Holomorphic Bloch spaces on the unit ball in C^n

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Abstract. This work is an introduction to anisotropic spaces of holomorphic functions, which have ω-weight and are generalizations of Bloch spaces on a unit ball. We describe the holomorphic Bloch space in terms of the corresponding L^∞_ω space. We establish a description of $(A^p(\omega))^*$ via the Bloch classes for all $0 < p \leq 1$.

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1. Introduction and basic constructions

Let C^n be the n-dimensional complex Euclidean space. For $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$ in C^n we define the inner product as follows:

$$\langle z, \zeta \rangle = z_1\overline{\zeta}_1 + \cdots + z_n\overline{\zeta}_n.$$

We write also: $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

Let $B^n = \{z \in C^n, |z| < 1\}$ be the unit ball in C^n and let $S^n = \{z \in C^n, |z| = 1\}$ be the boundary of B^n. We denote by $H(B^n)$ the set of holomorphic functions on B^n and by $H^\infty(B^n)$ the set of bounded holomorphic functions on B^n.

Let $f \in H(B^n)$, then $f(z) = \sum_m a_m z^m$ $(z \in B^n)$, where the summation is over all multi-indices $m = (m_1, \dots, m_n)$, each m_k is a nonnegative integer and $z^m = z_1^{m_1} \dots z_n^{m_n}$. Putting $f_k(z) = \sum_{|m|=k} a_m z^m$ for each $k \geq 0$, $|m| = m_1 + \cdots + m_n$, then the Taylor series of f has the following form

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \tag{1}$$

which is called the homogeneous expansion of f. It is clear that each f_k is a homogeneous polynomial of degree k.

An important notion in the study of holomorphic function spaces is the notion of fractional differential operators. In this paper we consider one type of them. For a holomorphic function f with homogeneous expansion (1) and for $\alpha > -1$

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we define the fractional differential as follows:

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(z), \quad z \in B^n,$$

and the inverse operator $D^{-\alpha}$ is defined in the standard sense:

$$D^{-\alpha} D^\alpha f(z) = f(z).$$

It is not difficult to show that

$$f(z) = \int_0^1 Df(rz)\, dr. \tag{2}$$

The Bloch space plays a very important role in classical geometric function theory. The one-dimensional case of the holomorphic Bloch space is well investigated (see [2], [3]). The aim of this paper is the study of the Bloch space on the unit ball in C^n. There are several possible ways for a generalization of the holomorphic Bloch space to higher dimensions (see [11], [12]). We give a new generalization of them and consider the weighted case which is new also in the one-dimensional case. Note that the polydisc case has already been investigated (see for example [7], [13]).

Let S be the class of all non-negative measurable functions ω on $(0,1)$ for which there exist positive numbers M_ω, q_ω, m_ω, ($m_\omega, q_\omega \in (0,1)$) such that

$$m_\omega \le \frac{\omega(\lambda r)}{\omega(r)} \le M_\omega,$$

for all $r \in (0,1)$ and $\lambda \in [q_\omega, 1]$. For properties of functions from S, see [10]. Using the results of [10], one can prove the following lemma.

Lemma 1.1. *Let $\omega \in S$. Then there exist bounded measurable functions η and ε so that*

$$\omega(x) = \exp\left\{ \eta(x) + \int_x^1 \frac{\varepsilon(u)}{u}\, du \right\}, \quad t \in (0,1),$$

and

$$-\alpha_\omega = \frac{\log m_\omega}{\log q_\omega^{-1}} \le \varepsilon(t) \le \frac{\log M_\omega}{\log q_\omega^{-1}} \le \beta_\omega, \quad t \in (0,1).$$

Next we assume that $\eta(x) = 0$ for $x \in (0,1)$.

Besides, for any functions f and g, the notation $f \preceq g$ ($f \succeq g$) will mean that $|f(z)| \le C|g(z)|$ ($|g(z)| \le C|f(z)|$) and the notation $f \asymp g$ will mean that $C_1|f(z)| \le |g(z)| \le C_2|f(z)|$ for some positive constants C, C_1, C_2 independent of z.

Remark 1.2. Note that it is not difficult to show that if $1-|z| \asymp 1-|w|$ then $\omega(1-|z|) \asymp \omega(1-|w|)$.

One of the applications is the description of the $(A^p(\omega))^*$ in case $0 < p \leq 1$ via Bloch spaces. Here $A^p(\omega)$ is the ω-generalization of $A^p(\alpha)$ space in the case of unit ball in C^n and is defined as the class of holomorphic functions f for which

$$\|f\|^p_{A^p(\omega)} = \int_{B^n} |f(z)|^p \omega(1-|z|)\, d\nu(z) < +\infty,$$

where $d\nu(z)$ is volume measure on B^n, normalized so that $\nu(B^n) = 1$ and $0 < \beta_\omega < 1$.

In particular, if $\omega(t) = t^\alpha$, then we have $A^p(\omega) = A^p(\alpha)$ (see [6], [5]). In this case we have a generalization of the Djrbashian's formula:

$$f(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_{B^n} \frac{(1-|\zeta|^2)^\alpha f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+1+\alpha}}\, d\nu(\zeta) \tag{3}$$

(for proof see [5, Theorem 6.1]).

The corresponding space of measurable functions will be denoted by $L^p(\omega)$.

It is known that $A^p(\omega)$ is a Banach space if $p \geq 1$ and a complete metric space with distance $\rho(f,g) = \|f-g\|^p_{A^p(\omega)}$ if $0 < p < 1$.

Definition 1.3. Let $f \in H(B^n)$, $\omega \in S$ and $0 < \alpha_\omega < 1$. A function f belongs to the Bloch space $B^n_\omega \equiv B_\omega$ if

$$M_f = \sup_{z \in B^n} \left\{ \frac{(1-|z|^2)}{\omega(1-|z|)} |Df(z)| \right\} < +\infty. \tag{4}$$

Notice that, in view of our definition of Df, $\|f\|_{B_\omega} = M_f$ is indeed a norm. (We do not have to add $|f(0)|$.) This follows from the fact that here $Df = 0$ implies $f = 0$ for holomorphic f. It is easy to see that B_ω is a Banach space with respect to the norm $\|\cdot\|$.

As in the case of a polydisc, one can see that if $n = 1$ and $\omega(t) = t^{1-s}$, then we have the Bloch space of one variable (for details see [7, Proposition 1.5]).

We need the following lemmas to prove the main results.

Lemma 1.4. *The following properties of D^m are evident:*

1. $DD^\alpha f(z) = D^{\alpha+1} f(z)$;
2. $D^m(1-\langle z,\zeta\rangle)^{-\alpha} \preceq (1-\langle z,\zeta\rangle)^{-\alpha-m}$;
3. $Df = Rf(z) + f(z)$, *where* $Rf(z) = \sum_{k=1}^n z_k \frac{\partial f(z)}{\partial z_k}$.

It is clear that $R(1-\langle z,\zeta\rangle)^{-\alpha} = \alpha\langle z,\zeta\rangle(1-\langle z,\zeta\rangle)^{-\alpha-1}$.

Lemma 1.5. *Let $\omega \in S$, $\alpha + 1 - \beta_\omega > 0$, and $\beta - \alpha > \alpha_\omega$. Then*

$$\int_{B^n} \frac{(1-|\zeta|^2)^\alpha \omega(1-|\zeta|)}{|1-\langle z,w\rangle|^{\beta+n+1}}\, d\nu(\zeta) \preceq \frac{\omega(1-|z|^2)}{(1-|z|^2)^{\beta-\alpha}}\,.$$

Proof: Let σ be the surface measure on S^n normalized so that $\sigma(S^n)=1$. The formula

$$\int_{B^n} f(z)\,d\nu(z) = 2n\int_0^1 r^{2n-1}\,dr\int_{S^n} f(r\zeta)\,d\sigma(\zeta) \tag{5}$$

shows the relation of both measures (for the proof see [12, p. 9] or [9, p. 13]).

By (5) for $\beta>0$ we get

$$\begin{aligned}
&\int_{B^n}\frac{(1-|\zeta|^2)^\alpha\omega(1-|\zeta|)}{|1-\langle z,\zeta\rangle|^{\beta+n+1}}\,d\nu(\zeta)\\
&= 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha\omega(1-r)\,dr\int_{S^n}\frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{\beta+n+1}}\\
&\le 2n\int_0^1 r^{2n-1}\frac{(1-r^2)^\alpha\omega(1-r)}{(1-r|z|)^{\beta+1}}\,dr.
\end{aligned}$$

In the last inequality we have used Theorem 1.12 from [12].

The problem is to estimate the last one-dimensional integral. Using the proof of Lemma 1.6 [7] and putting $a=\alpha$, $b-1=\beta+1$, we get

$$\int_0^1\frac{(1-r^2)^\alpha\omega(1-r)}{(1-r|z|)^{\beta+1}}\le C\frac{(1-|z|)^\alpha\omega(1-|z|)}{(1-|z|)^\beta}$$

if $\alpha+1-\beta_\omega>0$, $\beta-\alpha>\alpha_\omega$, which proves our lemma. □

2. Description theorems in B_ω

Lemma 2.1. *Let $\beta>-1$ and $f\in H(B^n)$, $f\in A^1(\beta)$. Then $(1-|z|^2)Df(z)\in L^1(\beta)$.*

Proof: Let $f\in A^1(\beta)$. By Theorem 2.16 from [12] we have $(1-|z|^2)Rf(z)\in L^1(\beta)$. It is clear, that the function $(1-|z|^2)f(z)$ also belongs to the space $L^1(\beta)$. Then by Lemma 1.4 we get $(1-|z|^2)Df(z)\in L^1(\beta)$. □

Corollary 2.2. *Let $f\in B_\omega$ and $\beta>\beta_\omega$. Then $Df\in A^1(\beta)$.*

Lemma 2.3. *Let $f\in B_\omega$, $\beta>\beta_\omega$, then*

$$|f(z)|\preceq\int_{B^n}\frac{(1-|\zeta|^2)^\beta|Df(\zeta)|}{|1-\langle z,\zeta\rangle|^{\beta+n}}\,d\nu(\zeta). \tag{6}$$

Proof: If $\beta>\beta_\omega$, then $Df\in A^1(\beta)$ hence the integral in (6) is convergent. Using (2) and (3) we get

$$\begin{aligned}
f(z) &= C(\beta,n)\int_0^1\int_{B^n}\frac{(1-|\zeta|^2)^\beta}{(1-r\langle z,\zeta\rangle)^{\beta+1+n}}Df(\zeta)\,d\nu(z)\,dr\\
&= C(\beta,n)\int_{B^n}(1-|\zeta|^2)^\beta Df(\zeta)\int_0^1\frac{dr}{(1-r\langle z,\zeta\rangle)^{\beta+1+n}}\,d\nu(z)
\end{aligned}$$

and the proof is finished. □

Lemma 2.4. *Let $f \in B_\omega$ and $\beta > \beta_\omega$. Then $f \in A^1(\beta - 1)$.*

PROOF: Using Lemma 2.3 for $\gamma > \beta_\omega$ and $\gamma - \beta > 0$ we get

$$\begin{aligned}
&\int_{B^n} |f(z)|(1-|z|^2)^{\beta-1}\, d\nu(z) \\
&\preceq \int_{B^n} |Df(\zeta)|(1-|\zeta|^2)^{\gamma} \int_{B^n} \frac{(1-|z|^2)^{\beta-1}}{|1-\langle z,\zeta\rangle|^{\gamma+n}}\, d\nu(z)\, d\nu(\zeta) \\
&\preceq \int_{B^n} |Df(\zeta)|(1-|\zeta|^2)^{\beta}\, d\nu(\zeta) < \infty,
\end{aligned}$$

by Corollary 2.2. □

Let $L^\infty_\omega = L^\infty_\omega(B^n)$ be the class of measurable functions on B^n, for which

$$\|f\|_{L^\infty_\omega} = \sup_{z\in B^n} \{|f(z)|\omega^{-1}(1-|z|^2)\} < +\infty.$$

Proposition 2.5. *A holomorphic function f belongs to B_ω if and only if the function $(1-|z|)Df(z)$ belongs to L^∞_ω.*

The next theorem gives a description of the analytic part of L^∞_ω.

Theorem 2.6. *Let $f \in H(B^n)$, $\alpha > \alpha_\omega + 1$, $k \in \mathbb{N}$. Then $(1-|z|^2)^\alpha D^k f(z) \in L^\infty_\omega$ if and only if $(1-|z|^2)^{\alpha-1} D^{k-1} f(z) \in L^\infty_\omega$.*

PROOF: Let $g(z) = (1-|z|^2)^\alpha D^k f(z)$ and $g \in L^\infty_\omega$. Taking β sufficiently large, using Lemmas 2.3 and 1.5, we get

$$\begin{aligned}
|D^{k-1}f(z)| &\preceq \int_{B^n} \frac{(1-|\zeta|^2)^\beta}{|1-\langle z,\zeta\rangle|^{n+\beta}} |D^k f(\zeta)|\, d\nu(\zeta) \\
&\le \sup_{z\in B^n}\left\{|D^k f(z)| \frac{(1-|\zeta|^2)^\alpha}{\omega(1-|\zeta|^2)}\right\} \int_{B^n} \frac{(1-|\zeta|^2)^{\beta-\alpha}}{|1-\langle z,\zeta\rangle|^{n+\beta}} \omega(1-|\zeta|)|\, d\nu(\zeta) \\
&\preceq \|g\|_{L^\infty_\omega} \frac{\omega(1-|z|)}{(1-|z|)^{\alpha-1}}
\end{aligned}$$

and, hence,

$$\sup_{z\in B^n}\left\{|D^{k-1} f(z)| \frac{(1-|\zeta|^2)^{\alpha-1}}{\omega(1-|\zeta|^2)}\right\} < \infty,$$

which proves that the function $h(z) = (1-|z|^2)^{k-1} D^{\alpha-1} f(z)$ belongs to the space L^∞_ω.

Conversely, let $h \in L^\infty_\omega$. Then, using Lemma 1.4 we get

$$|D^k f(z)| \preceq \int_{B^n} \frac{(1-|\zeta|^2)^\beta}{|1-\langle z,\zeta\rangle|^{n+\beta+2}} D^{k-1} f(\zeta)\, d\nu(\zeta).$$

Repeating the argument of the first part of the proof, we finish the proof of the theorem. □

Using Theorem 2.6 one can give an another characterization of B_ω.

Theorem 2.7. *A function f belongs to B_ω if and only if*

$$\sup_{z\in B^n} \left\{ \frac{(1-|\zeta|^2)^k}{\omega(1-|\zeta|)} |D^k f(z)| \right\} < \infty,$$

for $\alpha > \alpha_\omega$.

3. Bounded projections and inverse operators

Let us consider the following operator

$$Q_\alpha f(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_{B^n} \frac{f(\zeta)\, d\nu(\zeta)}{(1-\langle z,\zeta\rangle)^{\alpha+n}} \quad (\alpha > 0).$$

Theorem 3.1. *Let $\alpha > \beta_\omega$. Then the map Q_α is bounded from $L^\infty_{\widetilde{\omega}}$ to B_ω, where $\widetilde{\omega}(t) = t^{\alpha-1}\omega(t)$. Moreover Q_α is surjective.*

Proof: Let $f \in L^\infty_{\widetilde{\omega}}$. We show that the function $F(z) = Q_\alpha f(z)$ belongs to the space B_ω. Using Lemma 1.5 we get

$$|DF(z)| \preceq \|f\|_{L^\infty_{\widetilde{\omega}}} \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-1}\omega(1-|\zeta|)}{|1-\langle z,\zeta\rangle|^{\alpha+n+1}}\, d\nu(\zeta) \preceq \|f\|_{L^\infty_{\widetilde{\omega}}} \frac{\omega(1-|z|)}{(1-|z|^2)}$$

which shows that $F \in B_\omega$ and Q_α is a bounded operator from $L^\infty_{\widetilde{\omega}}$ to B_ω. Next we show that Q_α is onto: for any $f \in B_\omega$ there exists a function $\phi \in L^\infty_{\widetilde{\omega}}$ such that $f(z) = Q_\alpha \phi(z)$ $(z \in B^n)$.

To this end we consider first the function $h(z) = (1-|z|^2)^\alpha Df(z)$ which belongs to $L^\infty_{\widetilde{\omega}}$. Then by Theorem 2.6 the function $\phi(z) = \alpha^{-1}(1-|z|^2)^{\alpha-1} f(z)$ belongs to $L^\infty_{\widetilde{\omega}}$, too. We have

$$Q_\alpha \phi(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-1} f(\zeta)}{(1-\langle z,\zeta\rangle)^{\alpha+n}}\, d\nu(\zeta).$$

Further, by Lemma 2.4 we get $f \in A^1(\alpha-1)$ if $\alpha > \beta_\omega$ and therefore $f(z) = Q_\alpha h(z)$, $z \in B^n$. □

If we consider the integral operator

$$P_\alpha f(z)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\int_{B^n}\frac{(1-|\zeta|^2)^{\alpha-1}f(\zeta)}{(1-\langle z,\zeta\rangle)^{\alpha+n}}\,d\nu(\zeta)\quad(\alpha>0),$$

then we have the following analogue of Theorem 3.1.

Theorem 3.2. *Let $\alpha>\beta_\omega$. Then P_α is a bounded operator from L^∞_ω to B_ω and if $\alpha>\beta_\omega$ then P_α is onto.*

Proof: The first part of the proof is similar to that of Theorem 3.1. To prove that the map is onto we take the function

$$\phi(z)=(1-|z|^2)\int_{B^n}\frac{(1-|\zeta|^2)^{\alpha-1}f(\zeta)}{(1-\langle z,\zeta\rangle)^{\alpha+n+1}}\,d\nu(\zeta),\quad f\in B_\omega$$

and show first that $\phi\in L^\infty_\omega$. To this end we use Lemma 2.3 and 1.4. Then

$$\int_{B^n}\frac{(1-|\zeta|^2)^{\alpha-1}d\,\nu(\zeta)}{(1-\langle\zeta,w\rangle)^{m+n}(1-\langle z,\zeta\rangle)^{\alpha+n+1}}\preceq\frac{1}{(1-\langle z,w\rangle)^{m+n+1}}\,.$$

Next for sufficient large $m\in\mathbb{N}$ we get

$$\begin{aligned}\frac{\phi(z)}{(1-|z|^2)^\alpha}&\preceq\left|\int_{B^n}\frac{(1-|\zeta|^2)^{\alpha-1}}{(1-\langle z,\zeta\rangle)^{\alpha+n+1}}\int_{B^n}\frac{(1-|w|^2)^m Df(w)}{(1-\langle\zeta,w\rangle)^{m+n}}\,d\nu(w)\,d\nu(\zeta)\right|\\&\le\int_{B^n}(1-|w|^2)^m|Df(w)|\left|\int_{B^n}\frac{(1-|\zeta|^2)^{\alpha-1}\,d\nu(\zeta)\,d\nu(w)}{(1-\langle\zeta,w\rangle)^{m+n}(1-\langle z,\zeta\rangle)^{\alpha+n+1}}\right|\\&\preceq\int_{B^n}\frac{(1-|w|^2)^m|Df(w)|}{|1-\langle z,w\rangle|^{m+n+1}}\,d\nu(w).\end{aligned}$$

By Lemma 1.5 we have

$$|\phi(z)|\le\|f\|_{B_\omega}(1-|z|^2)\int_{B^n}\frac{(1-|w|^2)^{m-1}\omega(1-|w|)}{|1-\langle z,w\rangle|^{m+n+1}}\,d\nu(w)\preceq\|f\|_{B_\omega}\omega(1-|z|).$$

Therefore $\phi\in L^\infty_\omega$. Next we show that $P_\alpha(\phi(z))\equiv f(z)$. We have

$$\begin{aligned}P_\alpha(\phi(z))&=C(\alpha,n)\int_{B^n}\frac{(1-|w|^2)^\alpha}{(1-\langle z,w\rangle)^{n+\alpha}}\int_{B^n}\frac{(1-|\zeta|^2)^{\alpha-1}f(\zeta)\,d\nu(\zeta)}{(1-\langle w,\zeta\rangle)^{n+\alpha+1}}\,d\nu(w)\\&=C(\alpha,n)\int_{B^n}(1-|\zeta|^2)^{\alpha-1}f(\zeta)\overline{\int_{B^n}\frac{(1-|w|^2)^\alpha\,d\nu(w)}{(1-\langle\zeta,w\rangle)^{n+\alpha+1}(1-\langle w,z\rangle)^{n+\alpha}}}\,d\nu(\zeta)\\&=C(\alpha,n)\int_{B^n}\frac{(1-|\zeta|^2)^{\alpha-1}f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+\alpha}}\,d\nu(\zeta)=f(z),\end{aligned}$$

where $C(\alpha,n)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$.

For the last equality we have used (3). $\square$

The next problem in which we are interested is the following: our aim is to find the inverse operator of P_α which maps B_ω to L^∞_ω. Furthermore, if this is the case, whether $P_\alpha(P^-_\alpha(f))(z) = f(z)$ $(z \in B^n)$ for all $f \in B_\omega$. The solution of this problem is positive. We consider the general operator

$$R_{\alpha,\beta} f(z) = (1-|z|^2)^\beta \int_{B^n} \frac{(1-|w|^2)^{\alpha-1} f(\zeta)}{(1-\langle z,\zeta\rangle)^{\alpha+\beta+n}}\, d\nu(\zeta), \quad \alpha+\beta > -1.$$

The following theorem holds.

Theorem 3.3. *Let $\alpha > \beta_\omega$ and $\beta > \alpha_\omega$. Then*

(a) *$P_\alpha R_{\alpha,\beta}(f)(z) \equiv f(z)$ $(z \in B^n)$ for all $f \in B_\omega$;*

(b) *the operator $R_{\alpha,\beta}$ is bounded from B_ω to L^∞_ω, and there exist constants $C_1(\omega)$, $C_2(\omega)$ such that*

$$C_1(\omega)\|f\|_{B_\omega} \le \|R_{\alpha,\beta} f\|_{L^\infty_\omega} \le C_2(\omega)\|f\|_{B_\omega}; \tag{7}$$

(c) *$f \in B_\omega$ if and only if $R_{\alpha,\beta} f \in L^\infty_\omega$.*

Proof: (a) We show that $P_\alpha R_{\alpha,\beta} f(z) = f(z)$, $z \in B^n$. To this end let us calculate $P_\alpha R_{\alpha,\beta} f(z)$ using the Fubini theorem:

$$\begin{aligned}
P_\alpha R_{\alpha,\beta} f(z) &= C(\alpha,n) \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha+\beta-1}}{(1-\langle z,\zeta\rangle)^{\alpha+n}} \int_{B^n} \frac{(1-|w|^2)^{\alpha-1} f(w)}{(1-\langle \zeta,w\rangle)^{\alpha+\beta+n+1}}\, d\nu(w)\, d\nu(\zeta) \\
&= C(\alpha,n) \int_{B^n} (1-|w|^2)^{\alpha-1} f(w) \overline{\int_{B^n} \frac{(1-|\zeta|^2)^{\alpha+\beta-1} d\nu(\zeta)}{(1-\langle w,\zeta\rangle)^{\beta+\alpha+n}(1-\langle \zeta,z\rangle)^{\alpha+n}}}\, d\nu(w) \\
&= \int_{B^n} \frac{(1-|w|^2)^{\alpha-1} f(w}{(1-\langle z,w\rangle)^{\alpha+n}}) d\nu(w) = f(z), \quad \alpha > \beta_\omega.
\end{aligned}$$

(We have used Lemma 2.4 and (3)).

(b) Let $f \in B_\omega$. Theorem 3.1 implies that there exists a function $\phi \in L^\infty_{\tilde\omega}$ such that $Q_\alpha \phi(z) = f(z)$ $(z \in U^n)$. Then by Fubini theorem, we get

$$\begin{aligned}
R_{\alpha,\beta} f(z) &= C(\alpha,n)(1-|z|^2)^\beta \int_{B^n} \phi(w) \int_{B^n} \frac{(1-|\zeta|^2)^{\alpha-1} d\nu(\zeta)\, d\nu(w)}{(1-\langle z,\zeta\rangle)^{\alpha+\beta+n}(1-\langle \zeta,w\rangle)^{\alpha+n}} \\
&= (1-|z|^2)^\beta \int_{B^n} \frac{\phi(w) d\nu(w)}{(1-\langle z,w\rangle)^{\alpha+\beta+n}}.
\end{aligned}$$

Therefore

$$\frac{|R_{\alpha,\beta} f(z)|}{\omega(1-|z|)} \preceq \|\phi\|_{L^\infty_{\tilde\omega}} (1-|z|^2)^\beta \int_{B^n} \frac{(1-|w|^2)^{\alpha-1}\omega(1-|w|)}{|1-\langle z,w\rangle|^{\alpha+\beta+n}}\, d\nu(w) \preceq \|\phi\|_{L^\infty_{\tilde\omega}};$$

in the last inequality we have used also Lemma 1.5.

So there exists a constant $C_2(\omega)$, such that

$$\frac{|R_{\alpha,\beta}f(z)|}{\omega(1-|z|)} \leq C_2(\omega)\|\phi\|_{L^\infty_\omega}\omega(1-|z|) \tag{8}$$

which shows that $R_{\alpha,\beta} \in L^\infty_\omega$.

Further, by Theorem 3.2 there exists $C_0(\omega) > 0$ such that

$$\|f\|_{B_\omega} = \|P_\alpha R_{\alpha,\beta}f\|_{B_\omega} \leq C_0(\omega)\|R_{\alpha,\beta}f\|_{L_\infty}.$$

Taking $C_1(\omega) = C_0^{-1}(\omega)$ we get

$$\|R_{\alpha,\beta}f\|_{L_\infty} \geq C_1(\omega)\|f\|_{B_\omega}. \tag{9}$$

By (8) and (9) we get the proof of (7).

(c) The proof follows from (7). □

Remark 3.4. Notice that the Bloch space B_ω is not separable. If we consider the subspace of B_ω of all functions $f \in B_\omega$ for which

$$\lim_{|z|\to 1-0} \frac{(1-|z|)}{\omega(1-|z|)}|Df(z)| = 0,$$

then we get a new separable space of holomorphic functions, called little Bloch space B^0_ω.

The little Bloch space is of independent interest (see [1], [4], [12]). Using standard arguments one can prove that

Proposition 3.5. *The following statements are true:*

(a) B^0_ω *is closed subspace of* B_ω*;*
(b) *the set of polynomials is dense in* B^0_ω.

In this paper we do not discuss other properties of this space. Based on the results of this paper we intend to write a separate paper about holomorphic weighted little Bloch spaces.

4. Linear continuous functionals on $A^p(\omega)$

In this section we describe the duals of $A^p(\omega)$ in terms of holomorphic Bloch space in the case if $0 < p \leq 1$. We need to establish the following lemmas before proving the duality result.

Lemma 4.1. *Let* $\omega \in S$, $f \in A^p(\omega)$, $0 < p < \infty$. *Then*

$$|f(z)| \preceq \frac{\|f\|_{A^p(\omega)}}{\omega^{1/p}(1-|z|)(1-|z|)^{(n+1)/p}}, \quad z \in B^n.$$

Proof: Let $z \in B^n$ and let $B_z^n(r)$ be the ball with the center z and radius $r = (1-|z|)/2$. If $w \in B_z^n(r)$, then

$$|w| \leq |w-z| + |z| \leq \frac{1-|z|}{2} + |z| = \frac{1+|z|}{2} < 1$$

which shows that $B_z^n(r) \subset B^n$. The function $|f|^p$ is subharmonic and we have

$$|f(z)|^p \leq \frac{1}{|B_z^n(r)|} \int_{B_z^n(r)} |f(w)|^p \, d\nu(w).$$

On the other hand it is not difficult to show that $1-|z| \asymp 1-|w|$. Then by Remark 1.2 we get also $\omega(1-|z|) \asymp \omega(1-|w|)$. Using the last fact we get

$$|f(z)|^p \omega(1-|z|) \preceq \frac{1}{|B_z^n(r)|} \int_{B_z^n(r)} |f(w)|^p \omega(1-|z|) \, d\nu(w) \leq \frac{\|f\|^p_{A^p(\omega)}}{|B_z^n(r)|}.$$

We have $|B_z^n(r)| \asymp (1-|z|)^{n+1}$. Then we get

$$|f(z)| \leq \frac{\|f\|_{A^p(\omega)}}{(1-|z|)^{(n+1)/p} \omega^{1/p}(1-|z|)}.$$

□

Lemma 4.2. *Let $\omega \in S$, $f \in A^p(\omega)$, $0 < p \leq 1$. Then*

$$\left(\int_{B^n} |f(z)| \frac{\omega^{1/p}(1-|z|)}{(1-|z|^2)^{(n+1)(1-1/p)}} \, d\nu(z) \right)^p \leq \int_{B^n} |f(z)|^p \omega(1-|z|) \, d\nu(z).$$

Proof: We have $|f(z)| = |f(z)|^p |f(z)|^{1-p}$. Then using Lemma 4.1, we get

$$|f(z)| \preceq \frac{\|f\|^{1-p}_{A^p(\omega)} |f(z)|^p}{\omega^{(1-p)/p}(1-|z|)(1-|z|)^{(1-p)(n+1)/p}}.$$

Therefore

$$|f(z)| \frac{\omega^{1/p}(1-|z|)}{(1-|z|)^{(n+1)(1-1/p)}} \preceq |f(z)|^p \|f\|^{1-p}_{A^p(\omega)} \omega(1-|z|),$$

and the integration gives us the proof of Lemma 4.2. □

The following theorem describes the continuous linear functionals on $A^p(\omega)$ in the case $0 < p \leq 1$.

Theorem 4.3. *Let $0 < p \leq 1$, $\omega \in S$. Then the dual of $A^p(\omega)$ under the pairing*

$$\langle f, g \rangle = \int_{B^n} f(z) \overline{g(z)} (1-|z|^2)^\alpha \, d\nu(z) \tag{10}$$

is isomorphic to B_{ω^*}, *where* $\omega^*(t) = \omega^{1/p}(t)t^{(n+1)(1/p-1)-\alpha}$ *and* $\alpha > \alpha_\omega/p + (n+1)(1/p-1)$.

Proof: Let Φ be a bounded linear functional on $A^p(\omega)$. Then using Lemma 4.2 we have

$$\left(\int_{B^n} |f(z)|\Omega(1-|z|)\,d\nu(z)\right)^p \leq \int_{B^n} |f(z)|^p\omega(1-|z|)\,d\nu(z),$$

where $\Omega(t) = \omega^{1/p}(t)t^{(n+1)(1/p-1)}$ and hence we get that Φ is also a bounded linear functional on $A^1(\Omega)$. As before we can regard $A^1(\Omega)$ as a subspace of $L^1(\Omega)$. Then by the Hahn-Banach theorem Φ can be regarded as element of $(L^1(\Omega))^*$. Next, we use the Riesz theorem: there exists a function $G \in L_\infty(B^n)$ such that

$$\Phi(f) = \int_{B^n} f(\zeta)\overline{G(\zeta)}\Omega(1-|\zeta|)\,d\nu(\zeta)$$

with $\|\Phi\| = \|G\|_{L_\infty(B^n)}$.

By Lemma 2.4 we have: if $\alpha > \max\{\alpha_\omega/p + (n+1)/(1/p-1),\ \beta_\omega - 1\}$ then $f \in A^1(\alpha)$. Therefore writing (3) for f and using also Fubini theorem, we get

$$\Phi(f) = \int_{B^n} (1-|t|^2)^\alpha f(t) \int_{B^n} \overline{G(\zeta)}\frac{\Omega(1-|\zeta|)\,d\nu(\zeta)}{(1-\langle\zeta,t\rangle)^{\alpha+n+1}}\,d\nu(t).$$

Let

$$g(t) = \int_{B^n} \overline{G(\zeta)}\frac{\Omega(1-|\zeta|)\,d\nu(\zeta)}{(1-\langle\zeta,t\rangle)^{\alpha+n+1}};$$

we show that $g \in B_{\omega^*}$. Using Lemmas 1.5, 4.2 we get

$$\begin{aligned}
|D^m g(t)| &\leq \int_{B^n} |G(\zeta)|\frac{\omega^{1/p}(1-|\zeta|)(1-|\zeta|)^{(n+1)(1/p-1)}}{|1-\langle\zeta,t\rangle|^{\alpha+n+m+1}}\,d\nu(\zeta)\\
&\leq \|G\|_{L_\infty(B^n)} \int_{B^n} \frac{\omega^{1/p}(1-|\zeta|)(1-|\zeta|)^{(n+1)(1/p-1)}}{|1-\langle\zeta,t\rangle|^{\alpha+n+m+1}}\,d\nu(\zeta)\\
&\leq \|G\|_{L_\infty(B^n)} \left(\int_{B^n} \frac{\omega(1-|\zeta|)d\nu(\zeta)}{|1-\langle\zeta,t\rangle|^{(\alpha+n+m+1)p}}\right)^{1/p}\\
&\preceq \|G\|_{L_\infty(B^n)} \frac{\omega^{1/p}(1-|t|)}{(1-|t|)^{\alpha+m-(n+1)(1/p-1)}}.
\end{aligned}$$

So we get

$$|D^m g(t)|\frac{(1-|t|)^m}{\omega^*(1-|t|)} \preceq \|G\|_{L_\infty(B^n)},$$

where $\omega^*(t) = \omega^{1/p}(t)t^{(n+1)(1/p-1)-\alpha}$, which shows that $g \in B_{\omega^*}$ and the functional Φ has the form

$$\Phi(f) = \int_{B^n} f(t)\overline{g(t)}(1-|t|^2)^\alpha \, d\nu(t).$$

Furthermore, there exists a constant $C_1 > 0$ such that

$$C_1\|g\|_{B_{\omega^*}} \le \|\Phi\|. \tag{11}$$

Conversely, let $\Phi(f)$ be defined by (10). We will show that Φ is a bounded functional on $A^p(\omega)$ and $g \in B_{\omega^*}$. By Theorem 3.1 there exists a function $h \in L^\infty_{\widetilde{\omega}}$ where $\widetilde{\omega}(t) = \omega^*(t)t^{\beta-1}$ $(\beta > \beta_\omega + 1)$ such that $Q_\beta(h)(z) = g(z)$. Then we get

$$\begin{aligned} I &\equiv \int_{B^n} (1-|\zeta|^2)^\alpha f(\zeta)\overline{\int_{B^n} \frac{h(t)d\nu(t)}{(1-\langle\zeta,t\rangle)^{n+\beta}}}\, d\nu(\zeta) \\ &= \int_{B^n} \overline{h(t)} \int_{B^n} \frac{(1-|\zeta|^2)^\alpha f(\zeta)}{(1-\langle t,\zeta\rangle)^{n+\beta}}\, d\nu(\zeta)\, d\nu(t). \end{aligned}$$

Therefore

$$|I| \le \|h\|_{L^\infty_{\widetilde{\omega}}} \int_{B^n} (1-|\zeta|^2)^\alpha |f(\zeta)| \int_{B^n} \frac{\omega^*(1-|t|)(1-|t|^2)^{\beta-1}}{|1-\langle t,\zeta\rangle|^{n+\beta}}\, d\nu(t)\, d\nu(\zeta).$$

Without loss of generality, we can take

$$\beta > \max\{\alpha_\omega/p - \alpha + (n+1)(1/p-1)+1,\ \beta_\omega/p + \alpha - (n+1)(1/p-1)\}.$$

Then by Lemma 1.5 we have

$$\begin{aligned} |I| &\le \|h\|_{L^\infty_{\widetilde{\omega}}} \int_{B^n} (1-|\zeta|^2)^{(n+1)(1/p-1)}\omega^{1/p}(1-|\zeta|)|f(\zeta)|\, d\nu(\zeta) \\ &\le \|h\|_{L^\infty_{\widetilde{\omega}}} \left(\int_{B^n} |f(\zeta)|^p \omega(1-|\zeta|)\, d\nu(\zeta)\right)^{1/p} = \|h\|_{L^\infty_{\widetilde{\omega}}} \|f\|_{A^p(\omega)}. \end{aligned}$$

Using the fact that $\|h\|_{L^\infty_{\widetilde{\omega}}} \le \|g\|_{B^*_\omega}$ we get

$$|\Phi(f)| \le C\|g\|_{B^*_\omega}\|f\|_{A^p(\omega)}. \tag{12}$$

Further, it is easy to show that the linear functional Φ is continuous on $A^p(\omega)$ if and only if

$$\|\Phi\| = \sup_{\|f\|_{A^p(\omega)}\le 1} |\Phi(f)\| < +\infty.$$

Then by (12) we get that $\Phi(f)$ is continuous on $A^p(\omega)$ and hence bounded. Furthermore there exists a constant $C_2 > 0$ such that

$$\|\Phi\| \le C_2\|g\|_{B^*_\omega}. \tag{13}$$

Using the inequalities (11) and (13) we finish the proof of our theorem. $\square$

Proposition 4.4. *Let $\widetilde{\omega}(t) = t^{-\alpha}\omega(t), \alpha > \max\{\alpha_\omega - 1,\ \beta_{\widetilde{\omega}} - 1\}$, $g \in B_{\widetilde{\omega}}$. Then there exists a function $G \in B_\omega$ such that*

$$g(z) = \int_{B^n} \frac{G(\zeta)\, d\nu(\zeta)}{(1 - \langle z, \zeta\rangle)^{\alpha+n+1}}\,. \tag{14}$$

Proof: Let $g \in B_{\widetilde{\omega}}$. Then the function $g_1(z) = (1-|z|^2)^{\alpha+1}Dg(z)$ belongs to the space L^∞_ω and, by Theorem 2.6, the function $g_2(z) = (1-|z|^2)^\alpha g(z)$ also belongs to L^∞_ω and $\|g_2\|_{L^\infty_\omega} \preceq \|g_1\|_{L^\infty_\omega}$. Taking

$$G(\zeta) = \int_{B^n} \frac{(1-|t|^2)^\alpha g(t)}{(1-\langle \zeta, t\rangle)^{n+1}}\, d\nu(t)$$

we get

$$\begin{aligned}
&\int_{B^n} \frac{G(\zeta)d\nu(\zeta)}{(1-\langle z,\zeta\rangle)^{\alpha+n+1}} \\
&\quad = \int_{B^n} (1-|t|^2)^\alpha g(t) \int_{B^n} \frac{d\nu(\zeta)\, d\nu(t)}{(1-\langle \zeta, t\rangle)^{n+1}(1-\langle z,\zeta\rangle)^{\alpha+n+1}} \\
&\quad = \int_{B^n} \frac{(1-|t|^2)^\alpha g(t)}{(1-\langle \zeta, t\rangle)^{\alpha+n+1}}\, d\nu(t) = g(z),
\end{aligned}$$

if $\alpha > \beta_{\widetilde{\omega}} - 1$. It remains to prove that $G \in B_{\widetilde{\omega}}$. We have

$$|DG(\zeta)| \preceq \|g_2\|_{L^\infty_\omega} \int_{B^n} \frac{\omega(1-|t|)\, d\nu(t)}{|1-\langle \zeta, t\rangle|^{n+2}} \le C\|g\|_{L^\infty_\omega} \frac{\omega(1-|\zeta|)}{(1-|\zeta|^2)}\,.$$

Hence $G \in B_\omega$. $\square$

Using Proposition 4.4 we have a new description of the space $A^p(\omega)$:

Theorem 4.5. *Let $0 < p \le 1$, $\omega \in S$. Then the dual of space $A^p(\omega)$ under the pairing*

$$\langle f, g\rangle = \int_{B^n} f(t)\overline{G(t)}\, d\nu(t)$$

is isomorphic to B_{ω^*}, where $\omega^*(t) = \omega^{1/p}(t)t^{(n+1)(1/p-1)}$.

Proof: Using Theorem 4.3 it is sufficient to prove that

$$\int_{B^n} f(t)\overline{g(t)}(1-|t|^2)^\alpha\, d\nu(t) = \int_{B^n} f(t)\overline{G(t)}\, d\nu(t).$$

To this end we use Proposition 4.4. We have with (3)

$$\int_{B^n} f(t)(1-|t|^2)^\alpha \overline{\int_{B^n} \frac{G(\zeta)\,d\nu(\zeta)}{(1-\langle t,\zeta\rangle)^{\alpha+n+1}}}\,d\nu(t)$$
$$= \int_{B^n} \overline{G(\zeta)} \int_{B^n} \frac{(1-|t|^2)^\alpha f(t)\,d\nu(t)}{(1-\langle \zeta,t\rangle)^{\alpha+n+1}}\,d\nu(\zeta) = \int_{B^n} f(t)\overline{G(t)}\,d\nu(t).$$

□

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