x_2)$. What is more, $\widehat{g}(x_1) \geq \inf_{b \in B} d(x_1, b) \geq s$, so $e_y\big|_A = \text{const} \geq s$, which means that $y \in \Delta(A, s)$.

(U4): By (U2), we know that the map is well defined. Fix $f, g \in E_r(A)$. Take $x \in S_{\mathbb{U}_r}(A, f)$. Then, for any $y \in S_{\mathbb{U}_r}(A, g)$ and each $a \in A$, $|f(a) - g(a)| = |d(x, a) - d(y, a)| \leq d(x, y)$ and hence $\|f - g\| \leq \operatorname{dist}_d(x, S_{\mathbb{U}_r}(A, g))$. In order to prove the converse inequality, first we shall show that

$$\tilde{g}(x) = \|f - g\| \quad \text{and} \quad \tilde{g}\big|_A = g \tag{2.7}$$

defines a Katětov map $\tilde{g}$ on $A \cup \{x\}$. For any $a \in A$ we have: $d(a, x) = f(a) \leq g(a) + \|f - g\|$ and $d(a, x) = f(a) \geq g(a) - \|f - g\|$. Moreover, $\|f - g\| \leq f(a) + g(a) = d(x, a) + g(a)$. The last three inequalities imply that $\big|\|f - g\| - g(a)\big| \leq d(x, a) \leq \|f - g\| + g(a)$ and thus the condition (2.7) is indeed satisfied. Now since $\|f - g\| \leq r$ (because the images of f and g are included in $[0, r]$) and $A \in \mathcal{O}(\mathbb{U}_r)$, so there is $y \in \mathbb{U}_r$ such that $d(y, a) = g(a)$ for $a \in A$ and $d(x, y) = \|f - g\|$. We have obtained $y \in S_{\mathbb{U}_r}(A, g)$ satisfying $d(x, y) = \|f - g\|$, which shows that $\operatorname{dist}_d(x, S_{\mathbb{U}_r}(A, g)) = \|f - g\|$. The same argument proves that $\operatorname{dist}_d(y, S_{\mathbb{U}_r}(A, f)) = \|f - g\|$ for each $y \in S_{\mathbb{U}_r}(A, g)$ and therefore $\operatorname{dist}_d(S_{\mathbb{U}_r}(A, f), S_{\mathbb{U}_r}(A, g)) = \|f - g\|$. It is clear that the suitable family of common spheres consists of pairwise disjoint nonempty sets which cover $\mathbb{U}_r$.

Now let $E = \{e_a\big|_A \colon a \in A\} \subset E_r(A)$ and $\psi_0 \colon E \ni e_a\big|_A \mapsto a \in A$. Then ψ_0 is isometric. Since $A \in \mathcal{O}(\mathbb{U}_r)$, $E_r(A)$ is separable and $\operatorname{diam} E_r(A) \leq r$, hence, by (U6), there is an isometric map $\psi \colon E_r(A) \to \mathbb{U}_r$ such that $\psi\big|_E = \psi_0$. Let $a \in A$ and $f \in E_r(A)$. We have $d(a, \psi(f)) = d(\psi(e_a\big|_A), \psi(f)) = \|e_a\big|_A - f\| = f(a)$, which shows that $\psi(f) \in S_{\mathbb{U}_r}(A, f)$.

(U5): Let $\{r_n\}_{n \geq 0}$ be a sequence of distinct real numbers such that $r_0 = 0$ and the set $R = \{r_n \colon n \geq 0\}$ is a dense subset of I. We shall build, using induction, a sequence of isometric maps $(\varphi_{r_n})_{n=0}^{\infty}$, with $\varphi_0 = \mathrm{id}_{\mathbb{U}_r}$, which satisfies the condition (2.5) for $t, s \in R$. Put $\varphi_0 = \mathrm{id}_{\mathbb{U}_r}$ and assume that the maps φ_{r_j} are defined for $j = 0, \dots, n-1$, where $n \geq 1$, in such a way that (2.5) is fulfilled for $t, s \in R_n = \{r_0, \dots, r_{n-1}\}$. Let $D = \{d_n \colon n \geq 1\}$ be a dense subset of $\mathbb{U}_r$. We shall define an isometric map $\Phi\colon D \to \mathbb{U}_r$ such that

$$d(\Phi(x), \varphi_t(x)) = |r_n - t| \qquad \text{and} \qquad \operatorname{dist}_d(\Phi(x), A) \geq r_n \tag{2.8}$$

for each $x \in D$ and $r \in R_n$. Suppose that Φ is defined (and satisfies the above conditions) for $x \in D_n = \{d_k\colon\ 0 < k < n\}$ with $n \geq 1$. Let $Y = \{\varphi_t(x)\colon\ t \in R_n\}$ and put $g\colon Y \ni \varphi_t(x) \mapsto |t - r_n| \in \mathbb{R}_+$. Observe that g is a Katětov map. What is more, $(e_x \circ \Phi^{-1}) \cup g \in E(\Phi(D_n) \cup Y)$. So, the map $f\colon \Phi(D_n) \cup Y \to \mathbb{R}_+$ defined by $f(z) = d(x, \Phi^{-1}(z))$ for $z \in \Phi(D_n)$ and $f(\varphi_t(x)) = |t - r_n|$ for $t \in R_n$ is well defined and Katětov. Moreover, f is upper bounded by r. Thus $\widehat{f} \wedge r$ is an inner Katětov map which extends f. Since A is central, hence there is $w \in \mathbb{U}_r$ such that

$$\widehat{f}(u) \wedge r = d(u, w) \tag{2.9}$$

for $u \in A \cup D_n \cup Y$. This yields that the formula $\Phi(x) = w$ extends Φ to an isometric map from $D_n \cup \{x\}$ into $\mathbb{U}_r$ in such a way that $d(\Phi(x), \varphi_t(x)) = |r_n - t|$ for $t \in R_n$. So, it remains to check that $\operatorname{dist}_d(\Phi(x), A) \geq r_n$ or, equivalently (thanks to (2.9)), that $\widehat{f}(a) \geq r_n$ for each $a \in A$. Take $a \in A$ and recall that $\widehat{f}(a) = \inf_{z \in \Phi(D_n) \cup Y}(f(z) + d(z, a))$. If $z = \varphi_t(x)$ with $t \in R_n$, then $f(z) + d(z, a) \geq |r_n - t| + t \geq r_n$. On the other hand, if $z = \Phi(y)$ for some $y \in D_n$, then $f(z) + d(z, a) \geq d(\Phi(y), a) \geq \operatorname{dist}_d(\Phi(y), A) \geq r_n$, where the last inequality follows from (2.8) for y. This implies that $\widehat{f}(a) \geq r_n$.

Having the map $\Phi\colon D \to \mathbb{U}_r$ satisfying (2.8), it remains to define the map φ_{r_n} as the unique extension of Φ.

Thus the sequence $(\varphi_{r_n})_n$ has been constructed. Finally put $\varphi_t(x) = \lim_{r_n \to t} \varphi_{r_n}(x)$ for each $t \in I$ and $x \in \mathbb{U}_r$. It is easy to check that the family $(\varphi_t)_{t \in I}$ defined in such a way satisfies (2.5). $\square$

The point (U1) says that Urysohn space have fractal properties. A special case of (U2) (when A is finite) was done by Melleray [13, §4.2]. Melleray has also shown in [12] that if A is a (nonempty) Heine-Borel subset of $\mathbb{U}$, then for each $M > 0$ the set $\{x \in \mathbb{U}\colon\ \operatorname{dist}_d(x, A) \geq M\}$ is isometric to $\mathbb{U}$ — this is related to our property (U1).

Another facts on 'spherical' geometry of Urysohn spaces are proved below.

2.8 Proposition. (a) *Let $r, s \in (0, +\infty]$. Let u and v be two arbitrary elements of $\mathbb{U}_r$ and $\mathbb{U}_s$, respectively. If p and q are two positive (finite) numbers such that $p \leq \frac{1}{2}r$ and $q \leq \frac{1}{2}s$, then the balls $\bar{B}_{\mathbb{U}_r}(u, p)$ and $\bar{B}_{\mathbb{U}_s}(v, q)$ are Λ-isometric.*

(b) *If $r < +\infty$ and s and t are two different numbers from the interval $[\frac{1}{2}r, r]$, then the closed balls $\bar{B}_{\mathbb{U}_r}(a, s)$ and $\bar{B}_{\mathbb{U}_r}(a, t)$ (where $a \in \mathbb{U}_r$ is arbitrary) are **not** Λ-isometric.*

Proof: To prove (a), use the back-and-forth method, starting with $\varphi(u) = v$.

To see (b), suppose, for the contrary, that $s < t$ and that there exists a Λ-isometry $\varphi\colon \bar{B}_{\mathbb{U}_r}(a, t) \to \bar{B}_{\mathbb{U}_r}(a, s)$. First of all, by (U2), $\operatorname{diam} \bar{B}_{\mathbb{U}_r}(a, t) =$

$\operatorname{diam} \bar{B}_{\mathbb{U}_r}(a,s) = r$ and therefore φ is an isometry. Let $b = \varphi^{-1}(a)$ and $q = (d(a,b)+t) \wedge r \geq t$. It is easy to see that the formulas $a \mapsto t$ and $b \mapsto q$ define a Katětov map on $\{a,b\}$ and hence there is $z \in \mathbb{U}_r$ such that $d(a,z) = t$ and $d(b,z) = q$. But then $z \in \bar{B}_{\mathbb{U}_r}(a,t)$ and $d(\varphi(z),a) = d(\varphi(z),\varphi(b)) = d(z,b) \geq t > s$, which denies the connection $\varphi(z) \in \bar{B}_{\mathbb{U}_r}(a,s)$. $\square$

Now we shall give two examples dealing with central subsets and common spheres.

2.9 Example. Let $r \in (0,+\infty]$. The assumption that $A \in \mathcal{O}(\mathbb{U}_r)$ in (U2) is essential: if a and b are two distinct points of $\mathbb{U}_r$ and $s = \frac{1}{2}d(a,b)$, then for $A = S_{\mathbb{U}_r}(\{a,b\},s)$ one has $S_{\mathbb{U}_r}(A,s) = \{a,b\}$ and $S_{\mathbb{U}_r}(A \cup \{a\},s) = \emptyset$, although the constant map 's' is a Katětov map on $A \cup \{a\}$.

Also the set $\Delta(A,s)$ in (U3) cannot be replaced by the set $P = \{x \in \Delta(A,s) : e_x \leq 2s \text{ on } A\}$, i.e. the set P is not finitely injective in general: if $A = \{a\}$ and $0 < s \leq \frac{1}{4}r$, then $P = \bar{B}_{\mathbb{U}_r}(a,2s) \setminus B_{\mathbb{U}_r}(a,s)$ is not finitely injective.

Similarly as central sets one may define *absolutely central* spaces: a separable metric space is *absolutely central* if each isometric copy of it in any Urysohn space is central. Let $\mathcal{AO}$ denote the class of all absolutely central (separable) metric spaces. Melleray [12] has proved the following

2.10 Theorem. *$\mathcal{AO}$ coincides with the class of precompact spaces.*

The above statement means that for every separable non-precompact metric space D there is a subset D' of the unbounded Urysohn space $\mathbb{U}$ which is isometric to D and is not central. This is however immediate for spaces whose Katětov hulls are non-separable — every isometric copy of such a metric space in any Urysohn space is not 0-central (which follows from (U0)). So, one may ask which metric spaces can have central (isometric) copies in the unbounded Urysohn space. Our next aim is to give such a characterization. The main result of the paper is:

2.11 Theorem. *A metric space X can be isometrically embedded as a central subset of $\mathbb{U}$ iff X has the collinearity property.*

The necessity follows from Theorem 1.2 and (U0). The proof of the sufficiency will be preceded by the next few lemmas. The first of them can easily be deduced from the note in [13, Definition 6.8]:

2.12 Lemma. *For every nonempty space (A,d_A) with separable Katětov hull there exists an unbounded separable finitely injective space $(\widetilde{A},\widetilde{d}\,)$ such that $(A,d_A) \subset (\widetilde{A},\widetilde{d}\,)$ and $A \in \mathcal{O}(\widetilde{A}\,)$.*

Now we want to show that if A is a central subset of Z, then A is central in the completion of Z as well. For simplicity, we fix the situation.

From now on, (X,d) is a complete metric space and Z and A are such nonempty subsets of X that Z is dense in X and

$$S_X(A \cup B, f) \neq \emptyset \tag{2.10}$$

for every finite subset B of Z and any $f \in E^i(X)$. Additionally, put $r = \operatorname{diam} X$ (we do not assume that X is unbounded). Under these assumptions we state and prove the next three lemmas.

2.13 Lemma. *For any $x_1, \ldots, x_p \in X$, each $\varepsilon > 0$ and $f \in E^i(X)$ there exists $z \in X$ such that $f(x) - \varepsilon \leq d(z, x) \leq f(x) + \varepsilon$ for every $x \in A \cup \{x_1, \ldots, x_p\}$.*

Proof: Since Z is dense in X, there are points $z_1, \ldots, z_p$ in Z such that $d(x_j, z_j) \leq \frac{\varepsilon}{4}$ for $j = 1, \ldots, p$. Put $g(x) = (f(x) + \frac{\varepsilon}{2}) \wedge r$ $(x \in X)$. Since $g \in E^i(X)$ and by (2.10), there is $z \in X$ such that $d(z, x) = g(x)$ for $x \in A \cup \{z_1, \ldots, z_p\}$. Observe that if $a \in A$, then $f(a) \leq g(a) = d(a, z) \leq f(a) + \varepsilon$. Finally, if $j \in \{1, \ldots, p\}$, then

$$\begin{aligned} f(x_j) - \varepsilon &\leq g(x_j) - \frac{\varepsilon}{2} \leq g(z_j) + d(x_j, z_j) - \frac{\varepsilon}{2} \leq g(z_j) - \frac{\varepsilon}{4} = d(z_j, z) - \frac{\varepsilon}{4} \\ &\leq d(z_j, x_j) + d(x_j, z) - \frac{\varepsilon}{4} \leq d(x_j, z) \leq d(x_j, z_j) + d(z_j, z) \leq \frac{\varepsilon}{4} + g(z_j) \\ &\leq \frac{\varepsilon}{4} + g(x_j) + d(x_j, z_j) \leq f(x_j) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = f(x_j) + \varepsilon. \end{aligned}$$

□

2.14 Lemma. *If $x_1, \ldots, x_p, z \in X$, $f \in E^i(X)$ and $\varepsilon > 0$ are such that $0 < f(x) - \frac{1}{8}\varepsilon \leq d(x, z) \leq f(x) + \varepsilon$ for each $x \in A \cup \{x_1, \ldots, x_p\}$, then there is $z' \in X$ such that $0 < f(x) - \frac{1}{8}\varepsilon' \leq d(x, z') \leq f(x) + \varepsilon'$ for $x \in A \cup \{x_1, \ldots, x_p\}$ and $d(z, z') \leq \varepsilon'$, where $\varepsilon' = \frac{3}{4}\varepsilon$.*

Proof: First of all, observe that $z \notin B$, where $B = A \cup \{x_1, \ldots, x_p\}$. (Indeed, for $x = z$ one of the two inequalities $0 < f(x) - \frac{1}{8}\varepsilon \leq d(x, z)$ is impossible.) Now define $g\colon B \cup \{z\} \to \mathbb{R}_+$ by the formulas: $g(x) = f(x) + \frac{1}{2}\varepsilon$ for $x \in B$ and $g(z) = \frac{5}{8}\varepsilon$. The map g is easily Katětov on B. What is more, if $x \in B$, then $f(x) > \frac{1}{8}\varepsilon$ and hence $|g(x) - g(z)| = f(x) - \frac{1}{8}\varepsilon \leq d(x, z) \leq f(x) + \varepsilon \leq g(x) + g(z)$. Thus g is a Katětov map and therefore, by Lemma 2.13 applied for $x_1, \ldots, x_p, z$, $\frac{1}{32}\varepsilon$ and $\widehat{g} \wedge r$, there is $z' \in X$ such that $g(x) \wedge r - \frac{1}{32}\varepsilon \leq d(x, z') \leq g(x) + \frac{1}{32}\varepsilon$ for $x \in B \cup \{z\}$. This means that for $x \in B$, $0 < f(x) - \frac{1}{8}\varepsilon < f(x) - \frac{1}{8}\varepsilon' \leq g(x) \wedge r - \frac{1}{32}\varepsilon \leq d(x, z') \leq g(x) + \frac{1}{32}\varepsilon \leq f(x) + \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon = f(x) + \varepsilon'$ and $d(z, z') \leq g(z) + \frac{1}{32}\varepsilon \leq \varepsilon'$. □

2.15 Lemma. *For every $f \in E^i(X)$ and each finite subset B of X, the common sphere $S_X(A \cup B, f)$ is nonempty.*

Proof: If $\delta_X(f) = 0$, then $f \in \mathfrak{e}(X)$ and hence easily $S_X(A \cup B, f) \neq \emptyset$. So, we may assume that $\delta_X(f) > 0$. Put $\varepsilon_n = (\frac{3}{4})^n \delta_X(f) > 0$. By Lemma 2.13, there exists $z_1 \in X$ such that $f(x) - \frac{1}{8}\varepsilon_1 \leq d(x, z_1) \leq f(x) + \varepsilon_1$ for $x \in A \cup B$. Observe that $f(x) \geq \frac{1}{2}\delta_X(f) > \frac{1}{8}\varepsilon_1$ for each $x \in A \cup B$. Now making use of Lemma 2.14 and induction, we obtain points $z_2, z_3, z_4, \ldots$ of X such that

$$0 < f(x) - \frac{1}{8}\varepsilon_n \leq d(x, z_n) \leq f(x) + \varepsilon_n \tag{2.11}$$

and $d(z_n, z_{n+1}) \leq \varepsilon_{n+1}$ for each $x \in A \cup B$ and $n \geq 1$. Since $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$, so the sequence $(z_n)_n$ is fundamental. By the completeness of X, there is $z \in X$ such that $\lim_{n\to+\infty} d(z_n, z) = 0$. Now letting $n \to +\infty$ in (2.11), we obtain $d(x, z) = f(x)$ for each $x \in A \cup B$, which finishes the proof. □

2.16 Corollary. *If A is a central subset of a metric space X, then A is central in the completion of X.*

Now to prove the 'if' part of Theorem 2.11, it suffices to combine Theorem 1.2 with Lemma 2.12 and Corollary 2.16. Note also that, in particular, we have obtained an alternative proof of the theorem of Urysohn (Theorem 2.3).

Having in mind Theorem 2.10, the next result is rather surprising.

2.17 Proposition. *Let (X, d) be a complete metric space and let B be a central subset of X. If A is a nonempty subset of X which is isometrically embeddable in $\mathbb{R}$ and $f \in E(X)$ is such that $l_A(f) = 0$, then the common sphere $S_X(A \cup B, f)$ is nonempty.*

Proof: Thanks to Corollary 2.6, we may assume that A is unbounded, and closed. Let $\tilde{A}$ be a subset of $\mathbb{R}$ which is isometric to A and such that $0 \in \tilde{A}$, $\sup \tilde{A} = +\infty$ and $\inf \tilde{A} \in \{-\infty, 0\}$. Let $\psi\colon A \to \tilde{A}$ be an isometry. Put $b = \psi^{-1}(0)$, $K_n = \bar{B}_X(b, n) \cap A$ and $F_n = S_X(B \cup K_n, f)$ $(n \geq 1)$. By Corollary 2.6, each F_n is nonempty, closed and bounded. What is more,

$$F_n \supset F_{n+1}. \tag{2.12}$$

We shall show that $(F_n)_n$ is a fundamental sequence with respect to the Hausdorff distance. Thanks to (2.12), it is enough to estimate the numbers $\mathrm{dist}_d(x, F_n)$ for $x \in F_m$ and $m < n$. Let $g = f \circ \psi^{-1} \in E(\tilde{A})$. First we will show that g has an extension of the form (1.2) with $a = \gamma = 0$ which is a Katětov map. Indeed, if $\inf \tilde{A} = -\infty$, then it is enough to apply Theorem 1.3(ii) for $\widehat{g}$ (recall that in that case $\gamma = l_{\tilde{A}}(g) = l_A(f) = 0$). On the other hand, if $\inf \tilde{A} = 0$, then we may apply Theorem 1.3(i) for $\widehat{g}\big|_{\mathbb{R}_+}$ to conclude that $g(x) = x + \int_0^{+\infty} u(t) \operatorname{sgn}(t - x)\, \mathrm{d}t$ for each $x \in \tilde{A}$ and some integrable function $u\colon \mathbb{R}_+ \to [0, 1]$. So, if $w\colon \mathbb{R} \to [0, 1]$ is equal to 0 on $(-\infty, 0)$ and coincides with u on $\mathbb{R}_+$, then w is integrable and

$$g(x) = |x| + \int_{-\infty}^{+\infty} w(t) \operatorname{sgn}(t) \operatorname{sgn}(t - x)\, \mathrm{d}t \tag{2.13}$$

for $x \in \tilde{A}$. Thus we have shown that in both the cases g has an extension of the form (2.13). And, what is important,

$$\int_{-\infty}^{0} w(t)\, \mathrm{d}t = 0 \qquad \text{if} \quad \inf \tilde{A} = 0. \tag{2.14}$$

Now for $n \geq 1$ put $b_n = \max(\tilde{A} \cap [0, n])$ and $a_n = \min(\tilde{A} \cap [-n, 0])$. Observe that the sequence $(b_n)_n$ tends to $+\infty$ and so does $(-a_n)_n$ if $\inf \tilde{A} \neq 0$.

Let $n > m \geq 1$ and $x \in F_m$. Note that $x \in S_X(B \cup K_n, e_x)$ and hence

$$\operatorname{dist}_d(x, F_n) \leq \operatorname{dist}_d(S_X(B \cup K_n, e_x), S_X(B \cup K_n, f)). \tag{2.15}$$

Further, the proof of (U4) shows that

$$\operatorname{dist}_d(S_X(B \cup K_n, e_x), S_X(B \cup K_n, f)) = \|e_x\big|_{B\cup K_n} - f\big|_{B\cup K_n}\|. \tag{2.16}$$

Since $x \in F_m$, so $x \in S_X(B, f)$ and therefore $e_x\big|_B = f\big|_B$. Thus $\|e_x\big|_{B\cup K_n} - f\big|_{B\cup K_n}\| = \|e_x\big|_{K_n} - f\big|_{K_n}\|$. This, combined with (2.15) and (2.16), yields

$$\operatorname{dist}_d(x, F_n) \leq \|e_x\big|_{K_n} - f\big|_{K_n}\|. \tag{2.17}$$

Now put $h = e_x \circ \psi^{-1} \in E(\tilde{A})$. Observe that

$$h(x) = g(x) \qquad \text{for } \ x \in \tilde{A} \cap [-m, m], \tag{2.18}$$

because $x \in S_X(K_m, f)$, and

$$\|e_x\big|_{K_n} - f\big|_{K_n}\| \leq \|h - g\|. \tag{2.19}$$

We shall show that

$$|h(x) - g(x)| \leq 2\int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t - 2\int_{a_m}^{b_m} w(t)\,\mathrm{d}t \qquad (x \in \tilde{A}). \tag{2.20}$$

Let $x \in \tilde{A}$. By (2.18), we may assume that $|x| > m$ and then $\sup\{\big|g(x) - |x - y|\big|\colon\ y \in \tilde{A} \cap [-m, m]\} \leq g(x), h(x) \leq \inf\{g(z) + |x - z|\colon\ z \in \tilde{A} \cap [-m, m]\}$. First assume that $x > m$. Substituting $y = a_m$ and $z = b_m$, we conclude that

$$|h(x) - g(x)| \leq g(b_m) + |x - b_m| - \big|g(a_m) - |x - a_m|\big|.$$

Since $a_m \leq 0 \leq b_m \leq m < x$ and thanks to (2.13), we have

$$|h(x) - g(x)| \leq g(a_m) + g(b_m) + a_m - b_m = 2\int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t - 2\int_{a_m}^{b_m} w(t)\,\mathrm{d}t,$$

which proves the inequality (2.20). The case of $x < -m$ is similar (substitute $y = b_m$ and $z = a_m$).

Now (2.17), (2.19) and (2.20) ensure us that

$$\operatorname{dist}_d(F_m, F_n) \leq 2\int_{-\infty}^{+\infty} w(t)\,\mathrm{d}t - 2\int_{a_m}^{b_m} w(t)\,\mathrm{d}t.$$

But the right-hand side expression tends to 0 if $m \to +\infty$. Indeed, if $\inf \tilde{A} = -\infty$, then $\lim_{n\to+\infty} a_n = -\lim_{n\to+\infty} b_n = -\infty$, while if $\inf \tilde{A} = 0$, then, by (2.14), this expression is equal to $2\int_0^{+\infty} w(t)\,\mathrm{d}t - 2\int_0^{b_m} w(t)\,\mathrm{d}t$, which tends to 0 as well. So, $(F_n)_n$ is a fundamental sequence and therefore there is a nonempty and closed subset F of X such that $\lim_{n\to+\infty} \mathrm{dist}_d(F_n, F) = 0$. By (2.12), $F = \bigcap_{n=1}^{\infty} F_n$, which simply implies that $F \subset S_X(A \cup B, f)$. $\square$

We end the paper with the following

Question. Does the assertion of Proposition 2.17 remain true if we replace the assumption that A *is isometrically embeddable in* $\mathbb{R}$ by A *has the collinearity property*?

References

[1] Aronszajn N., Panitchpakdi P., *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439.

[2] Cameron P.J., Vershik A.M., *Some isometry groups of Urysohn space*, Ann. Pure Appl. Logic **143** (2006), no. 1–3, 70–78.

[3] Gao S., Kechris A.S., *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), no. 766, 78 pp.

[4] Hahn H., *Reelle Funktionen I*, Akademische Verlagsgesellschaft, Leipzig, 1932.

[5] Holmes M.R., *The universal separable metric space of Urysohn and isometric embeddings thereof in Banach spaces*, Fund. Math. **140** (1992), 199–223.

[6] Holmes M.R. *The Urysohn space embeds in Banach spaces in just one way*, Topology Appl. **155** (2008), no. 14, 1479–1482.

[7] Huhunaišvili G.E., *On a property of Urysohn's universal metric space* (Russian), Dokl. Akad. Nauk USSR (N.S.) **101** (1955), 332–333.

[8] Isbell J.R. *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964), 65–76.

[9] Kalton N., *Extending Lipschitz maps into $\mathcal{C}(K)$-spaces*, Israel J. Math. **162** (2007), 275–315.

[10] Katětov M. *On universal metric spaces*, in General Topology and its Relations to Modern Analysis and Algebra, VI (Prague, 1986), Z. Frolík (Ed.), Helderman Verlag, Berlin, 1988, pp. 323–330.

[11] Kirk W., Sims B., Eds., *Introduction to hyperconvex spaces*, in Handbook of Metric Fixed Point Theory, Chapter 13, Kluwer Academic Publishers, Dordrecht, 2001.

[12] Melleray J. *On the geometry of Urysohn's universal metric space*, Topology Appl. **154** (2007), 384–403.

[13] Melleray J., *Some geometric and dynamical properties of the Urysohn space*, Topology Appl. **155** (2008), no. 14, 1531–1560.

[14] Pestov V. *Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon*, University Lecture Series **40**, AMS, Providence, RI, 2006.

[15] Urysohn P.S. *Sur un espace métrique universel*, Bull. Sci. Math. **51** (1927), 43–64, 74–96.

[16] Uspenskij V.V., *On the group of isometries of the Urysohn universal metric space*, Comment. Math. Univ. Carolin. **31** (1990), no. 1, 181–182.

[17] Uspenskij V.V., *The Urysohn universal metric space is homeomorphic to a Hilbert space*, Topology Appl. **139** (2004), no. 1–3, 145–149.

[18] Uspenskij V.V., *On subgroups of minimal topological groups*, Topology Appl. **155** (2008), 1580–1606.

Jagiellonian University, Institute of Mathematics, ul. Łojasiewicza 6, 30-348 Kraków, Poland

Email: piotr.niemiec@uj.edu.pl

(*Received* March 18, 2009, *revised* May 15, 2009)