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# About remainders in compactifications of homogeneous spaces

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*Abstract.* We prove a dichotomy theorem for remainders in compactifications of homogeneous spaces: given a homogeneous space $X$, every remainder of $X$ is either realcompact and meager or Baire. In addition we show that two other recent dichotomy theorems for remainders of topological groups due to Arhangel'skii cannot be extended to homogeneous spaces.

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## 1. Introduction

Remainders in compactifications of topological spaces have been a popular topic in the last few years; this is basically due to the fact that topological groups are much more sensitive to the properties of their remainders than topological spaces in general. For instance, Arhangel'skii has recently proved two dichotomy theorems about remainders in compactifications of topological groups (see [2] and [3]):

**Theorem A.** *Every remainder of a topological group is either Lindelöf or pseudocompact.*

**Theorem B.** *Every remainder of a topological group is either $\sigma$-compact or Baire.*

In this note we shall concentrate our attention on remainders in compactifications of homogeneous spaces. Clearly, this is the natural step towards a possible extension of Arhangel'skii's results. We obtain a further dichotomy by showing that every remainder of a homogeneous space is either realcompact and meager or Baire. Such a dichotomy seems weaker than the previous ones but there is a good reason for this: none of the Arhangel'skii's dichotomy theorems can be generalized to the case of homogeneous spaces. We shall prove this in Section 3 by constructing one counterexample for both results. In the last part we will provide some upper bounds for the cardinality of a homogeneous space whose remainder is a countable union of metrizable spaces.

## 2. Preliminary results

By 'space' we always mean a Tychonoff topological space. Recall that a space is Baire if the intersection of a sequence of open and dense subsets is a dense subset; we say that a space is meager if it can be represented as the union of a sequence of nowhere dense subsets. For all undefined notions we refer to [7].

We will sometimes refer to a classical and well-known result which is due to Henriksen and Isbell: a space $X$ is of countable type if and only if some (every) of its remainders is Lindelöf (see [9]). Recall that a space $X$ is of countable type if for every compact subset $H \subseteq X$ there exists a compact subset $K \subseteq X$ such that $H \subseteq K$ and $K$ has countable character in $X$. Similarly, we say that a space $X$ is of point-countable type if for every $x \in X$ there exists a compact subset $K \subseteq X$ such that $x \in K$ and $K$ has countable character in $X$.

However, we will often use the following characterization of spaces of point-countable type: a space $X$ is of point-countable type if and only if for some (every) compactification $\gamma X$ of $X$, the space $X$ can be represented as the union of a family of $G_\delta$-subsets of $\gamma X$ (see for instance [1] or [7, Exercise 3.5.G(a)]).

The first theorem generalizes somehow the necessary condition of the Henriksen-Isbell theorem.

**Theorem 2.1.** *Let $X$ be a nowhere locally compact space. If $X$ is of point-countable type, every remainder of $X$ is realcompact.*

Proof: Fix any compactification $\gamma X$ of $X$. As $X$ is nowhere locally compact, $\gamma X$ is a compactification of the remainder $Y = \gamma X \setminus X$ as well. Since $X$ is of point-countable type, it can be represented as the union of a family of $G_\delta$-subsets of $\gamma X$. Of course we may assume these $G_\delta$-subsets to be closed. This implies that for every point $x \in X$ there exists a continuous function $f\colon \gamma X \to I$ such that $f(x) = 0$ and $f(y) > 0$, for every $y \in Y$. Since $\gamma X \leq \beta Y$, we have that the Čech-Stone compactification $\beta Y$ of $Y$ has the same property, thus $Y$ is realcompact by [7, Theorem 3.11.10]. $\square$

It is not clear to us whether the converse of the preceding theorem is true. However, the requirement that a single remainder is realcompact is definitely not strong enough to force the space to be of point-countable type. Below we describe a nowhere locally compact space which is not of point-countable type but which has a realcompact remainder.

*Example* 2.2. Let $Y$ be the discrete space of cardinality $\omega_1$ and let $\omega Y$ be its-one point compactification. Consider the realcompact, nowhere locally compact space $Y^{\omega_1}$ and let $Z = (\omega Y)^{\omega_1}$. The space $X = Z \setminus Y^{\omega_1}$ is dense in $Z$ and it is clearly nowhere locally compact; moreover it is $G_\delta$-dense in $Z$. This implies that $X$ cannot contain $G_\delta$-subsets of $Z$. Hence $X$ is not of point-countable type.

## 3. A dichotomy theorem and a counterexample

**Lemma 3.1.** *Let $X$ be a space and $\gamma X$ a compactification of $X$. If the remainder $Y = \gamma X \setminus X$ is not Baire, there is a non-empty $G_\delta$-subset of $\gamma X$ that is contained in $X$.*

Proof: Let $\{U_n : n \in \omega\}$ be a family of dense open subsets of $Y$ such that $\bigcap\{U_n : n \in \omega\}$ is not dense in $Y$ and for every $n \in \omega$, let $V_n$ be the maximal open subset of $\gamma X$ satisfying $U_n = Y \cap V_n$, that is, $V_n$ is just the union of all open sets $V \subseteq \gamma X$ such that $U_n = Y \cap V$.

Observe that each $V_n$ is dense in $\gamma X$ because any non-empty open set $W \subseteq \gamma X$ is either contained in $V_n$ or satisfies the condition $W \cap Y \neq \emptyset$. It then follows that the set $G = \bigcap\{V_n : n \in \omega\}$ is dense in $\gamma X$, because $\gamma X$ is Baire. If $W$ is a non-empty open subset of $\gamma X$ satisfying $W \cap \bigcap\{U_n : n \in \omega\} = \emptyset$, then $W \cap G$ is a $G_\delta$-subset in $\gamma X$ and $W \cap G \subseteq X$. □

The failure of the Baire property is equivalent to the existence of some non-empty open meager subset. On the other hand, in the lattice of open sets meagerness is preserved only downward and so the property of being meager for the whole space is in general strictly stronger than not being Baire.

The next result, which is our dichotomy, shows, among other things, that this is not the case for remainders of homogeneous spaces.

**Theorem 3.2.** *The remainder of a homogeneous space is either Baire or meager and realcompact.*

Proof: Let $X$ be a homogeneous space and $\gamma X$ a compactification of $X$. Assume that the remainder $Y = \gamma X \setminus X$ is not Baire. First of all, observe that $X$ cannot be locally compact, therefore $Y$ is dense in $\gamma X$. With the same notation adopted in the proof of Lemma 3.1, we see that there exists a non-empty open set $W \subseteq \gamma X$ and a $G_\delta$-subset $G = \bigcap\{V_n : n \in \omega\} \subseteq \gamma X$ such that $W \cap G$ is dense in $W$ and $W \cap G \subseteq X$. Clearly, $W \cap G$ is Čech-complete.

Since, by homogeneity, every non-empty open subset of $X$ intersects a homeomorphic copy of $W \cap X$, we see that every non-empty open set $U \subseteq X$ contains a non-empty open set with a dense Čech-complete subspace. Let $\mathcal{U}$ be a maximal pairwise disjoint family of open sets in $X$ having a dense Čech-complete subspace. The previous assertion guarantees that $\bigcup \mathcal{U}$ is dense in $X$. For any $U \in \mathcal{U}$, let $Z_U$ be a dense Čech-complete subspace of $U$ and $V_U$ an open subset of $\gamma X$ such that $U = V_U \cap X$.

Since $Z_U$ is dense in $V_U$ and a Čech-complete space is a $G_\delta$-subset in all of its compactifications (see [7, Theorem 3.9.1]), we conclude that $Z_U$ is also a $G_\delta$-subset in $V_U$. As $X$ is dense in $\gamma X$, the family $\{V_U : U \in \mathcal{U}\}$ consists of pairwise disjoint subsets and this in turn ensures that the set $Z = \bigcup\{Z_U : U \in \mathcal{U}\}$ is a $G_\delta$-subset in $\bigcup\{V_U : U \in \mathcal{U}\}$, and hence in $\gamma X$. Therefore, $\gamma X \setminus Z$ is a countable union of closed subsets with empty interior. Since $Y$ is dense in $\gamma X$ and $Y \subseteq \gamma X \setminus Z$, we conclude that $Y$ is meager.

By Theorem 2.1, to prove that $Y$ is realcompact it suffices to prove that $X$ is of point-countable type. Fix a homeomorphism $h\colon X \to X$ and consider the subspace $h(Z)$ of $X$. It is easy to see from the construction of $Z$ that $h(Z)$ is a $G_\delta$-subset of $\gamma X$. Since the space $X$ is homogeneous it follows that $X$ is the union of $G_\delta$-subsets of $\gamma X$. This implies that $X$ is of point-countable type. □

The question arises whether the two Arhangel'skii's dichotomy theorems can be fully extended to the class of homogeneous spaces. The answer is no in both cases, as we show in the next example.

We describe a homogeneous space of point-countable type but not of countable type which has a dense Čech-complete subspace. Homogeneous spaces of point-countable but not of countable type were presented in [4], however the following is basically [5, Theorem 5.5].

*Example* 3.3. Consider the Cantor set $2^\omega$, the space $\omega_1{+}2$ endowed with the order topology, and topologize the set $X = 2^\omega \times (\omega_1{+}2)$ as follows: basic open neighbourhoods of points of the form $(t, \alpha)$, where $t \in 2^\omega$ and $\alpha \in \omega_1{+}1$, are of the form $\{t\} \times U$, where $U$ is an open subset of $\omega_1{+}1$ and $\alpha \in U$. Basic open neighbourhoods of points of the form $(t, \omega_1{+}1)$, are given by:

$$V(t) = V \times (\omega_1{+}2) \setminus (\{t\} \times (\omega_1{+}1)),$$

where $V$ is an open neighbourhood of $t$ in $2^\omega$.

It is easy to verify that the space $X$ is compact and zero-dimensional. We now let $Y = X \setminus (2^\omega \times \{\omega_1\})$. The subspace $Y$ is dense in $X$ and it is easy to see from the construction that it is first-countable. Observe that the remainder $X \setminus Y$ of $Y$ is homeomorphic to the discrete space of cardinality $2^\omega$, which is not Lindelöf. Therefore, by the Henriksen-Isbell theorem, the space $Y$ is not of countable type. So, the space $Y^\omega$ is not of countable type, while it is of point-countable type since it is even first-countable. In particular, since $Y$ is first-countable and zero-dimensional, it follows from a result by Dow and Pearl (see [6]) that $Y^\omega$ is homogeneous.

Consider now the dense subspace $Z$ of $Y$ given by $2^\omega \times \omega_1$. Since $\omega_1$ is a locally compact space, it follows that $Z$ is locally compact as well, and then it is Čech-complete. Therefore we have that $Z^\omega$ is dense in $Y^\omega$ and it is Čech-complete. This completes the proof.

The special properties of the space in the above example, together with the Henriksen-Isbell theorem and the proof of Theorem 3.2, imply that for any compactification $\gamma Y^\omega$ of $Y^\omega$ the remainder $\gamma Y^\omega \setminus Y^\omega$ is a non-Lindelöf meager space. In particular this remainder is neither Baire nor Lindelöf. Since pseudocompactness implies Bairness and $\sigma$-compactness implies Lindelöfness, this clearly shows the impossibility to extent the two Arhangel'skiĭ's dichotomy theorems to homogeneous spaces.

However, it is reasonable to ask whether in Theorem 3.2 we can substitute the Baire property with the pseudocompactness property, i.e., we could ask:

**Question 3.4.** *Let $X$ be a homogeneous space and let $\gamma X$ be a compactification of $X$. Is it true that the remainder $\gamma X \setminus X$ is either pseudocompact or realcompact and meager?*

A successful step in this direction could be to prove Lemma 3.1 under the assumption that the remainder is not pseudocompact; unfortunately we were not able to prove that, even under the assumption that the space $X$ is homogeneous. If $X$ is a topological group, things change radically. This is stated in the following result which is due to Arhangel'skii; it is an immediate corollary of [3, Lemma 2.1 and Lemma 2.2].

**Proposition 3.5.** *Let $G$ be a topological group and let $\gamma G$ be a compactification of $G$. If the remainder $\gamma G \setminus G$ is not pseudocompact, the space $G$ is of point-countable type.*

Observe that a Baire space can have a $G_\delta$-point without a countable base of neighbourhoods, while any pseudocompact space is first-countable at any $G_\delta$-point. More generally, in a pseudocompact space any compact $G_\delta$-subset has a countable base of neighbourhoods. As a sort of partial answer to Question 3.4, we can prove the following:

**Theorem 3.6.** *Let $X$ be a homogeneous space and $\gamma X$ a compactification of $X$. If $C$ is a compact $G_\delta$-subset in the remainder $Y = \gamma X \setminus X$, then either $C$ is first-countable in $Y$ or $Y$ is realcompact.*

Proof: Assume that $C$ is not first-countable in $Y$ and let $\{U_n : n \in \omega\}$ be a family of open sets in $Y$ such that $\bigcap\{U_n : n \in \omega\} = C$. By induction, for any $n \in \omega$ we may find an open set $V_n$ in $\gamma X$ such that $C \subseteq V_{n+1} \subseteq \overline{V}_{n+1} \subseteq V_n \subseteq U_n \cup X$. The set $F = \bigcap\{V_n : n \in \omega\}$ is a closed $G_\delta$-subset of $\gamma X$ and $F \cap Y = C$. Of course, we cannot have $F = C$ because this would imply that $C$ would have a countable local base in $\gamma X$, and a fortiori in $Y$. So $F \neq C$, hence $F \setminus C$ is a non-empty $G_\delta$-subset in $\gamma X$ which actually lies in $X$. Clearly, $F \setminus C$ contains a non-empty closed $G_\delta$-subset $K$ of $\gamma X$. By compactness, $K$ has a countable local base in $\gamma X$ and so even in $X$. The homogeneity of $X$ ensures that $X$ can be covered by compact sets of countable character, and the regularity of $\gamma X$ implies in turn that each of these compact sets has countable character in $\gamma X$. So, we see that $X$ is actually covered by a family of $G_\delta$-subsets of $\gamma X$. By the remark we made at the beginning of Section 2, we get that $X$ is a space of point-countable type and, by Theorem 2.1, this suffices to conclude that $Y$ is realcompact. □

## 4. About homogeneous spaces having a remainder that is the union of metrizable subspaces

In this last section we provide some upper bounds for the cardinality of a homogeneous space whose remainder is the union of countably many metrizable spaces.

**Theorem 4.1.** *Let $X$ be a homogeneous space and $\gamma X$ a compactification of $X$. If the remainder $Y = \gamma X \setminus X$ is the union of countably many metrizable subspaces, then either $X$ is of point-countable type or $\pi\chi(X) = \omega$.*

Proof: If $X$ is not of point-countable type, the last paragraph of the proof of Theorem 3.2 implies that $Y$ is a Baire space. As $Y$ is the union of countably many (metrizable) subspaces, there exists one of them, call it $Z$, that is not nowhere dense in $Y$, i.e., there exists a non-empty open set $V \subseteq Y$ such that $Z \cap V$ is dense in $V$. Let $U$ be an open subset of $\gamma X$ such that $U \cap Y = V$. Since $X$ is not of point-countable type, it is nowhere locally compact, therefore $Y$ is dense in $\gamma X$. Then the subset $A = U \cap Z$ is dense in $U$. The last assertion implies that each point of $A$ has a countable base in $U$ (see [11, 2.7(a)]) and therefore in $\gamma X$.

Let us denote by $B$ the $\omega$-closure of $A$ (in $\gamma X$), that is $B = \bigcup\{\overline{S}^{\gamma X} : S \subseteq A \ \&\ |S| \leq \omega\}$. We claim that we cannot have $B \subseteq Y$. Indeed, if we assume $B \subseteq Y$, we have that the countably compact subspace $B$ is the union of countably many metrizable subspaces. Since a metrizable space is a $D$-space it follows from a result by Gerlits, Juhász and Szentmiklóssy (see [8]), that $B$ is compact. This implies that $U \subseteq \overline{A} = B$, which is a contradiction with the fact that $X$ is dense in $\gamma X$.

So $B \cap X \neq \emptyset$, and we may consequently fix a point $x \in X$ and a subset $C \subseteq A$ such that $|C| \leq \omega$ and $x \in \overline{C}$. The fact that $\gamma X$ is first-countable at each point of $C$ ensures that $x$ has a countable $\pi$-base in $\gamma X$ and therefore also in $X$, since $X$ is dense in $\gamma X$. As $X$ is homogeneous we have that $\pi\chi(X) = \omega$. □

**Corollary 4.2.** *Let $X$ be a homogeneous space and $\gamma X$ a compactification of $X$. If the remainder $\gamma X \setminus X$ is the union of countably many metrizable subspaces, then either $X$ is of point-countable type or $|X| \leq 2^{c(X)}$.*

Proof: To get the desired inequality it is enough to refer to the formula $|X| \leq 2^{\pi\chi(X)\cdot c(X)}$, which is due to Ismail ([10]), and then apply the preceding theorem. □

**Theorem 4.3.** *Let $X$ be a homogeneous space and $\gamma X$ a compactification of $X$. If the remainder $\gamma X \setminus X$ is the union of countably many metrizable subspaces, then either $\gamma X$ can be mapped onto $I^{c(X)^+}$ or $|X| \leq 2^{c(X)}$.*

Proof: Assume that $\gamma X$ cannot be mapped onto $I^{c(X)^+}$. Then, from a result by Šapirovskiĭ (see [12]), we have that the set $A$ of points at which the space $\gamma X$ has $\pi$-character not exceeding $c(X)$ is dense in $\gamma X$. If $C$ is a countable subset of $A$ and $p \in \overline{C}$ it is easy to see that the $\pi$-character of $\gamma X$ at $p$ does not exceed $c(X)$; therefore $p \in A$, and this shows that $A$ is $\omega$-bounded (in $\gamma X$).

We claim that $A \cap X \neq \emptyset$. This is clearly true if $X$ is open in $\gamma X$, i.e., if $X$ is locally compact. So, assume that $X$ is not locally compact. By the same argument of the preceding theorem we cannot have that $A \subseteq \gamma X \setminus X$, otherwise $A$ would be compact and hence closed in $\gamma X$. This would imply $A = \gamma X \setminus X$, which is a contradiction with the fact that $X$ is not locally compact. So $A \cap X \neq \emptyset$.

Since the space $\gamma X$ has $\pi$-character not exceeding $c(X)$ at some point $x$ of $X$, and since $X$ is dense in $\gamma X$ we have that also $X$ has $\pi$-character not exceeding $c(X)$ at $x$. The homogeneity of $X$ and the Ismail formula $|X| \leq 2^{\pi\chi(X)\cdot c(X)}$, imply that $|X| \leq 2^{c(X)}$. □

Notice that the cardinality of a homogeneous space is, in general, not bounded by $2^{c(X)}$. For instance, we may consider the (even) topological group $X = \Sigma 2^{\mathfrak{c}^+}$. Observe at first that $|X| \geq \mathfrak{c}^+$. Moreover, the space $2^{\mathfrak{c}^+}$ is c.c.c. and since $X$ is dense in $2^{\mathfrak{c}^+}$, the space $X$ is c.c.c. as well.

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